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MULTIPLICATIVE FUNCTIONS WITH SMALL INCREMENT I.

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Abstract. We prove that if f is a completely multiplicative function and

$$\sum_{n\leq x}\frac{|f(n+1)-f(n)|}{n}=O(\log x),$$

then either

$$\sum_{n\leq x}\frac{|f(n)|}{n}=O(\log x)\quad\text{or}\quad f(n)=n^{\sigma+it}\quad 0<\sigma\leq 1, t\in\mathbb{R}.$$

1. Introduction

Let, as usual, $\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$ be the set of positive integers, real and complex numbers, respectively. Let $\mathcal{M}$, $\mathcal{M}^*$ be the set of complex-valued multiplicative (completely multiplicative) functions. We say that $f\in\mathcal{M}_1$ (resp. $\mathcal{M}_1^*$), if $f\in\mathcal{M}$ (resp. $\mathcal{M}^*$) and $|f(n)|=1$ for every $n\in\mathbb{N}$. Let $\mathcal{A}$, $\mathcal{A}^*$ be the set of real-valued additive (completely additive) functions.

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Let E be the set of those arithmetical function $e(n)$ $(n \in \mathbb{N})$, for which

$$\sup_{x \geq 1} \frac{1}{\log x} \sum_{n \leq x} \frac{|e(n)|}{n} < \infty. \tag{1.1}$$

Our purpose in this short paper is to prove the next assertion.

Theorem 1. *Let $f \in \mathcal{M}^*$, $\delta(n) = f(n+1) - f(n)$ $(n \in \mathbb{N})$. Assume that $\delta \in E$ and that $f \notin E$. Then*

$$f(n) = n^{\sigma + it}, \quad 0 < \sigma \leq 1, t \in \mathbb{R}.$$

The proof is based upon an important theorem due to O. Klurman which is cited now as

Lemma 1. (O. Klurman [4]) *If $f \in \mathcal{M}_1$ and*

$$\lim_{x \to \infty} \frac{1}{\log x} \sum_{n \leq x} \frac{|f(n+1) - f(n)|}{n} = 0, \tag{1.2}$$

then

$$f(n) = n^{it} \quad (n \in \mathbb{N}), \quad \tau \in \mathbb{R}.$$

A weaker assertion has been proved in [3], namely that, if $f \in \mathcal{M}$ and

$$\lim_{x \to \infty} \frac{1}{x} \sum_{n \leq x} |f(n+1) - f(n)| = 0,$$

then either

$$\lim_{x \to \infty} \frac{1}{x} \sum_{n \leq x} |f(n)| = 0 \quad \text{or} \quad f(n) = n^{\sigma + it} \quad 0 < \sigma \leq 1, t \in \mathbb{R}.$$

Remark 1. O. Klurman proved Lemma 1 for $f \in \mathcal{M}_1^*$, but using the method of Mauclaire and Murata [5] one can prove that if (1.2) holds for $f \in \mathcal{M}_1$, then $f \in \mathcal{M}_1^*$.

Remark 2. In [1] and [6] the following assertion has been proved. Assume that $f, g \in \mathcal{M}$, for which

$$\sum_{n \leq x} |g(n+K) - f(n)| = O(x),$$

where K is an arbitrary positive integer. Then, either

$$\sum_{n\le x}|g(n)|=O(x)\quad\text{and}\quad\sum_{n\le x}|f(n)|=O(x),$$

or

$$f(n)=n^sU(n),g(n)=n^sV(n)\quad 0<\mathrm{Re}\,s\le 1,$$

furthermore

$$V(n+K)=U(n)\quad\text{for every}\quad n\in\mathbb{N}.$$

Lemma 2. *If $f\in\mathcal{M}^*$ and* (1.2) *holds, then either*

$$\lim_{x\to\infty}\frac{1}{\log x}\sum_{n\le x}\frac{|f(n)|}{n}=0,$$

or

$$f(n)=n^{\sigma+it}\quad(n\in\mathbb{N}),\quad 0<\sigma\le 1,t\in\mathbb{R}.$$

Proof. Let

$$\Delta_Kf(n)=\max_{|k|\le K}|f(n+k)-f(n)|.$$

From (1.2) we have that

(1.3) $$\lim_{x\to\infty}\frac{1}{\log x}\sum_{n\le x}\frac{|\Delta_Kf(n)|}{n}=0$$

for every fixed K.

Assume first that there exists such a prime power $Q=q^\ell$ for which $|f(Q)|=$ $=\rho<1$.

Let

$$n=Qn_1+a_0,\quad\text{if}\quad 0\le a_0<Q,q\nmid n_1$$

and

$$n=(a_0+Q)+Q(n_1-1),\quad\text{if}\quad 0\le a_0<Q,q\mid n_1.$$

Then

$$|f(n)|\le|\Delta_Kf(n)|+\rho|f(n_1)|$$

in the first case, and

$$|f(n)|\le|\Delta_Kf(n)|+\rho|f(n_1)|+|\Delta_Kf(n_1)|$$

in the second case. Thus

$$s(x):=\sum_{n\le x}\frac{|f(n)|}{n}\le\rho\sum_{\substack{n_1\le\frac{x}{Q}\\ q\nmid n_1}}\frac{|f(n_1)|}{n_1}+\rho\sum_{\substack{n_1\le\frac{x}{Q}\\ q|n_1}}\frac{|f(n_1)|}{n_1}+\epsilon(x)\log x,$$

where $\epsilon(x)\searrow 0$ as $x\to\infty$.

Thus

$$s(x)\le\rho s(\frac{x}{Q})+\epsilon(x)\log x \tag{1.4}$$

for every $x\ge Q$. This implies that $|f(n)|\ge 1$ for every $n\in\mathbb{N}$.

If $Q^r\le x<Q^{r+1}$, then

$$s(x)<\rho^{r-1}s(Q)+\sum_{\ell=0}^{r-1}\rho^{\ell}\epsilon(\frac{x}{Q^{\ell}})\log\frac{x}{Q^{\ell}},$$

consequently

$$\frac{s(x)}{\log x}\to 0\quad(x\to\infty).$$

Let us assume now that $F(n)=|f(n)|\ge 1$ ($n\in\mathbb{N}$). Let $u(n)=\log F(n)$. Then $u(n)\ge 0$, $u\in\mathcal{A}^*$,

$$|u(n+1)-u(n)|\le|F(n+1)-F(n)|\le|f(n+1)-f(n)|,$$

consequently

$$\lim_{x\to\infty}\frac{1}{\log x}\sum_{n\le x}\frac{|u(n+1)-u(n)|}{n}=0.$$

Hence, applying a theorem of Kátai [2] we obtain that $u(n)=\sigma\log n$. Then

$$F(n)=n^{\sigma},F(n+1)-F(n)=\sigma n^{\sigma-1}+O(n^{\sigma-2})$$

and

$$\sum_{n\le x}\frac{|F(n+1)-F(n)|}{n}=o(\log x)$$

implies that $\sigma<1$. Let $f(n)=n^{\sigma}g(n)$, $g\in\mathcal{M}_1^*$. If $g\in\mathcal{M}_1^*$, then

$$\liminf_{x\to\infty}\frac{1}{\log x}\sum_{n\le x}\frac{|g(n+1)-g(n)|}{n}=0,$$

and Lemma 1 implies Lemma 2. ∎

2. Proof of Theorem 1

Let
$$h(x)=\sum_{n\le x}\frac{|f(n)|}{n}.$$

Assume first that there exists a prime q such that $|f(q)|=\rho<1$. From (1.1) we obtain that
$$\sum_{n\le x}\frac{|\Delta_K f(n)|}{n}=O(\log x),$$

if K is fixed, and so
$$h(x)\le\sum_{n\le x}\frac{q|f(q)||f(n_1)|}{qn_1}+O(\log x).$$

Then
$$h(x)\le\rho h(\frac{x}{q})+B\log x\quad(x\ge 2q),$$

whence we obtain that $h(x)=O(\log x)$, $f\in E$.

We can assume that $|f(q)|\ge 1$ holds for every prime q.

Let $F(n):=|f(n)|$. Then $F(n+1)-F(n)\in E$. Let q_1,q_2 be primes, $F(q_1)=q_1^{\lambda_1}$, $F(q_2)=q_2^{\lambda_2}$. We shall prove that $\lambda_1=\lambda_2$.

Assume indirectly that $\lambda_1<\lambda_2$.

It is clear that
$$h(x)=q_1^{\lambda_1}h(\frac{x}{q_1})+B_1\log x,\quad B_1\quad\text{is bounded,}$$

$$h(x)=q_2^{\lambda_2}h(\frac{x}{q_2})+B_2\log x,\quad B_2\quad\text{is bounded.}$$

Hence
$$h(x)=q_1^{k_1\lambda_1}h(\frac{x}{q_1^{k_1}})+O(q_1^{k_1\lambda_1}\log x)$$

and
$$h(x)=q_2^{k_2\lambda_2}h(\frac{x}{q_2^{k_2}})+O(q_2^{k_2\lambda_2}\log x).$$

If k_1, k_2 are appropriately chosen, then
$$q_2^{k_2}<q_1^{k_1}<q_2^{k_2+1},q_1^{k_1\lambda_1}<\frac{1}{4}q_2^{k_2\lambda_2},$$

so

$$h(\frac{x}{q_1^{k_1}}) = h(x)q_1^{-k_1\lambda_1} + O(\log x)$$

and

$$h(\frac{x}{q_2^{k_2}}) = h(x)q_2^{-k_2\lambda_2} + O(\log x),$$

consequently

$$0 \le h(\frac{x}{q_2^{k_2}}) - h(\frac{x}{q_1^{k_1}}) = \Big(\frac{1}{q_2^{k_2\lambda_2}} - \frac{1}{q_1^{k_1\lambda_1}}\Big)h(x) + O(\log x),$$

whence

$$-\frac{3}{4}h(x) + O(q_1^{k_1\lambda_1}\log x) \ge 0.$$

Thus $h(x) = O(\log x)$, which contradicts to our assumption.

Hence we have $F(n) = n^\lambda$. If $\lambda = 0$, then $f \in E$. Since $F(n+1) - F(n) \in E$, we obtain that $0 < \lambda \le 1$. Let

$$f(n) = n^\lambda U(n), U \in \mathcal{M}_1^*.$$

Then

$$|U(n+1) - U(n)| = \Big|\frac{f(n+1)}{(n+1)^\lambda} - \frac{f(n)}{(n)^\lambda}\Big| \le \frac{|f(n+1) - f(n)|}{n^\lambda} + \Big|1 - (1 + \frac{1}{n})^\lambda\Big|.$$

Thus

$$\sum_{n\le x}\frac{|U(n+1) - U(n)|}{n} \le \sum_{n\le x}\frac{|f(n+1) - f(n)|}{n^{\lambda+1}} + c\sum_{n\le x}\frac{1}{n^2}$$

the right hand side is bounded as $x \to \infty$. The conditions of Lemma 1 hold for U, thus $U(n) = n^{it}, t \in \mathbb{R}$.

This completes the proof of Theorem 1. ■

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