

MULTIPLICATIVE FUNCTIONS WITH SMALL INCREMENT II.

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Abstract. We prove that if f is a multiplicative function satisfying the relations

$$\overline{\lim}_{n \rightarrow \infty} \frac{1}{\log x} \sum_{n \leq x} \frac{|f(n)|}{n} = \infty, \quad \overline{\lim}_{n \rightarrow \infty} \frac{1}{\log x} \sum_{n \leq x} \frac{|f(n+K) - f(n)|}{n} < \infty,$$

then there are real numbers σ, t with $(0 < \sigma \leq 1)$ and a Dirichlet character $\chi \pmod{K}$ such that $f(n) = n^{\sigma+it} \chi(n)$.

1. Introduction

Let, as usual, \mathcal{P} , \mathbb{N} , \mathbb{R} , \mathbb{C} be the set of primes, positive integers, real and complex numbers, respectively. Let \mathcal{M} , \mathcal{M}^* be the set of complex-valued multiplicative (completely multiplicative) functions. We say that $f \in \mathcal{M}_1$ (resp. \mathcal{M}_1^*), if $f \in \mathcal{M}$ (resp. \mathcal{M}^*) and $|f(n)| = 1$ for every $n \in \mathbb{N}$. For an arithmetical function f , $f : \mathbb{N} \rightarrow \mathbb{C}$ let

$$\Delta_K f(n) := f(n+K) - f(n), \quad K \in \mathbb{N},$$

where $K \in \mathbb{N}$.

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In [1] we proved the following assertion.

Theorem A. *Let $f \in \mathcal{M}$, assume that*

$$(1.1) \quad \sum_{n \leq x} |\Delta_K f(n)| = O(x),$$

then, either

$$(1.2) \quad \sum_{n \leq x} |f(n)| = O(x),$$

or

$$f(n) = n^{\sigma+it} \chi(n) \quad \text{for every } n \in \mathbb{N}, (n, K) = 1,$$

where $0 < \sigma \leq 1, t \in \mathbb{R}$ and χ is a Dirichlet character (mod K).

In [1] and [7] the following result is proved.

Theorem B. *Assume that the functions $f, g \in \mathcal{M}$ and a number $K \in \mathbb{N}$ satisfy*

$$\sum_{n \leq x} |g(n+K) - f(n)| = O(x).$$

Then, either

$$\sum_{n \leq x} |g(n)| = O(x), \quad \sum_{n \leq x} |f(n)| = O(x)$$

or there are functions $F, G \in \mathcal{M}$ and a complex number s such that

$$f(n) = n^s F(n), \quad g(n) = n^s G(n), \quad 0 < \Re s \leq 1$$

and $G(n+K) - F(n) = 0$ for every $n \in \mathbb{N}$.

Let \mathcal{M}_K be the set of those multiplicative functions f , for which $f(n) = 0$ if $(n, K) > 1$.

Our purpose in this paper is to investigate those $f \in \mathcal{M}_K$ for which

$$(1.3) \quad \overline{\lim}_{n \rightarrow \infty} \frac{1}{\log x} \sum_{n \leq x} \frac{|f(n)|}{n} = \infty,$$

and

$$(1.4) \quad \overline{\lim}_{n \rightarrow \infty} \frac{1}{\log x} \sum_{n \leq x} \frac{|\Delta_K f(n)|}{n} < \infty.$$

Theorem 1. *Let $f \in \mathcal{M}_K$ be such a function for which (1.3), (1.4) hold. Then*

$$f(n) = n^{\sigma+it}\chi(n), \quad 0 \leq \sigma \leq 1, t \in \mathbb{R}$$

and χ is a Dirichlet character (mod K).

Remark 1. This theorem for $K = 1$ was proved in [3].

Remark 2. The proof is based upon a theorem of O. Klurman [5] which is referred here as

Lemma 1. *Let $f \in \mathcal{M}^*$, $|f(n)| = \begin{cases} 1 & \text{if } (n, K) = 1 \\ 0 & \text{if } (n, K) > 1 \end{cases}$*

and

$$(1.5) \quad \frac{1}{\log 2x} \sum_{n \leq x} \frac{|\Delta_K f(n)|}{n} \rightarrow 0 \quad (x \rightarrow \infty).$$

Then

$$f(n) = n^{i\tau}\chi(n) \quad \text{for every } n \in \mathbb{N}, \tau \in \mathbb{R},$$

where χ is a Dirichlet character (mod K). If f is such a function, then (1.5) holds true.

Remark 3. By using the method of J.-L. Maucclair and L. Murata [6], we have

Lemma 2. *If $f \in \mathcal{M}$,*

$$|f(n)| = \begin{cases} 1 & \text{if } (n, K) = 1 \\ 0 & \text{if } (n, K) > 1 \end{cases}$$

and (1.5) holds, then $f \in \mathcal{M}^*$, consequently

$$f(n) = n^{i\tau}\chi(n) \quad \text{for every } n \in \mathbb{N}, \tau \in \mathbb{R}$$

where χ is a Dirichlet character (mod K).

2. Proof of Lemma 2

Let q be a prime, $q \nmid K$. It is enough to prove that $f(q^k) = f(q)^k$ for every q and $k = 1, 2, \dots$

Let

$$S_\ell(q) = q^{\ell-1} + \cdots + q + 1 \quad (\ell = 1, 2, \dots).$$

Thus

$$S_\ell(q) = qS_{\ell-1}(q) + 1 \quad (\ell = 2, 3, \dots).$$

Since $|f(m)| = 1$ if $(m, qK) = 1$, therefore

$$(2.1) \quad \lim_{x \rightarrow \infty} \frac{1}{\log x} \sum_{\substack{m \leq x \\ (m, Kq) = 1}} \frac{|f(m)|}{m} = c_q > 0.$$

Let $A \in \mathbb{N}$. Since

$$|\Delta_{AK}f(n)| \leq \sum_{j=0}^{A-1} |\Delta_K f(m + jK)|,$$

therefore

$$(2.2) \quad \lim_{x \rightarrow \infty} \frac{1}{\log x} \sum_{\substack{m \leq x \\ (m, Kq) = 1}} \frac{|f(q^k m + KS_k(q)) - f(q^k)f(m)|}{mq} = 0.$$

Furthermore,

$$\begin{aligned} & f(q^\ell m + KS_\ell(q)) - f(q^\ell m + K(S_\ell(q) - 1)) = \\ & = f(q^\ell m + KS_\ell(q)) - f(q)f(q^{\ell-1}m + KS_{\ell-1}(q)). \end{aligned}$$

Applying this for $\ell = 1, \dots, k$, we obtain that

$$\frac{1}{\log x} \sum_{\substack{m \leq x \\ (m, Kq) = 1}} \frac{|f(q^k m + KS_k(q)) - f(q)^k f(m)|}{mq} \rightarrow 0,$$

which by (2.2) implies that

$$|f(q^k) - f(q)^k| \sum_{\substack{m \leq x \\ (m, Kq) = 1}} \frac{|f(m)|}{m} = O(\log x),$$

and this by (2.1) implies that $f(q^k) = f(q)^k$ for every $k \in \mathbb{N}$.

3. Proof of Theorem 1

If f is a solution, then $F(n) := |f(n)|$ is a solution also.

Let $p \in \mathcal{P}, (p, K) = 1, Q = p^\alpha$. For some $n \in \mathbb{N}$, let $n \equiv \ell_n K \pmod{Q}$, $\ell_n \in \{0, \dots, Q-1\}$ and let n_1 be defined by the relation $n = Qn_1 + \ell_n K$. Then

$$(3.1) \quad |F(n) - F(Qn_1)| = |\Delta_{K\ell_n} F(Qn_1)|.$$

If $p|n_1$, then $(p, n_1 - K) = 1$, and so

$$\begin{aligned} F(Qn_1) &= F(Q(n_1 - K)) + \Delta_{QK} F(Q(n_1 - K)) = \\ &= F(Q)F(n_1 - K) + \Delta_{QK} F(Q(n_1 - K)) = \\ &= F(Q)F(n_1) - F(Q)\Delta_K F(n_1 - K) + \Delta_{QK} F(Q(n_1 - K)), \end{aligned}$$

so, hence and from (3.1), we get

$$(3.2) \quad \begin{aligned} |F(n) - F(Q)F(n_1)| &\leq F(Q)|\Delta_K F(n_1 - K)| + \\ &\quad + |\Delta_{QK} F(Q(n_1 - K))| + |\Delta_{K\ell_n} F(Qn_1)|. \end{aligned}$$

Let us assume that Q is fixed. $n_1 = m$ at the mapping $n \rightarrow n_1$ occurs for Q distinct n , furthermore $n_1 \leq n/Q$ and $n_1 \geq 1$ if $n \geq QK$.

Since

$$|\Delta_{Ks} F(m)| \leq \sum_{j=0}^{s-1} |\Delta_K F(m + jK)|,$$

from (3.2) we obtain that

$$(3.3) \quad \frac{1}{\log x} \sum_{n \leq x} \frac{1}{n} |F(n) - F(Q)F(n_1)| < C$$

if $x \geq 2$. C may depend on Q .

Let

$$E_F(x) = \sum_{n \leq x} \frac{|F(n)|}{n}.$$

From (3.3) we obtain that

$$(3.4) \quad E_F(x) \leq F(Q)E_F\left(\frac{x}{Q}\right) + c_1,$$

and

$$(3.5) \quad F(Q)E_F\left(\frac{x - QK}{Q}\right) - C \leq E_F(x).$$

If $F(Q) < 1$, then from (3.4) we obtain that $\limsup E_F(x) < \infty$, which contradicts our assumption. Thus $F(n) \geq 1$ for every n coprime to K .

Let $F(Q) = Q^\eta$, $\eta \geq 0$. If $\eta = 0$, then $E_F(x) = O(\log x)$ easily follows from (3.4).

Assume that $\eta > 0$. Let $A > C + 1$, x_0 be so large that $E_F(x_0) \geq A$. Let

$$x_{\nu+1} = Qx_\nu + QK \quad (\nu = 1, 2, \dots).$$

Thus from (3.5)

$$E_F(x_{\nu+1}) \geq Q^\eta E_F(x_\nu) - E_F(x_{\nu+1}) \geq Q^\eta E_F(x_\nu) - C \quad (\nu = 1, 2, \dots),$$

and this implies that

$$E_F(x_{\nu+1}) \geq c_2 Q^{(\nu+1)\eta},$$

where c_2 is a positive constant, and so

$$E_F(x) > c_3 x^\eta$$

with some positive constant c_3 if x is large.

On the other hand, from (3.4) we obtain that

$$E_F(x) \leq c_4 x^\eta$$

with some positive constant c_4 if x is large. Hence we obtain that

$$F(Q) = Q^\eta \quad \eta \in \mathbb{R}, \eta \geq 0,$$

consequently η is a constant,

$$F(n) = \begin{cases} n^\eta & \text{if } (n, K) = 1 \\ 0 & \text{if } (n, K) > 1. \end{cases}$$

If $\eta = 0$, from (3.4) we obtain that $E_f(x) = O(\log x)$.

Since $\Delta_K f(n) \asymp n^{\eta-1}$, therefore $\eta \leq 1$.

Let us write $g(n) = f(n) \cdot n^{-\eta}$. Then $g(n)$ is multiplicative, $|g(n)| = 1$ if $(n, K) = 1$ and $g(n) = 0$, if $(n, K) > 1$.

We have

$$\begin{aligned} \Delta_K f(n) &= (n+K)^\eta g(n+K) - n^\eta g(n) = \\ &= n^\eta \Delta_K g(n) + \left[(n+K)^\eta - n^\eta \right] g(n+K), \end{aligned}$$

and so

$$\sum_{n \leq x} \frac{n^\eta |\Delta_K g(n)|}{n} \leq \sum_{n \leq x} \frac{|\Delta_K f(n)|}{n} + \sum_{n \leq x} \frac{C n^{\eta-1}}{n} = O(\log x).$$

Consequently we obtain that

$$\frac{1}{\log x} \sum_{\substack{n \leq x \\ (n, K)=1}} \frac{|\Delta_K g(n)|}{n} \rightarrow 0 \quad (x \rightarrow \infty).$$

From Lemma 1, 2 we obtain that

$$g(n) = n^{i\tau} \chi(n), \quad \tau \in \mathbb{R},$$

where χ is a Dirichlet character (mod K).

Theorem 1 is proved. ■

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