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MODELLING THE ECOSYSTEM OF THE EASTER ISLAND WITH DELAY DIFFERENTIAL EQUATIONS

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Abstract. The invasive species model presents the ecosystem of the Easter Island and describes the connections between three species: people, trees and rats. In 2008, Basener, Brooks, Radin and Wiandt presented an article, in which they created a mathematical model [1], [2]. The model was investigated in [6], too. In this work we suggest a new model in which discrete delay is involved. We investigate the equilibrium points and their stability.

1. Introduction

The Easter Island is located in the Pacific Ocean, at the southeastern point of the Polynesian Triangle. Primarily, it is famous for its culture and its approximately 900 stone statues, so called Moais. The Polynesian arrived to the island around 400–700, where they built their civilization and used the ample resources of the island. The human population used the trees for building houses, creating canoes and for transportation of the statues.

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In the later centuries the population greatly increased, which caused irreversible changes in the ecosystem of the island. Due to the deforestation trees disappeared, while the agriculture declined due to the soil erosion. These resulted in scarce availability of food.

According to Terry Hunt (an American anthropologist), the main reason why the forests could not heal properly was the Polynesian rat. Most probably the Polynesian rats arrived to the island with the first Polynesian settlers. Its main nourishment was the fruit of the palm trees. All the above reasons caused the disaster of the Easter Island's ecosystem.

Mathematical models were created to model this phenomenon, one of them is the invasive species model. However this model has a serious disadvantage: it does not represent the reality exactly, as it does not consider that a certain amount of time is needed for the seeds to become a full-grown tree. To overcome this, in this paper we introduce discrete delay and compare the stability of the original and the new model.

2. Invasive species model

In 2008, Basener et al. presented an article, in which they created a mathematical model for this phenomenon [1]. Using the notation $P(t)$, $T(t)$ and $R(t)$ at the given time instant for the people, trees and rats, respectively, the invasive species model is a system of differential equations for these unknown functions, which describe the relations between them. We note that we measure P in people, the units for T is the amount of trees that would support one human, and for R it is the number of rats that would be supported by one tree unit.

In the following we describe each equation in this system.

- It is assumed that the growth rate of the human population is defined by the logistic equation, where the carrying capacity is the amount of the trees. Hence, the equation has the form

$$\frac{dP}{dt} = aP\left(1 - \frac{P}{T}\right), \tag{2.1}$$

  where the non-negative parameter a shows the growth rate of the human population.

- The growth rate of the rat population is defined also by the logistic equation, where the carrying capacity is the amount of the trees again. Hence, the equation has the form

$$\frac{dR}{dt} = cR\left(1 - \frac{R}{T}\right), \tag{2.2}$$

where the non-negative parameter c represents the growth rate of the rat population.

- The rats eat the seeds of the trees and the humans decrease the amount of the trees, too. Therefore, the equation for the rats has the form

$$\frac{dT}{dt} = \frac{b}{1+fR}T\left(1 - \frac{T}{M}\right) - hP, \tag{2.3}$$

where the non-negative parameter f shows the effect of the rats, b is the growth rate of the trees, and h is the harvest by the human population. The parameter M denotes the carrying capacity of the trees. We note that linear harvesting is considered in this model.

The equations (2.1)–(2.3) yield a system of nonlinear ordinary differential equations, which is frequently called dynamical system.

In the following we analyse some qualitative properties for the above system. First we define the equilibrium of the dynamical system. Roughly speaking, it is a value of the state variables where the state variables do not change. In other words, an equilibrium is a solution that does not change with time. This implies the following definition.

Let $f : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be a given function.

Definition 2.1. When for the solution $x(t)$ of the dynamical system

$$\frac{dx}{dt} = f(t, x) \tag{2.4}$$

there exists a constant vector $x^* \in \mathbb{R}^n$ such that $x(t) = x^*$ for all t, then the vector x^* is called *equilibrium point* of the system.

Clearly, the equilibrium can be defined from the equation $f(t, x^*) = 0$. Therefore, for the invasive species model the equilibrium point $(P^*, R^*, T^*) \in$ $\in \mathbb{R}^3$ of the system (2.1)–(2.3) can be defined as the solutions of the following system of algebraic equations:

$$\begin{aligned} aP^*\left(1 - \frac{P^*}{T^*}\right) &= 0, \\ cR^*\left(1 - \frac{R^*}{T^*}\right) &= 0, \\ \frac{b}{1+fR^*}T^*\left(1 - \frac{T^*}{M}\right) - hP^* &= 0. \end{aligned} \tag{2.5}$$

Hence, following the results of [1], [2], by an easy computation we can define the equilibrium points of the system (2.5), and we get the following four points:

$$\mathcal{P}_1(0,0,M), \tag{2.6}$$

$$\mathcal{P}_2(0,M,M), \tag{2.7}$$

$$\mathcal{P}_3\left(\frac{(b-h)M}{b}, 0, \frac{(b-h)M}{b}\right), \tag{2.8}$$

$$\mathcal{P}_4\left(\frac{(b-h)M}{b+fhM}, \frac{(b-h)M}{b+fhM}, \frac{(b-h)M}{b+fhM}\right). \tag{2.9}$$

We note that only the point $\mathcal{P}_4$ is an inner point of the solution domain, while in the other cases the equilibrium points are on the boundary. This means, that for the points $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$ at least one species is not present in the interaction. The point $\mathcal{P}_4$ is the most interesting one, because in this case all the three populations (people, trees and rats) live together.

From a qualitative point of view, one of the most important properties of a dynamical system is its stability, which means the following. The dynamical system (2.4) is well-posed if we add an initial condition to this system, i.e., we fix the initial state of the system. (This condition is necessary to guarantee the uniqueness of the solution.) The stability theory addresses the stability of solutions of differential equations and of trajectories of dynamical systems under small perturbations of initial conditions. This means that some given solution is called stable if the system always returns to it after small disturbances of the initial condition. Otherwise, i.e., if the system moves away from the equilibrium after small disturbances, then the solution is called unstable.

Our aim is to investigate the stability property of the equilibrium points. In [1] it is shown that for the dynamical system (2.1)–(2.3) the equilibrium points $\mathcal{P}_1, \mathcal{P}_2$ and $\mathcal{P}_3$ are unstable, while $\mathcal{P}_4$ point is conditionally stable.

In the next sections we introduce a new model, we define its equilibrium points, and we will investigate their stability properties.

3. The discrete delay system

The deficiency of the system (2.1)–(2.3) is that it does not reflect the reality exactly, therefore we cannot consider it as an adequate model for the invasive species phenomenon. The problem is the following. The amount of the seeds consumed by the rats does not decrease the amount of the trees immediately, but those seeds will not become a full-grown tree. Hence, the effect of the rats appears after a certain amount of time. This motivates to change the system (2.1)–(2.3) by using discrete delay for modeling of the island's ecosystem.

In the sequel, we incorporate in the system (2.1)–(2.3) the above-mentioned discrete delay in order to build up a more reliable model for the island's ecosystem. This means that the delay differential equations have been used in the modeling. The rodents eat the fruits of the trees and the seeds too, so the rats prevent the regeneration of the trees. We denote by τ the time necessary for a seed to become a full-grown tree. Hence, by modeling this process with a delay differential equation, the third equation in the system (2.1)–(2.3), i.e., the equation for the trees, has the new form

$$\frac{dT(t)}{dt} = \frac{b}{1+fR(t-\tau)}T(t-\tau)\left(1-\frac{T(t-\tau)}{M}\right) - hP(t). \tag{3.1}$$

With this modification the new mathematical model is the following system of equations:

$$\frac{dP}{dt} = aP\left(1-\frac{P}{T}\right), \tag{3.2}$$

$$\frac{dR}{dt} = cR\left(1-\frac{R}{T}\right), \tag{3.3}$$

$$\frac{dT(t)}{dt} = \frac{b}{1+fR(t-\tau)}T(t-\tau)\left(1-\frac{T(t-\tau)}{M}\right) - hP(t). \tag{3.4}$$

The system (3.2)–(3.4) yields a delay ODE system (DODE). Next we define the equilibrium points of this system and we investigate also the stability of these points.

First we give the definition of a DODE system. In the same way as it was given for ODE system (2.4), we give the definition as follows.

Let $f_d : \mathbb{R}^{2n+1} \to \mathbb{R}^n$ be a given function.

Definition 3.1. When for the solution $x(t)$ of the dynamical system

$$\frac{dx}{dt} = f_d(t, x(t), x(t-\tau)) \tag{3.5}$$

there exists a constant vector $x^* \in \mathbb{R}^n$ such that $x(t) = x^*$ for all t, then the vector x^* is called *equilibrium point* of the system.

The equilibrium points are independent of time, hence, the equilibrium points can be defined by solving the system of algebraic equations $f(t, x^*, x^*) = 0$.

Let us define the equilibrium points for the DODE system (3.2)–(3.4). As one can see, these points can be defined by solving the algebraic system (2.5). This means that the discrete delay does not change the equilibrium points, i.e., the following statement is true.

Proposition 3.1. *The equilibrium points of the DODE system* (3.2)–(3.4) *and the ODE system* (2.1)–(2.3) *are the same.*

Remark. Is this statement true for any DODE system, when f_d in (3.5) is an arbitrary function? As one can easily see, the corresponding algebraic equations are the same for a system with delay and without delay if and only if the coordinate functions f_d^i of f_d $(i = 1, 2, \dots, n)$ depend only either on $x(t)$ or $x(t-\tau)$, i.e., they have the form $f_d^i(t, x(t))$ or $f_d^i(t, x(t-\tau))$. For the DODE system (3.2)–(3.4) this requirement is satisfied, and this fact causes the identity of the equilibrium points.

4. Stability analysis

In this section we investigate the stability of the equilibrium points for DODE system (3.2)–(3.4). As usual, first we linearize the delay equation system around the equilibrium points and then examine the roots of the corresponding characteristic equation. The real part of the roots of the characteristic equation show the stability of the given equilibrium point: the given point is stable when $Re\lambda \le 0$ for all roots. Otherwise, it is unstable.

The linerisation of the delay system (3.2)–(3.4) at the point $\mathcal{P}(P(t), R(t), T(t))$ has the form

$$(4.1) \qquad \frac{d}{dt}\begin{pmatrix} P(t) \\ R(t) \\ T(t) \end{pmatrix} = A_1 \begin{pmatrix} P(t) \\ R(t) \\ T(t) \end{pmatrix} + A_2 \begin{pmatrix} P(t-\tau) \\ R(t-\tau) \\ T(t-\tau) \end{pmatrix},$$

where $A_1(\mathcal{P})$ and $A_2(\mathcal{P})$ are 3×3 Jacobian matrices of the system and the latter matrix depends on the delay parameter τ, too.

The Jacobian matrices at the point $\mathcal{P}(P(t), R(t), T(t))$ are the following:

$$(4.2)\quad A_1 = \begin{pmatrix} a - \frac{2aP}{T} & 0 & \frac{aP^2}{T^2} \\ 0 & c - \frac{2cR}{T} & \frac{cR^2}{T^2} \\ -h & 0 & 0 \end{pmatrix},$$

$$(4.3)\quad A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{-bfT(t-\tau)}{(1+fR(t-\tau))^2}\left(1 - \frac{T(t-\tau)}{M}\right) & \frac{b}{1+fR(t-\tau)} - \frac{2bT(t-\tau)}{(1+fR(t-\tau))M} \end{pmatrix}$$

which means that the characteristic equation of the system (4.1) is

$$(4.4)\qquad \det\left(\lambda I - A_1 - e^{-\lambda\tau} A_2\right) = 0.$$

We examine the characteristic equation and their eigenvalues at the equilibrium points $\mathcal{P}_i$, $(i = 1, 2, 3, 4)$, defined in (2.6)–(2.9).

4.1. The stability of $\mathcal{P}_1$

The first case shows the state of the island when there are no people and rats, and the trees are at their carrying capacity. At the equilibrium point $\mathcal{P}_1$ the Jacobian has the form

$$(4.5)\qquad A_1(\mathcal{P}_1) = \begin{pmatrix} a & 0 & 0 \\ 0 & c & 0 \\ -h & 0 & 0 \end{pmatrix}, \quad A_2(\mathcal{P}_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -b \end{pmatrix}.$$

Hence, the characteristic equation is

$$(4.6)\qquad \begin{vmatrix} \lambda - a & 0 & 0 \\ 0 & \lambda - c & 0 \\ h & 0 & \lambda + be^{-\lambda\tau} \end{vmatrix} = (\lambda - a)(\lambda - c)(\lambda + be^{-\lambda\tau}) = 0.$$

Since $\lambda_1 = a > 0$, the equilibrium point $\mathcal{P}_1$ is unstable.

4.2. The stability of $\mathcal{P}_2$

In this case we analyze the island before the arrival of the Polynesian settlements. There are no people on the island, only the rat population and trees. At the equilibrium point $\mathcal{P}_2$ the Jacobian matrices are the following:

$$(4.7)\qquad A_1(\mathcal{P}_2) = \begin{pmatrix} a & 0 & 0 \\ 0 & -c & c \\ -h & 0 & 0 \end{pmatrix}, \quad A_2(\mathcal{P}_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{b}{1+fM} \end{pmatrix}.$$

In this case the characteristic equation is

$$\text{(4.8)}\quad \begin{vmatrix} \lambda - a & 0 & 0 \\ 0 & \lambda + c & -c \\ h & 0 & \lambda + \frac{be^{-\lambda\tau}}{1+fM} \end{vmatrix} = (\lambda - a)\left[(\lambda + c)\left(\lambda + \frac{be^{-\lambda\tau}}{1+fM}\right)\right] = 0.$$

Since $\lambda_1 = a > 0$, the equilibrium point $\mathcal{P}_2$ is unstable.

4.3. The stability of $\mathcal{P}_3$

In this situation there are no rats on the island, only the human population and trees. We investigate the stability of the equilibrium point $\mathcal{P}_3$, the Jacobian matrices have the form

$$A_1(\mathcal{P}_3) = \begin{pmatrix} -a & 0 & a \\ 0 & c & 0 \\ -h & 0 & 0 \end{pmatrix}, \quad A_2(\mathcal{P}_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\frac{hfM(b-h)}{b} & 2h - b \end{pmatrix}.$$

The characteristic equation at the equilibrium point $\mathcal{P}_3$ is the following:

$$\text{(4.9)}\quad \begin{vmatrix} \lambda + a & 0 & -a \\ 0 & \lambda - c & 0 \\ h & \frac{h(b-h)Mf}{b}e^{-\lambda\tau} & \lambda - (2h-b)e^{-\lambda\tau} \end{vmatrix} =$$

$$= (\lambda - c)\left[(\lambda + a)(\lambda - (2h-b)e^{-\lambda\tau}) + ah\right] = 0.$$

The $\lambda_1 = c > 0$ is the eigenvalue of the characteristic equation, thus the equilibrium point $\mathcal{P}_3$ is unstable.

Proposition 4.1. *The equilibrium points $\mathcal{P}_1, \mathcal{P}_2$ and $\mathcal{P}_3$ of the DODE system* (3.2)–(3.4) *are unstable, and the discrete delay does not change the stability.*

4.4. The stability of $\mathcal{P}_4$

The last case is the most interesting both mathematically and ecologically, because it is the equilibrium that corresponds to the coexistence of all three biological populations: the people, the trees and the rats. The equilibrium point is conditionally stable in the case of system (2.1)–(2.3). For this point the Jacobian matrices have the form

$$A_1(\mathcal{P}_4) = \begin{pmatrix} -a & 0 & a \\ 0 & -c & c \\ -h & 0 & 0 \end{pmatrix}, \quad A_2(\mathcal{P}_4) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{bfF(F-M)}{(1+fF)^2 M} & \frac{b(M-2F)}{(1+fF)M} \end{pmatrix},$$

where

$$F := \frac{(b-h)M}{b+fhM} \tag{4.10}$$

denotes the equilibrium point.

The characteristic equation is in this case

$$P(\lambda) := \begin{vmatrix} \lambda + a & 0 & -a \\ 0 & \lambda + c & -c \\ h & -\frac{bfF(F-M)}{(1+fF)^2M}e^{-\lambda\tau} & \lambda - \frac{b(M-2F)}{M(1+fF)}e^{-\lambda\tau} \end{vmatrix} =$$

(4.11)

$$= \lambda^3 + \lambda^2(a+c) + \lambda(ac+ah) - \lambda^2 e^{-\lambda\tau}F_2 +$$

$$+\lambda e^{-\lambda\tau}(-cF_1 - cF_2 - aF_2) + e^{-\lambda\tau}(-acF_1 - acF_2) + ahc = 0,$$

where F_1 and F_2 denote the following:

$$F_1 = \frac{bfF(F-M)}{(1+fF)^2M}, \tag{4.12}$$

$$F_2 = \frac{b(M-2F)}{M(1+fF)}. \tag{4.13}$$

The characteristic equation of the point $\mathcal{P}_4$ is complicated and it depends on different parameters. Thus, to examine the stability, we fix those parameters which are determined from the original phenomenon, and we analyse the stability in dependence of those parameters which can be controlled, which are the parameters h and f, showing the effect of the people and rats. We can control these influences, thus we use a wide range of these parameter values to solve the characteristic equation and examine the real part of the roots.

Hence, we fix the parameters as $a = 0.03, b = 1, c = 10, M = 12000$ and the parameters may vary as $h \in [0, 0.5]$ and $f \in [0, 0.01]$, respectively.

We examined the stability domain with different values of parameters f and h. The following figures show the non-positive real part roots of the equation (4.11) (i.e., the stability of $\mathcal{P}_4$) with different delay parameters τ. In all figures the horizontal axis is h, the vertical axis is f.

The first picture shows the non-positive real part roots of the characteristic equation if $\tau = 0$. This case yields the system without delay, i.e., the ODE system (2.1)–(2.3). We note that this figure is identical to the figure given in [1] for this system.

Figure 2 shows the stability domain when the delay parameter is chosen as $\tau = 0.005$.

Finally, Figure 3 represents the stability domain when the delay parameter is increased significantly.

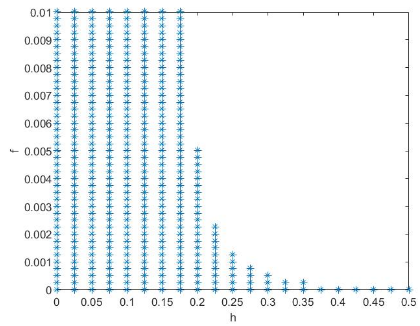

Figure 1. The non-positive real part roots of the characteristic equation with $\tau = 0$.

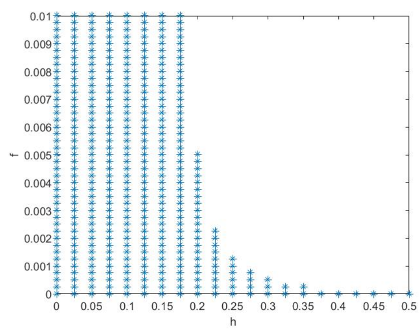

Figure 2. The non-positive real part roots of the characteristic equation with $\tau = 0.005$.

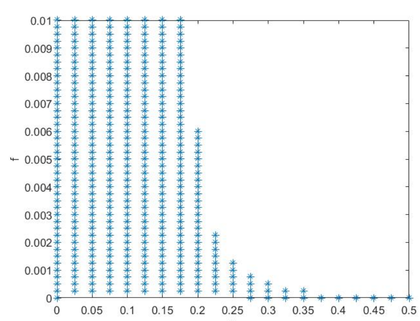

Figure 3. The non-positive real part roots of the characteristic equation with $\tau = 2.02$.

As we can see, by these given fixed parameters a, b, c and M, the stability domain (f, h) does not change considerably.

However, we can choose such parameter values ($a = 0.03, \quad b = 10, \quad c = 1,$ $M = 4000$), for which the stability domain of the DODE system and the ODE system can be different.

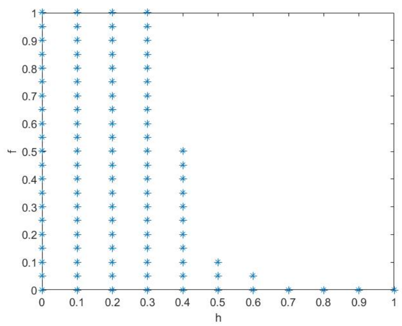

Figure 4. The non-positive real parts of the characteristic equation with $\tau = 0$.

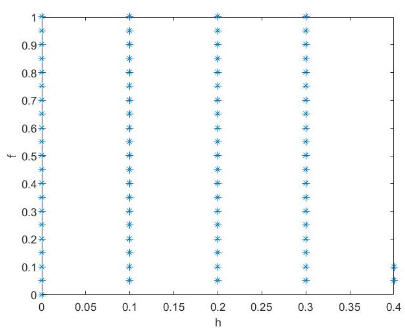

Figure 5. The non-positive real parts of the characteristic equation with $\tau = 2$.

Based on the above numerical experiments, the effect of the delay shows the following. For several choices of the parameters a, c, b and M the (f, h) stability map is the same, while for other choices the map is changed significantly. However for the equilibrium point P_4 we can find suitable parameter setting under which this point is stable.

5. Summary

The invasive species model describes the relationship between three species on the Easter Island. This process is both mathematically and ecologically interesting. We have modified the invasive species model with introducing a discrete delay differential equation. This new model represents the processes more accurately, this is why it is really important to analyse it. Our aim was to investigate the stability of this modified system. We found that the delay does not affect the instability of the equilibrium points $\mathcal{P}_1, \mathcal{P}_2$ and $\mathcal{P}_3$.

The case of point $\mathcal{P}_4$ is more complicated. If $\tau = 0$, we get the invasive species model (2.1)–(2.3), which is conditionally stable. In the previous section we consider the stability domain for different τ values and fixed parameters. The stability domains are considerably similar and there exist such parameters when the stability domains are different. With the introduction of τ the equilibrium point $\mathcal{P}_4$ can be stable.

However we still have open questions like: For which parameters and delay is the system stable? Has the system bifurcation for different parameter domains? Our plan in the future is to determine the effect of the delay and the stability domain. To be able to perform these, we would like to use other methods, e. g., Rouché's theorem [3]. Also we would like to use numerical methods to solve the DODE system. Using the operator splitting could be useful by separating the different components: in the DODE model the delayed and the non-delayed term. Our aim is to use different operator splitting techniques, especially the sequential and the Strang-Marchuk splittings [4], [5].

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