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LIMIT SETS OF GRAPH-DRIVEN ITERATED (MULTI)FUNCTION SYSTEMS

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Abstract. Our paper focuses on a new approach to the construction of graph sequences driven using iterated function systems (briefly IFSs) and iterated multifunction systems (briefly IMSs). We build self-similar graphs sequences generated initally based on an arbitrary graph. We analyze the graphs sequences gotten within the iterations of IFSs and IMSs. We specially analyze the density properties between those graphs which are in the same sequence and we also focus on graph sequences driven by Erdős–Rényi graphs.
The main aim of our paper is to analyze the limit sets of the graph-driven IFSs and IMSs. We characterize them using the Sierpinski carpet and the $[0,1]^2$ set which we interpret as limits of the constructed graph sequences.

1. Preliminaries and notations

The purpose of this paper is to present a connection between the fixed point theory and graph sequences. Using results on IMSs ([4], [9]) we construct graph sequences such that the adjacency matrices of the graphs will be generated by the iterations.

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Mandelbrot and Vicsek introduced a construction for directed and self-similar fractal sequences [7]. The aim of paper is to construct and analyze self-similar graph sequences using Erdős–Rényi graphs.

Let us note a simple graph $G(V, E)$, where V is the set of the nodes and $E \subseteq V \times V$ is the set of the undirected edges such that for all $(x, y) \in E$ there exist the (y, x) edge too, where $x \neq y$ and $x, y \in V$. Moreover, let us denote with $n = |V|$ the number of the nodes in the given G graph.

The Erdős–Rényi model (see [3], [5]) has two parameters: the n number of the nodes in the graphs and the p probability of the connections. So, the construction process fixes the nodes and adds all the edges with independently probability.

Based on results known of IMSs (see [4], [9]) let us use the following definitions: we refer to $f = (f_1, f_2, \dots, f_m)$ as an iterated function system (IFS), where the f_i's are singlevalued continuous self operators on a complete metric space X. The fractal operator generated by f is the following

$$T_f(Y) = \bigcup_{i=1}^{m} f_i(Y), \text{ for each } Y \in P_{cp}(X).$$

A fixed point of T_f is called a self-similar set of f, which is a fractal, if it has a non-integer Hausdorff dimension.

Moreover, for the contractions $f_1, f_2, \dots f_m$ T_F is also a contraction and it has an $A^* \in P_{cp}(X)$ fixed set. Moreover, if these contractions are similarity mappings, then A^* is a fractal (see [12]) and for any nonempty subset $A \subseteq X$, the $T_f^n(A) \to A^*$ as $n \to +\infty$.

If $F_1, F_2, \dots, F_m : X \to P_{cp}(X)$ are multivalued operators on the metric space, then the $F = \{F_1, F_2, \dots, F_m\}$ is an iterated multifunction system (IMS). If the F_i operators are upper semicontinuous, then the $T_F : P_{cp}(X) \to$ $\to P_{cp}(X)$ given by

$$T_F(Y) = \bigcup_{i=1}^{m} F_i(Y), \text{ for each } Y \in P_{cp}(X).$$

is called the fractal operator generated by the F.

We call a given element $x \in X$ as a fixed point of T if and only if $T(x) \in X$. Let us note the set of fixed point with $Fix(T) = \{x \in X | x \in T(x)\}$ which we call as fixed set too. For multivalued contractions the same results hold (see [8]).

Let be X the compact subset $[0, 1]^2$.

The aim of this paper is to present a new construction method for self-similar graph sequences using IFSs and IMSs. We init from an arbitrary graph and its adjacency matrix defined projected to $[0,1]^2$ and we generate the next graph using IFSs and IMSs as the followings.

Just like in [11], we construct graph sequences based initially on an arbitrary graph $G(V,E)$ with $n=|V|$ nodes such that the k^{th} element of the sequence is a graph with n^k nodes.

Let us consider a simple graph $G(V,E)$, where $n=|V|$ and let be $T:[0,1]^2\to[0,1]^2$. Our construction says that an undirected $(i,j)\in E$ edge exists if and only if $\left(\left[\frac{i-1}{n},\frac{i}{n}\right],\left[\frac{j-1}{n},\frac{j}{n}\right]\right)\subseteq T([0,1]^2)$ and $\left(\left[\frac{j-1}{n},\frac{j}{n}\right],\left[\frac{i-1}{n},\frac{i}{n}\right]\right)\subseteq$ $\subseteq T([0,1]^2)$, respectively.

We apply the $f_1,f_2,\dots f_m:[0,1]^2\to[0,1]^2$ mappings on the adjacency matrix and we get that an $(i,j)\in E$ exists at the k^{th} graph in the sequence generated by G if and only if $\left(\left[\frac{i-1}{n^k},\frac{i}{n^k}\right],\left[\frac{j-1}{n^k},\frac{j}{n^k}\right]\right)\subseteq T^k([0,1]^2)$ and $\left(\left[\frac{j-1}{n^k},\frac{j}{n^k}\right],\left[\frac{i-1}{n^k},\frac{i}{n^k}\right]\right)\subseteq T^k([0,1]^2)$, respectively.

Our paper focuses on the construction of such IFSs and IMSs, which generate adjacency matrices of graph sequences and we interpret the limit sets of these as the limit of the graph sequences. We define IFSs driven by graphs and we construct IMSs using set operations on these such that the iterations give the adjacency matrix of the graphs in the sequence.

The study of graph limits is known by testing homomorphisms in graphs sequences (see [6]). The main aim of this paper is to create a new connection between graph limits and fixed point theory. We know that the fixed sets of IMSs corresponding to self-similar networks introduced Barabási, Ravasz and Vicsek (see [1] and [10]) can be analyzed using the Cantor set (see [11]).

Let us consider a simple graph $G=(V,E)$ with $n=|V|$ nodes and let be f an IFS, which will generate the adjacency matrices of the sequences. Thus, let us note the graphs of the sequence as the followings.

$$f^1(G)=f(G),f^2(G),f^3(G),\dots,f^k(G),\dots.$$

Based on these results, we specially focus on graph sequences driven by Erdős–Rényi graphs. We note the elements of a sequence based on the IFS f and on an arbitrary Erdős–Rényi graph as the followings.

$$f^1(ER(n,p))=f(ER(n,p)),f^2(ER(n,p))\dots,f^k(ER(n,p)),\dots.$$

We use the concept for referring to the graph sequences generated using an arbitrary IMS F. For instance, we note the graphs of sequence based on the G graph as the followings.

$$F^1(G)=F(G),F^2(G),F^3(G),\dots,F^k(G),\dots.$$

If G is an $ER(n,p)$ Erdős-Rényi graph, then the graphs of the sequence are noted as

$$F^1(ER(n,p))=F(ER(n,p)),F^2(ER(n,p)),\dots,F^k(ER(n,p)),\dots.$$

We note that for each given $G = (V, E), n = |V|$ this construction always init from $f(G) = G$.

We analyze the density properties in the graph sequences and we also show that the limit sets of IFSs and IMSs can be described using the Sierpinski carpet and the $[0, 1]^2$ set.

2. Construction of iterated (multi)function systems driven by Erdős–Rényi graphs

We introduce the generation of the graph sequences using IFSs and IMSs. Firstly, we define those edge-directed functions which we use at the construction. We also introduce a class of functions which we call as diagonal-functions in the next. Secondly, we introduce the IFSs and the IMSs which generate graph sequences and we also focus on graph sequences driven by Erdős–Rényi graphs.

Based on the graph G, let consider the following edge-directed $f_{ij} : [0, 1]^2 \to \to [0, 1]^2$ mappings

$$f_{ij}(x, y) = \Big(\frac{i-1}{n}, \frac{j-1}{n}\Big) + \frac{1}{n}(x, y), \quad \forall (i, j) \in E.$$

Let define the following $f_i : [0, 1]^2 \to [0, 1]^2$ functions corresponding to the loops:

$$f_i(x, y) = \Big(\frac{i-1}{n}, \frac{i-1}{n}\Big) + \frac{1}{n}(x, y), \quad i = 1, 2, \ldots, n.$$

We refer to these the functions as diagonal-functions in the next.

Let consider the IFSs $f(G) = \{\{f_{ij} | (i, j) \in E\} \cup \{f_i | i = 1, 2, \ldots, n\}\}$ constructed by the f_{ij} and f_i functions and let be $T_{f(G)} : P_{cp}([0, 1]^2) \to: P_{cp}([0, 1]^2)$ the corresponding fractal operators as the followings.

$$T_{f(G)}(Y) = \Big(\bigcup_{(i,j)\in E} f_{ij}(Y)\Big) \cup \Big(\bigcup_{i=1}^{n} f_i(Y)\Big), \text{ for each } Y \in P_{cp}([0, 1]^2).$$

Let define and note the k^{th} element of the $F(G) : [0, 1]^2 \to [0, 1]^2$ graph-directed IMSs' sequences as the followings.

$$F(G)^k(x, y) = \bigcup_{i=1}^{n} \bigcup_{j=1}^{n} \Big[\Big(\frac{i-1}{n}, \frac{j-1}{n}\Big) + \frac{1}{n} F(G)^{k-1}(x, y)\Big].$$

We define the corresponding fractal operators $T_{F(G)} : P_{cp}([0,1]^2) \rightarrow P_{cp}([0,1]^2$ as

$$T_{F(G)}(Y) = F(Y).$$

In the next section we show that these graph-directed IFSs and IMSs generate the adjacency matrices of graph sequences on $[0,1]^2$. We base the sequence on arbitrary G graphs such that their first iteration always generates the adjacency matrix of G.

We analyze the density properties in the graph sequences generated for arbitrary graphs and we also focus on that case when the initial graph G is an Erdős-Rényi graph such that n denotes the number of the nodes and p is the independent probability of the edges.

We note the elements of the graphs sequences based on G and driven by the fractal operator of f and F as

$$f^1(G) = f(G), f^2(G), \dots, f^k(G), \dots,$$

and

$$F^1(G) = F(G), F^2(G), \dots, F^k(G), \dots, \text{ respectively.}$$

Thus, we focus specially on that cases, when the initially graph is an Erdős–Rényi graph.

3. Density properties in the IFS- and IMS-driven graph sequences

In this section we focus on the analyze of the density properties in the graph sequences driven by the defined graph-driven IFSs and IMSs.

Let be $G = (V, E)$ a simple graph such that $n = |V|$. Let define the density of G as $d(G) = \frac{|E|}{n^2}$.

We describe the density properties in the graph sequences corresponding to the iterated function system f, the iterated multifunction system F and we also show that the IFS and the IMS generate self-similar graph sequences.

Theorem 3.1. *For a given simple graph $G = (V, E), n = |V|$ the*

$$T^1_{f(G)}([0,1]^2), T^2_{f(G)}([0,1]^2), \dots, T^k_{f(G)}([0,1]^2), \dots$$

sequence generates the adjacency matrices of a graph sequence such that the density of the $f^k(G)$ graph given by $T^k_{f(G)}([0,1]^2)$ is

$$d(f^k(G)) = \frac{(n + |E|)^k}{(n^k)^2},$$

for all $n \in \mathbb{N}^*$.

Proof. We use mathematical induction for showing that $T^k_{f(G)}([0,1]^2)$ gives the adjacency matrix of $f^k(G)$ and we also use induction for showing that $d(f^k(G)) = \frac{(n+|E|)^k}{(n^k)^2}$ for all $n \in \mathbb{N}^*$.

The initial set is $[0,1]^2$ and we check that $T_{f(G)}([0,1]^2)$ generates the adjacency matrix of G. Here we apply the f on the $[0,1]^2$ set, so we get the following set as $T_{f(G)}([0,1]^2)$:

• For all $(i,j) \in E$, the f_{ij} gives us that $\left[\frac{i-1}{n}, \frac{j-1}{n}\right]$ little square which guarantees the existence of the edge (i,j). G is a simple graph, so the existence of the function f_{ji} and the corresponding (j,i) edge are also showed.

• The f_i functions from the IFS f gives the existence of the loops, which don't modify essentially the G graph.

Thus, the fractal operator given by the IFS f generates the adjacency matrix of the G graphs, which we note as $f(G)$ in the following.

Moreover, $d(f(G)) = n + |E|$, because the $f_i, i = 1,2,\dots,n$ mappings give us n degrees and we have $|E|$ edges. Thus, we showed that $T_{f(G)}([0,1]^2)$ corresponds to $f(G)$, whose density is $d(f(G)) = \frac{(n+|E|)^1}{(n^1)^2}$

Secondly, let suppose that $T^k_{f(G)}([0,1]^2)$ generates the adjacency matrix of the k^{th} element is the graph sequence and its density is equal with $d(f^k(G)) = = \frac{(n+|E|)^k}{(n^k)^2}$.

We show that $f^{k+1}(G)$ is generated by $T^{k+1}_{f(G)}([0,1]^2)$ such that it is constructed by using n replicas of $f^k(G)$ and some more edges between the nodes of these replicas. We also show that density of $f^{k+1}(G)$ is equal with $d(f^{k+1}(G)) = = \frac{(n+|E|)^{k+1}}{(n^{k+1})^2}$.

Based on that $T^k_{f(G)}([0,1]^2)$ generates the adjacency matrix of $f^k(G)$ we need show that T also generates from $T^k_{f(G)}([0,1]^2)$ a graph which we interpret as $f^{k+1}(G)$.

Let be $\left\{\left(\left[\frac{u-1}{n^k}, \frac{u}{n^k}\right], \left[\frac{v-1}{n^k}, \frac{v}{n^k},\right]\right), \left(\left[\frac{v-1}{n^k}, \frac{v}{n^k}\right], \left[\frac{u-1}{n^k}, \frac{u}{n^k},\right]\right)\right\} \subseteq T^k_{f(G)}([0,1]^2)$ such that $u, v \in \{1,2,\dots,n^k\}$, which means that (u,v) is an undirected edge in $f^k(G)$ such that it has n^k nodes.

Let apply the elements of the IFS f on these sets, where f_i gives us the

following sets:
(3.1)

$$f_i(\Big(\Big[\frac{u-1}{n^k},\frac{u}{n^k}\Big],\Big[\frac{v-1}{n^k},\frac{v}{n^k}\Big]\Big) =$$
$$= \Big(\frac{i-1}{n},\frac{i-1}{n}\Big) + \frac{1}{n}\Big(\Big[\frac{u-1}{n^k},\frac{u}{n^k}\Big],\Big[\frac{v-1}{n^k},\frac{v}{n^k},\Big] =$$
$$= \Big(\frac{i-1}{n},\frac{i-1}{n}\Big) + \Big(\Big[\frac{u-1}{n^{k+1}},\frac{u}{n^{k+1}}\Big],\Big[\frac{v-1}{n^{k+1}},\frac{v}{n^{k+1}},\Big]\Big) =$$
$$= \Big(\Big[\frac{n^k(i-1)+(u-1)}{n^{k+1}},\frac{n^k(i-1)+u}{n^{k+1}}\Big],\Big[\frac{n^k(i-1)+(v-1)}{n^{k+1}},\frac{n^k(i-1)+v}{n^{k+1}}\Big]\Big)$$

and
(3.2)

$$f_i(\Big(\Big[\frac{v-1}{n^k},\frac{v}{n^k}\Big],\Big[\frac{u-1}{n^k},\frac{u}{n^k}\Big]\Big) =$$
$$= \Big(\Big[\frac{n^k(i-1)+(v-1)}{n^{k+1}},\frac{n^k(i-1)+v}{n^{k+1}}\Big],\Big[\frac{n^k(i-1)+(u-1)}{n^{k+1}},\frac{n^k(i-1)+u}{n^{k+1}}\Big]\Big),$$
$$\text{for all } u,v \in \{1,2,\ldots,n^k\} \text{ and } i = 1,2,\ldots,= n.$$

It is easy to check that $n^k(i-1)+(u-1), n^k(i-1)+(v-1)\} \in \{0,1,\ldots,n^{k+1}-$ $-1\}$ and $\{n^k(i-1)+u, n^k(i-1)+v\} \in \{1,2,\ldots,n^{k+1}\}$, what means that f_i transforms the sets generated by the (u,v) edge from $f^k(G)$ to n edges from $f^{k+1}(G)$.

Let index the nodes of $f^k(G)$ as $1,2,\ldots,n^k$.

We highlight that the elements f_i transform the (u,v) edge from $f^k(G)$ to the $(n^k(i-1)+u, n^k(i-1)+v)$ edges, for all $u,v = 1,2,\ldots,n^k$ and $i = 1,2,\ldots,n$. For an arbitrary fixed f_i we get a replica of $f^k(G)$ which is now part of $f^{k+1}(G)$.

The self-similar property of the graph sequence is showed, because the using of the f_i transformations causes that the $(k+1)^{\text{th}}$ element of the graphs sequence is constructed by n replicas of the k^{th} element in the same sequence.

On the other hand, the functions f_{ij} transform the sets corresponding to the undirected (u,v) edge from $f^k(G)$ as the followings.

(3.3)

$$f_{ij}(\Big(\Big[\frac{u-1}{n^k},\frac{u}{n^k}\Big],\Big[\frac{v-1}{n^k},\frac{v}{n^k}\Big]\Big) =$$
$$= \Big(\frac{i-1}{n},\frac{j-1}{n}\Big) + \frac{1}{n}\Big(\Big[\frac{u-1}{n^k},\frac{u}{n^k}\Big],\Big[\frac{v-1}{n^k},\frac{v}{n^k},\Big] =$$
$$= \Big(\frac{i-1}{n},\frac{j-1}{n}\Big) + \Big(\Big[\frac{u-1}{n^{k+1}},\frac{u}{n^{k+1}}\Big],\Big[\frac{v-1}{n^{k+1}},\frac{v}{n^{k+1}},\Big]\Big) =$$
$$= \Big(\Big[\frac{n^k(i-1)+(u-1)}{n^{k+1}},\frac{n^k(i-1)+u}{n^{k+1}}\Big],\Big[\frac{n^k(j-1)+(v-1)}{n^{k+1}},\frac{n^k(j-1)+v}{n^{k+1}}\Big]\Big)$$

and
(3.4)

$$f_{ij}\Big(\Big(\Big[\frac{v-1}{n^k},\frac{v}{n^k}\Big],\Big[\frac{u-1}{n^k},\frac{u}{n^k}\Big]\Big)=$$
$$=\Big(\Big[\frac{n^k(i-1)+(v-1)}{n^{k+1}},\frac{n^k(i-1)+v}{n^{k+1}}\Big],\Big[\frac{n^k(j-1)+(u-1)}{n^{k+1}},\frac{n^k(j-1)+u}{n^{k+1}}\Big]\Big),$$
$$\text{for all } u,v\in\{1,2,\dots,n^k\} \text{ and } i=1,2,\dots,=n.$$

Just like in Equation 3.3 and Equation 3.4, the application of the f_{ij} mappings transform the edges from $f^k(G)$ to $f^{k+1}(G)$. A function f_{ij} from for all (u,v) in $f^k(G)$ edge generates n edges in $f^{k+1}(G)$.

Based on that $d(f^k(G))=\frac{(n+|E|)^k}{(n^k)^2}$ and $f^k(G)$ has n^k nodes we know that $f^k(G)$ has $(n+|E|)^k)$ edges. We know that for all edges from $f^k(G)$ correspond $(n+|E|)$ edges from $f^{k+1}(G)$ such that n new edges are given by the functions $f_i, i=1,2,\dots,m$ and $|E|$ new connections are given by $f_{ij},(i,j)\in E$ functions.

The density of $f^{k+1}(G)$ holds the following equation.

$$d(f^{k+1}(G))=\frac{(n+|E|)^{k+1}}{(n^{k+1})^2} \tag{3.5}$$

Based on that $G=(V,E), V=|E|$ is the initial simple graph, $n+|E|\leq n^2$. The equality holds if and only if G is a complete graph and then $d(f^k(G))=1$. Otherwise, the density of the k^{th} element in the sequence is as the followings.

$$\lim_{k\to+\infty} d(f^k(G))=\frac{(n+|E|)^k}{(n^2)^k}=0,\ \text{for all non complete } G \text{ graph.} \tag{3.6}$$

Thus, we based the graph sequences on an arbitrary graph and we generated a self-similar sequence using the IFS f. We also analyzed the density of the elements in the graph sequences and we showed that if we have a non-complete graph then limit is equal with 0. ∎

We present the first and the second graphs and the corresponding adjacency matrices of the following graph sequence.

Example 3.2. Let be $G=(V,E)$ a simple graph completed with all of the loops such that $V=\{1,2,3,4,5\}$ and $E=\{(1,2),(2,1),(4,5),(5,4)\}\cup\{(1,1),(2,2),(3,3),\ (4,4),(5,5)\}$. Let consider the f-driven graph sequence based on G on Figure 1.

Let be G an Erdős–Rényi graph with n nodes and let be the independent probability p. Thus, the next corollary is given.

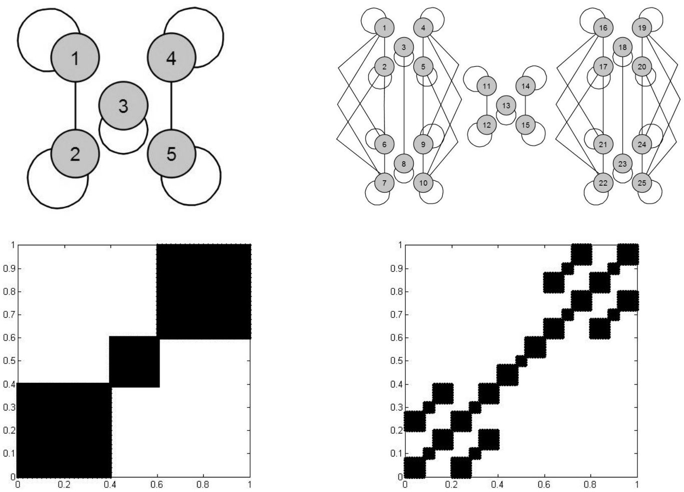

Figure 1. $f(G)$ and $f^2(G)$ from Example 3.2 (the graphs and the corresponding adjacency matrices).

Corollary 3.1. *For a given Erdős–Rényi graph* $G(V,E) = ER(n,p)$, $n = |V|$ *the* $T^1_{f(ER(n,p))}([0,1]^2)$, $T^2_{f(ER(n,p))}([0,1]^2)$, $\dots$, $T^k_{f(ER(n,p))}([0,1]^2), \dots$ *sequence generates the adjacency matrices of a graph sequence such that the expected value of the density of the graph* $f^k(ER(n,p))$ *given by* $T^k_{f(G)}([0,1]^2)$ *is* $E(d(f^k(ER(n,p)))) = \frac{\left(n+p\frac{n(n-1)}{2}\right)^k}{(n^k)^2}$, *for all* $n \in \mathbb{N}^*$.

Proof. Let consider that $G = ER(n,p)$ is an Erdős-Rényi graph such that it has n nodes and the probability of nodes is always independently p. Based on [2], we know that the expected number of edges in $ER(n,p)$ is $pC^2_n = p\frac{n(n-1)}{2}$. We also know that the existence of the edges is always independent in Erdős–Rényi graphs.

Thus, the expected value of the density in the graph sequence generated by $ER(n,p)$ is as follows.

$$E(d(f^k(ER(n,p)))) = \frac{\left(n + pC^2_n\right)^k}{(n^k)^2} = \frac{\left(n + p\frac{n(n-1)}{2}\right)^k}{(n^k)^2}. \tag{3.7}$$

■

Let us construct the iterated multifunction system F using Banach-type transformations and set operations on the f and the f' iteration function systems defined above.

The aim of this is to construct such IMSs, which generate graph sequences with different density properties. Let show that F generates sequences based on an arbitrary graph G such that the density of the k^{th} element is larger or equal then the density of the $k+1^{\text{th}}$ element.

Theorem 3.3. *For a given simple graph $G=(V,E), n=|V|$ the*

$$T^1_{F(G)}([0,1]^2), T^2_{F(G)}([0,1]^2), \dots, T^k_{F(G)}([0,1]^2), \dots$$

sequence generates the adjacency matrices of a graph sequence such that the density of the $F^k(G)$ graph given by $T^k_{F(G)}([0,1]^2)$ is

$$d(F^k(G)) = \frac{n+|E|}{n^2},$$

for all $n \in \mathbb{N}^*$.

Proof. For a given simple graph $G=(V,E), n=|V|$, the k^{th} iteration of $T_{F(G)}$ is constructed with$(n^{k-1})^2$ replicas of it's first iteration using Banach-type transformations and shiftings.

Firstly, we verify the following equation.

$$T^k_{F(G)}([0,1]^2) = \bigcup_{i=1}^{n}\bigcup_{j=1}^{n}\Big[\Big(\frac{i-1}{n},\frac{j-1}{n}\Big)+\frac{1}{n}T^{k-1}_{F(G)}(x,y)\Big] =$$

$$(3.8) \qquad = \bigcup_{i=1}^{n^{k-1}}\bigcup_{j=1}^{n^{k-1}}\Big[\Big(\frac{i-1}{n^{k-1}},\frac{j-1}{n^{k-1}}\Big)+\frac{1}{n^{k-1}}f(G)([0,1]^2)\Big],$$

for all $k \in \mathbb{N}\setminus\{0,1\}$ such that $F(G)([0,1]^2) = f(G)([0,1]^2)$.

This means that

(3.9)

$$T^k_{F(G)}([0,1]^2) = \bigcup_{i=1}^{n}\bigcup_{j=1}^{n}\Big[\Big(\frac{i-1}{n},\frac{j-1}{n}\Big)+\frac{1}{n}T^{k-1}_{F(G)}(x,y)\Big] =$$

$$= \bigcup_{i=1}^{n}\bigcup_{j=1}^{n}\Big[\Big(\frac{i-1}{n},\frac{j-1}{n}\Big)+\frac{1}{n}\Big[\bigcup_{i_2=1}^{n}\bigcup_{j_2=1}^{n}\Big[\Big(\frac{i_2-1}{n},\frac{j_2-1}{n}\Big)+\frac{1}{n}T^{k-2}_{F(G)}(x,y)\Big]\Big] =$$

$$= \bigcup_{i=1}^{n}\bigcup_{j=1}^{n}\Big[\Big(\frac{i-1}{n},\frac{j-1}{n}\Big)+\Big[\bigcup_{i_2=1}^{n}\bigcup_{j_2=1}^{n}\Big[\Big(\frac{i_2-1}{n^2},\frac{j_2-1}{n^2}\Big)+\frac{1}{n^2}T^{k-2}_{F(G)}(x,y)\Big]\Big] =$$

$$= \bigcup_{i=1}^{n^2}\bigcup_{j=1}^{n^2}\Big[\Big(\frac{i-1}{n^2},\frac{j-1}{n^2}\Big)+\frac{1}{n^2}T^{k-2}_{F(G)}(x,y)\Big].$$

Following the same transformation step by step, we get that

$$T^k_{F(G)}([0,1]^2) = \bigcup_{i=1}^{n^{k-1}} \bigcup_{j=1}^{n^{k-1}} \Big[\Big(\frac{i-1}{n^{k-1}}, \frac{j-1}{n^{k-1}}\Big) + \frac{1}{n^{k-1}} T_{F(G)}([0,1]^2)\Big] =$$

$$(3.10)\qquad = \bigcup_{i=1}^{n^{k-1}} \bigcup_{j=1}^{n^{k-1}} \Big[\Big(\frac{i-1}{n^{k-1}}, \frac{j-1}{n^{k-1}}\Big) + \frac{1}{n^{k-1}} F(G)([0,1]^2)\Big] =$$

$$= \bigcup_{i=1}^{n^{k-1}} \bigcup_{j=1}^{n^{k-1}} \Big[\Big(\frac{i-1}{n^{k-1}}, \frac{j-1}{n^{k-1}}\Big) + \frac{1}{n^{k-1}} f(G)([0,1]^2)\Big],$$

for all $n \in \mathbb{N}^*, k \in \mathbb{N} \setminus \{0,1\}$ such that $F(G)([0,1]^2) = f(G)([0,1]^2)$.

As we showed in the proof of Theorem 3.1, $f(G)([0,1]^2)$ gives the adjacency matrix of G. We also know that the graph $F^k(G)$ given by $T^k_{F(G)}([0,1]^2)$ has n^k nodes, so the $\frac{1}{n^k}T^k_{F(G)}([0,1]^2)$ projected to the $\Big(\Big[\frac{i-1}{n^k}, \frac{i}{n^k}\Big], \Big[\frac{j-1}{n^k}, \frac{i}{n^k}\Big]\Big)$ little squares also correspond to edges in $F^k(G)$, for all $i, j = 1, 2, \ldots, n^k$.

If (i,j) is an edge in $F^k(G)$, then $\Big(\Big[\frac{i-1}{n^k}, \frac{i}{n^k}\Big], \Big[\frac{j-1}{n^k}, \frac{i}{n^k}\Big]\Big) \subseteq T^k([0,1]^2)$. Based on that G is a simple graph we know that $f(G)([0,1]^2)$ is symmetric on the first bisector. This and Equation 3.8 follow that $\Big(\Big[\frac{j-1}{n^k}, \frac{j}{n^k}\Big], \Big[\frac{i-1}{n^k}, \frac{j}{n^k}\Big]\Big) \subseteq$ $\subseteq T^k([0,1]^2)$, so (i,j) is also an edge. Thus, $F^k(G)$ is an undirected graph.

The $T^1_{F(G)}([0,1]^2), T^2_{F(G)}([0,1]^2), \ldots, T^k_{F(G)}([0,1]^2), \ldots$ sequence gives a self-similar graph sequence with n^k nodes at in $T^k_{F(G)}([0,1]^2)$ and $\Big[\Big(\frac{i-1}{n^{k-1}}, \frac{i-1}{n^{k-1}}\Big) +$ $+\frac{1}{n^{k-1}} f(G)([0,1]^2)\Big] \subseteq T^k_{F(G)}([0,1]^2)$ generates a replica of $F(G)$, for all $i = 1, 2, \ldots, n^k$. Thus, the k^{th} element of the sequence contains n^k replicas of the first initial graph.

$F^k(G)$ is constructed by $(n^{k-1})^2$ shiftings of $F(G)$ and it has $(n^k)^2$ nodes. Based on $d(F(G)) = \frac{n+|E|}{n^2}$ we get that

$$(3.11)\qquad d(F^k(G)) = \frac{(n+|E|)(n^{k-1})^2}{(n^k)^2} = \frac{n+|E|}{n^2},$$

which means that the graph density is constant in the sequence generated by G. ■

Example 3.4. Let be $G = (V, E)$ a simple graph completed with all of the loops such that $V = \{1,2,3,4,5\}$ and $E = \{(1,2),(2,1),(4,5),(5,4)\} \cup$ $\cup\{(1,1),(2,2),(3,3),\ (4,4),(5,5)\}$. Let consider the F-driven graph sequence based on G on Figure 2.

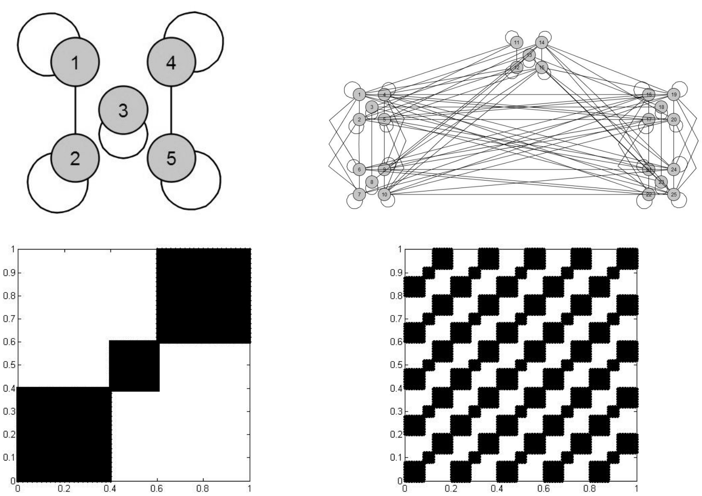

Figure 2. $F(G)$ and $F^2(G)$ from Example 3.4 (the graphs and the corresponding adjacency matrices).

Just like as above, be G an Erdős-Rényi graph with parameters n and p.

Corollary 3.2. *For a given simple graph* $G(V,E) = ER(n,p), n = |V|$ $T^1_{F(ER(n,p))}([0,1]^2, T^2_{F(ER(n,p))}([0,1]^2), \ldots, T^k_{F(ER(n,p))}([0,1]^2), \ldots$ *generates the adjacency matrices of a graph sequence such that the expected value of the density of the graph* $F^k(ER(n,p))$ *given by* $T^k_{F(G)}([0,1]^2)$ *is* $E(d(F^k(ER(n,p)))) = = \frac{\left(n+p\frac{n(n-1)}{2}\right)}{n^2}$ *for all* $n \in \mathbb{N}^*$.

Proof. Let us consider that $G = ER(n,p)$. So, based also on [2], the expected number of edges in $ER(n,p)$ is $pC_n^2 = p\frac{n(n-1)}{2}$ and the edges of these graphs are always independent.

The expected value of the k^{th} element in the graph sequence generated by $G = ER(n,p)$ is

$$E(d(F^k(ER(n,p)))) = \frac{\left(n+p\frac{n(n-1)}{2}\right)}{n^2}, \text{ for all } i \in \mathbb{N}^*. \tag{3.12}$$

■

Thus, in this section we constructed self-similar graph sequences using graph-driven IFSs and IMSs and we analyzed the density properties in the graph sequences. We focus on the description of the fixed sets generated by these IFSs' and the IMSs'.

4. Description of the IFSs' and the IMSs' limit sets using the Sierpinski carpet and the $[0,1]^2$ unit square

In this section we introduce the graph-driven version of the Sierpinski carpet, which we note as $SIER(G)$ for a given graph $G = (V, E)$.

Let consider the mappings $f_{ij} : [0,1]^2 \to [0,1]^2$ as

$$f_{ij}(x, y) = \Big(\frac{i-1}{n}, \frac{j-1}{n}\Big) + \frac{1}{n}(x, y), \text{ such that } (i, j) \in E \text{ and } i \neq j.$$

We complete the graph with all of the possible loops, so we define the mappings $f_i : [0,1]^2 \to [0,1]^2$ as

$$f_i(x, y) = \Big(\frac{i-1}{n}, \frac{i-1}{n}\Big) + \frac{1}{n}(x, y), \text{ such that } (i, i) \in E,$$

where f_i corresponds to the loop of the i^{th} node and let construct the graph-driven IFS $f(G) = \{\{f_{ij}|(i, j) \in E\} \cup \{f_i|(i, i) \in E\}\}$ using the presented functions.

Example 4.1. For instance, if $G = (V, E), V = \{1, 2, 3\}$ and $E = \{(1, 2), (1, 3), (2, 1), (2, 3), (3, 1), (3, 2), \} \cup \{(1, 1), (3, 3)\}$, then the corresponding $f(G)$ generates by iterations the classical Sierpinski carpet. Based on that the construction uses shiftings and Banach-type contractions, the IFS has a fixed point set , which is the Sierpinski carpet at this case.

Based on this example, we refer to the limit of these graph-driven generalization as the graph-driven extension of the Sierpinski carpet, which we note as $SIER(G)$ for a given $G = (V, E)$.

The main fixed point set result for these graphs-driven IFS f is as follows.

Theorem 4.2. $\lim\limits_{k\to+\infty} T^k_{f(G)}([0,1]^2) = SIER(G) = Fix(f(G))$.

Proof. $f(G)$ is constructed by the finite union of Banach-type operations, so the corresponding fractal operator $T_{f(G)}$ has an unique fixed point set. Based on Example 4.1, we know that f is a graph-directed generalization of the Sierpinski carpet. Thus, $T_{f(G)}$ has the $SIER(f(G))$ graph-driven unique fixed point set, which can approximated by iterations.

We interpret these limit sets as the limit of the f-driven graph sequence based on the arbitrary G graph, which we characterized using the graph-driven generalization of the Sierpinski carpet. ■

Let focus on the case, when G is an Erdős-Rényi graph. Thus, we also get that the fixed set of the graph sequence driven by a random graph can be also described using the graph-driven generalization of the Sierpinski carpet.

Corollary 4.1. $\lim_{k\to+\infty} f^k(ER(n,p)) = SIER(ER(n,p))$ *and* $\lim_{k\to+\infty} f^k(ER(n,p)) = Fix(f(ER(n,p)))$.

Proof. If $G = ER(n,p)$, then the corresponding iterated function system $f(ER(n,p))$ has the $SIER(ER(n,p))$ fixed set. ∎

For the constructed graph-driven IMS F corresponding to graph sequences with constant graph density the following fixed point theorem hold for any given G graph.

Theorem 4.3. $\lim_{k\to+\infty} F^k(G) = [0,1]^2$.

Proof. The k^{th} iteration of F is constructed by the union of $(n^{k-1})^2$ number of sets.

$$T^k_{F(G)}([0,1]^2) = \bigcup_{i=1}^{n^{k-1}} \bigcup_{j=1}^{n^{k-1}} \Big[\Big(\frac{i-1}{n^{k-1}}, \frac{j-1}{n^{k-1}}\Big) + \frac{1}{n^{k-1}} f(G)([0,1]^2)\Big], \tag{4.1}$$

for all $n, k \in \mathbb{N}^*$.

If $k \to +\infty$, then

$$\lim_{k\to+\infty} \Big[\Big(\frac{i-1}{n^{k-1}}, \frac{j-1}{n^{k-1}}\Big) + \frac{1}{n^{k-1}} f(G)([0,1]^2)\Big] = \Big(\frac{i-1}{n^{k-1}}, \frac{j-1}{n^{k-1}}\Big), \tag{4.2}$$

for all $i, j = 1, 2, \ldots, n^k - 1$.

Moreover, the crossing points of an $n^{k-1} \times n^{k-1}$ size grid is always included in $T^k_{F(G)}$ such that it is projected to $[0,1]^2$. Although $T^{k+1}_{F(G)}$ contains the skeleton of a $n^k \times n^k$ size grid projected to the unit square, the limit of the $T^1_{F(G)}([0,1]^2)$, $T^2_{F(G)}([0,1]^2)$, $\ldots, T^k_{F(G)}([0,1]^2), \ldots$ sequence never reachs the $[0,1]^2$ unit square.

If $k \to +\infty$, then $\lim_{k\to+\infty} T^k_{F(G)}([0,1]^2) = [0,1]^2$.

On the other hand, $T_{F(G)}([0,1]^2)$ generates the adjacency matrix of $G = F(G)$. This means, that the limit of the F-driven graph sequences is the unit square, but it is not a fixed set for the IMS F. ∎

We also confirm this result for Erdős–Rényi graphs too.

Corollary 4.2. $\lim_{k\to+\infty} F^k(ER(n,p)) = [0,1]^2$.

Proof. If G is an Erdős-Rényi graph such that it has n nodes and the probability of nodes is always independently p, the result of presented theorem hold for it. ∎

As a conclusion, we based our graph sequence constructions using a set of graph-driven IFS and a set of graph-driven IMSs. We showed that the results hold for arbitrary graphs. We characterized the limit of the graph sequences using the Sierpinski carpet and the unit square.

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