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ANALYSIS OF SOME CHARACTERISTIC PARAMETERS IN AN INVASIVE SPECIES MODEL

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Abstract. In this article we expand the model first proposed by Basener et al. in [3] describing the dynamics and ecological disaster of Easter Island. We examine how the choice of two parameters (the number of regions and the effect of rats on the reproduction of trees) changes the stability of the system. We state some propositions about the stability region, and then draw it using numerical methods. Then we compare our results to the original choice by Basener et al. in [3].

1. Introduction

In recent years numerous papers and books discussed the events that led to the demographic collapse on Easter Island (Rapa Nui) in the 16th and 17th century. Most of them visualized a scenario in which the reckless consumption of goods provoked the catastrophic events. Some of them even claimed that these events could happen globally, so our increasing growth will lead to the fall of humanity (see [1]).

However, in the early 2000s Hunt ([5], [6]) proposed a new theory in which the collapse was caused not only by people, but also the rats (originally brought by the settlers) which ate the seeds of the trees. Some even suppose that the rats

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were transported to the island for dietetic purposes – this model was studied in [9], but now we will neglect this effect.

The theories of Hunt were formalized by Basener et al. ([2]) the following way:

$$\begin{aligned} \frac{dP}{dt} &= aP\left(1-\frac{P}{T}\right), \\ \frac{dR}{dt} &= cR\left(1-\frac{R}{T}\right), \\ \frac{dT}{dt} &= \frac{b}{1+fR}T\left(1-\frac{T}{M}\right)-hP \end{aligned} \tag{1.1}$$

in which P, R, T denote the number of people, rats and trees, and a, b and c are the birth ratio for humans, the trees and the rats, respectively. Further, f is the destructive effect of the rats on trees, M is the maximum amount of trees which can live on the island, and h is the number of trees cut down by a person in a year.

In [3], Basener et al. used the following spatial invasive species model to represent the dynamics on Easter Island: they thought of Rapa Nui as an island which has a volcano in the middle, so they split the habitable coast into N regions and labelled them from 1 to N (Figure 1). This way, the neighbours of region s (if $1 < s < N$) are the regions with labels $s-1$ and $s+1$. Also, the region with label 1 and the one with label N are next to each other.

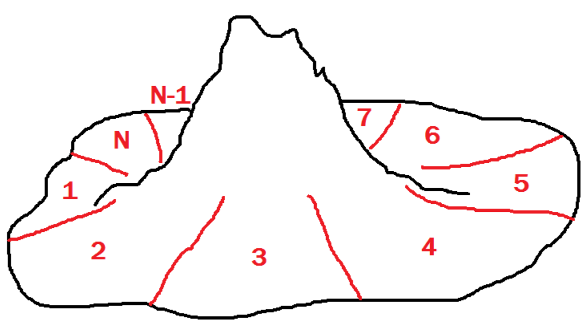

Figure 1. The split of the habitable coast into N regions.

They also introduced the parameters D_p and D_r to describe the diffusion of people and rats. This way, they got the equations

$$\begin{aligned}
\frac{dP^s}{dt} &= aP^s\left(1-\frac{P^s}{T^s}\right)+D_p(P^{s-1}-2P^s+P^{s+1}),\\
\frac{dR^s}{dt} &= cR^s\left(1-\frac{R^s}{T^s}\right)+D_r(R^{s-1}-2R^s+R^{s+1}),\\
\frac{dT^s}{dt} &= \frac{b}{1+fNR^s}T^s\left(1-\frac{T^s}{\frac{M}{N}}\right)-hP^s,
\end{aligned} \tag{1.2}$$

where P^s, R^s and T^s denote the numbers of the corresponding groups in region s ($s \in \{1, \ldots N\}$).

In their article, it turned out that if we increase either the constant D_p or the D_r, the system becomes unstable. However, one can suppose that the source of this instability comes from the system's asymmetry: the first two equations involve diffusion, while the third one does not. In a closed system like Easter Island, the movement of seeds (by the wind or animals) cannot be overlooked. For this reason we can modify the previous equation and add a term which corresponds to the diffusion of the trees, or any other resources:

$$\begin{aligned}
\frac{dP^s}{dt} &= aP^s\left(1-\frac{P^s}{T^s}\right)+D_p(P^{s-1}-2P^s+P^{s+1}),\\
\frac{dR^s}{dt} &= cR^s\left(1-\frac{R^s}{T^s}\right)+D_r(R^{s-1}-2R^s+R^{s+1}),\\
\frac{dT^s}{dt} &= \frac{b}{1+fNR^s}T^s\left(1-\frac{T^s}{\frac{M}{N}}\right)-hP^s+D_t(T^{s-1}-2T^s+T^{s+1}).
\end{aligned} \tag{1.3}$$

The coexistence equilibrium computed in [2] is also an equilibrium of this system (Eq. 1.3), which is

$$P_\varepsilon = R_\varepsilon = T_\varepsilon = \frac{1}{N}\frac{M(b-h)}{b+hMf}$$

for every one of the N systems.

If we linearize the system at this equilibrium, we get the following:

$$\begin{pmatrix} \dfrac{dP^s}{dt} \\ \dfrac{dR^s}{dt} \\ \dfrac{dT^s}{dt} \end{pmatrix} = \begin{bmatrix} -a & 0 & a \\ 0 & -c & c \\ -h & \dfrac{-fMh(b-h)}{b(1+fM)} & \dfrac{fMh-b+2h}{1+fM} \end{bmatrix} \begin{pmatrix} P^s \\ R^s \\ T^s \end{pmatrix} + \tag{1.4}$$

$$+ \begin{pmatrix} D_p(P^{s-1}-2P^s+P^{s+1}) \\ D_r(R^{s-1}-2R^s+R^{s+1}) \\ D_t(T^{s-1}-2T^s+T^{s+1}) \end{pmatrix}.$$

We decouple the equations using the Fourier transformation. For the first two equations (with functions x and y) we get the same results as [3], while the equation of the z function can be computed the following way:

$$\frac{dz_r}{dt} = \frac{1}{N}\sum_{s=1}^{N} e^{\frac{-2\pi irs}{N}} \frac{dT^s}{dt} =$$

$$= \frac{1}{N}\sum_{s=1}^{N} e^{\frac{-2\pi irs}{N}} \Big[-hP^s - \frac{fMh(b-h)}{b(1+fM)}R^s + \frac{fMh-b+2h}{1+fM}T^s +$$

$$+D_t(T^{s-1}-2T^s+T^{s+1})\Big].$$

Using the notations

$$\alpha := \frac{fMh(b-h)}{b(1+fM)},$$

$$\beta := \frac{fMh-b+2h}{1+fM}$$

we can rewrite the previous expression as follows:

$$\frac{1}{N}\sum_{s=1}^{N} e^{\frac{-2\pi irs}{N}} \left[-hP^s - \alpha R^s + \beta T^s + D_t(T^{s-1}-2T^s+T^{s+1})\right] =$$

$$= -h\frac{1}{N}\sum_{s=1}^{N} e^{\frac{-2\pi irs}{N}} P^s - \alpha\frac{1}{N}\sum_{s=1}^{N} e^{\frac{-2\pi irs}{N}} R^s + \beta\frac{1}{N}\sum_{s=1}^{N} e^{\frac{-2\pi irs}{N}} T^s +$$

$$+D_t\frac{1}{N}\sum_{s=1}^{N}e^{\frac{-2\pi irs}{N}}T^{s-1}-D_t\frac{2}{N}\sum_{s=1}^{N}e^{\frac{-2\pi irs}{N}}T^{s}+D_t\frac{1}{N}\sum_{s=1}^{N}e^{\frac{-2\pi irs}{N}}T^{s+1}=$$

$$=-hx_r-\alpha y_r+\beta z_r+D_te^{\frac{-2\pi ir}{N}}\frac{1}{N}\sum_{s=1}^{N}e^{\frac{-2\pi ir(s-i)}{N}}T^{s-1}-2D_tz_r+$$

$$+D_te^{\frac{2\pi ir}{N}}\frac{1}{N}\sum_{s=1}^{N}e^{\frac{-2\pi ir(s+i)}{N}}T^{s+1}=$$

$$=-hx_r-\alpha y_r+\beta z_r+D_te^{\frac{-2\pi ir}{N}}z_r-2D_tz_r+D_te^{\frac{2\pi ir}{N}}z_r=$$

$$=-hx_r-\alpha y_r+\left[\beta-2D_t\left(1-cos\frac{2\pi r}{N}\right)\right]z_r=$$

$$=-hx_r-\alpha y_r+\left[\beta-4D_tsin^2\frac{\pi r}{N}\right]z_r.$$

Thus, the decoupled system can be written as:

(1.5)

$$\begin{pmatrix}\frac{dx_r}{dt}\\ \frac{dy_r}{dt}\\ \frac{dz_r}{dt}\end{pmatrix}=\begin{pmatrix}-\left[a+4D_psin^2\frac{\pi r}{N}\right] & 0 & a\\ 0 & -\left[c+4D_rsin^2\frac{\pi r}{N}\right] & c\\ -h & -\alpha & \beta-4D_tsin^2\frac{\pi r}{N}\end{pmatrix}\begin{pmatrix}x_r\\ y_r\\ z_r\end{pmatrix}.$$

If we use the values of the constants from [2], which are $a = 0.03$, $b = 1$, $c = 10$, $M = 12000$ and $h = 0.25$, the matrix from equation (1.5) has the following form:

(1.6)

$$S_r:=\begin{pmatrix}-\left[0.03+4D_psin^2\frac{\pi r}{N}\right] & 0 & 0.03\\ 0 & -\left[10+4D_rsin^2\frac{\pi r}{N}\right] & 10\\ -0.25 & \frac{-2250f}{1+12000f} & \frac{6000f-1}{2+24000f}-4D_tsin^2\frac{\pi r}{N}\end{pmatrix}.$$

For example, if we choose the values $f = 0.001$ and $N = 10$ (like in [3]), we get

$$(1.7)\quad\begin{pmatrix}-\left[0.03+4D_psin^2\frac{\pi r}{10}\right] & 0 & 0.03\\ 0 & -\left[10+4D_rsin^2\frac{\pi r}{10}\right] & 10\\ -0.25 & -\frac{9}{52} & \frac{5}{26}-4D_tsin^2\frac{\pi r}{10}\end{pmatrix}$$

for every $r = 1, \dots 10$ region on the island.

The system is stable, if it is stable in every region. Thus, if we want to examine the stability of the system, we have to examine all the ten matrices. One region is stable if all the eigenvalues of the corresponding matrix have a negative real part. If this property holds for every region, then the system is stable, otherwise it is unstable.

In [3], the values $f = 0.001$ and $N = 10$ were chosen, and only the parameters of diffusion were changed. However, it is not clear how this choice affects the stability of the system. In the following pages, we will do this analysis: we fix the (D_p, D_r, D_t) triplet and only change the parameters f and N. This way we will search for those (f, N) pairs (for a fixed (D_p, D_r, D_t) triplet) where our system is stable and those for which it is unstable.

2. Choosing the parameters f and N

From now on, we call an (f, N) pair stable, if our system is stable at those values (with a fixed diffusion parameter triplet (D_p, D_r, D_t)). Similarly, we call it unstable if our system is unstable for those f and N values. This way, we define the function $g : \mathbb{N} \setminus \{0\} \to \mathbb{R}$ the following way:

$$g(N) := \{f : \forall \varepsilon > 0\, (f - \varepsilon, N) \text{ is stable and } (f + \varepsilon, N) \text{ is unstable}\}.$$

In other words, we will search for that value f for a fixed N where upon increasing f the system becomes unstable. We will assume that there is at most one such value for every N. Note that it may happen that $g(N) < 0$ or $g(N) = \infty$, which means that for some (D_p, D_r, D_t) triplets the system will always be either unstable, or stable. From now on, we will examine those cases when $0 \le g(N) < \infty$.

We also define the following parameter:

$$C_r := 4sin^2 \frac{\pi r}{N}.$$

It is clear that $C_r \in (0, 4)$. Now we state the following proposition:

Proposition 2.1. *For every fixed (D_p, D_r, D_t) triplet*

$$g(N) \longrightarrow \min_{n \in \mathbb{N}} g(n) \text{ as } N \to \infty.$$

The convergence of $g(N)$ means that if we increase N, after a sufficiently large N_0 the value f where the system changes stabilty will be almost the same for every value of N for which $N > N_0$. Also, because we know the limit, we can choose the value N to be $N = \min_{n\in\mathbb{N}} g(n)$ and do not have to use a large N (which would mean the calculation of the eigenvalues of N matrices).

For the proof of Proposition 2.1, we define the following sequence for every value of f:

$$a_N(f) := \left\{ \begin{array}{c} 1, \text{ if the system is stable for } (f, N) \\ -1, \text{ if the system is unstable for } (f, N) \end{array} \right\}.$$

Proposition 2.2. *For every $f \neq \min_{n\in\mathbb{N}} g(n)$, $a_N(f)$ converges as $N \to \infty$.*

Note that because a_N is a function with a discrete range, the convergence means that

$$\exists N_0 : \forall n, m > N_0 : a_n(f) = a_m(f).$$

Proof. (Proposition 2.2.) For the proof, we will use the Routh-Hurwitz criteria for stability (see Theorem 5.1 in the appendix). From this, we get that our system is stable if the functions A, B, C and D are positive on the interval $[0, 4]$ (for the definitions of these functions, see the appendix). When we change N, we examine several points from this interval, and if at all points the system is stable (all functions are positive), then we call it stable - otherwise, we call it unstable.

Now, we will separate three cases. In the first case, let us assume that all functions are positive on the whole interval $[0, 4]$. It means that if we take any number of points, the system is always stable. For these values of f $a_N(f) = 1$, so because it is a constant sequence, it converges.

In the second case, let us assume that for every point in the interval $[0, 4]$ there is always a function which is not positive. It means that if we take any number of points, the system is always unstable. For these values of f $a_N(f) = -1$, so it converges.

In the third case, let us assume that there are stable and unstable parts of the interval $[0, 4]$. Then, in this case, we will prove three lemmas:

Lemma 2.1. *Let us assume that $f \neq \min_{n\in\mathbb{N}} g(n)$. In this case, there exists an N for which there is an examined point where the system is unstable if and only if there is an unstable interval in $[0, 4]$.*

Lemma 2.2. *If there exists an N such that there is an examined point in the unstable part of the interval $[0, 4]$, then there is an n bigger than N for which there are two such points in the unstable part.*

Lemma 2.3. *If there exists an N such that there are two examined points in the unstable part of $[0, 4]$, then for every n bigger than N there is an examined point in the unstable interval.*

It is clear, that if all three lemmas are true, then by the definition of convergence, $a_N(f) \longrightarrow -1$ as $N \to \infty$, so the proposition holds.

First we prove the first lemma. Let us pick a point in the unstable interval and denote it by x. A point can only be an examination point, if its inverse image by the $4sin^2(\pi x)$ function is rational. If x satisfies this condition, then it is an examined point, and the first lemma holds. If its inverse image is irrational, then by the density of the irrational points on the real line, and because $4sin^2(\pi x)$ is continuous, there exists a rational point near x, for which its image is still in the unstable region.

Now we have to prove that it cannot happen that there are only isolated unstable points in the interval $[0, 4]$ for $f_0 \neq \min_{n\in\mathbb{N}} g(n)$. The isolated points can only occur when one or more of the functions has a minimum value 0 and it is positive otherwise. For the proof, we separate two cases. In the first case, for $f_0+\varepsilon_0$ our system becomes unstable and for $f_0-\varepsilon_0$ it is stable. The change of the stability is caused by the function with minimum value at zero, and all the other functions are positive. This means that $\exists n_0 : f = g(n_0)$. Now we show that in this case, $f = \min_{n\in\mathbb{N}} g(n)$. Let us assume that $\exists n_1$ for which $g(n_1) < g(n_0) = f$. This means that for $(f, n_1) : f \in (f_0 - \varepsilon_1, f_0 + \varepsilon_1)$ our system is unstable for every small ε_1. This means that our system is unstable at $(n_0, f_0 - \varepsilon_1)$ for every small ε_1, but if $\varepsilon_1 = \varepsilon_0$, it must be stable, so we got a contradiction.

In the second case let us assume that our system is stable in every small neighbourhood of f_0, but it is unstable at f_0. If the unstable point has a rational inverse image, this will mean that $g(N)$ is not well defined. On the other hand, if the unstable point has an irrational inverse image, then $g(N) < 0$, but we assumed that $g(N) \geq 0$. Thus we proved the first lemma.

Now we prove the second lemma. Let us pick two points in the unstable interval (because of the first lemma, we know that such an interval exists). If both points have rational inverse images, for example $\frac{a}{b}$ and $\frac{c}{d}$ then if we take the common denominator of the two points, for example $\frac{e}{(b,d)}$ and $\frac{h}{(b,d)}$, then if $n = (b, d)$, then these two points are examined points. If one of the points has an irrational inverse image, then as in the previous case, we can choose another point still in the unstable interval, which has a rational inverse image. Thus we proved the second lemma.

We prove the third lemma by contradiction. Let us assume that $\forall n : n > N$ there are no examined points in the interval. Because of the second lemma, we know that there exists an $N_1 \leq N$ for which there are two examined points in the unstable interval. Let us suppose that there are no other examined points between them, and denote them by y and z. Let us look at those inverse images of these points which are in the interval $[0, 0.5]$. Let us denote these inverse images by y' and z'. We know that their difference is $\frac{1}{N_1}$. Because of the assumption, we know that there is no such point which has an inverse image in $[y', z']$ for $\forall n : n > N$. It would mean that the inverse images of the examined points are farther than $\frac{1}{n}$ for every $n > N$ – however, it is a contradiction. ■

Proof. (Proposition 2.1.) We prove the proposition by contradiction. Let us assume that there is an $\varepsilon > 0$ for which $\forall N$ there exists an $n > N$ for which $|g(n) - \min_{n\in\mathbb{N}} g(n)| \geq \varepsilon$. Let us examine $a_N(\min_{n\in\mathbb{N}} g(n) + \varepsilon)$. By the previous proposition, we know that $a_N(\min_{n\in\mathbb{N}} g(n) + \varepsilon)$ converges. However, because of the first assumption, it cannot converge, so we get a contradiction. ■

It is clear that the $f = \min_{n\in\mathbb{N}} g(n)$ case is the one when there are only isolated unstable points on the interval $[0, 4]$. Depending on their location, we separate three cases.

In case (A), the only unstable point is at $C_r = 0$. In this case $g(N)$ is a constant function, because we found the unstable point for $N = 1$, so it will not change.

In case (B), the only unstable point is at $C_r = 4$. In this case $g \to g(2)$, and $g(N) = g(2)$ for every even N.

In case (C), we assume that one function, or more functions have a minimum value zero in the interval. If all the points have irrational inverse images, then we will never find them, so in this case $a_N(\min_{n\in\mathbb{N}} g(n)) = 1$. If at least one of them has a rational inverse image (let us denote it by $\frac{a}{b}$), then the sequence $g(N)$ will be eventually periodic with period b. If more of them has a rational inverse image, then the sequence will have more periods, having the smallest denominator as prime period.

As we can see, the minimum of the function $g(N)$ may have different values depending on the (D_p, D_r, D_t) triplet. However, we can state the following proposition about the maximum of this function.

Proposition 2.3. *For every (D_p, D_r, D_t) triplet*

$$\max_{n\in\mathbb{N}} g(n) = g(1).$$

Proof. We prove this by contradiction. Let us assume that $\exists k : g(k) > g(1)$. We know that for this k, the $(g(1) + \varepsilon, k)$ pair will be a stable point (where $\varepsilon > 0$ is a small value). But this is a contradiction, because the point $C_r = 0$ is an examined point for every $n \in \mathbb{N}$, so the $(g(1) + \varepsilon, 1)$ point must have been stable, but it was not. ■

If we want to compute $g(1)$, we will have to search for that one f value, for which (at least) one of the stability functions has a root at $C_r = 0$. Because of the form of the functions B, C and D, the value of $g(1)$ will be the same for every (D_p, D_r, D_t) triplet. This way we get the following three inequalities for those values of f for which the system is stable:

$$-0.3f_2 - 0.3f_1 + 0.075 > 0,$$

$$-10f_1 - 10.03f_2 + 0.3075 > 0,$$

$$(-10f_1 - 10.03f_2 + 0.3075)(10.03 - f_2) + -0.3f_2 - 0.3f_1 + 0.075 > 0,$$

where

$$f_1 := \frac{-2250f}{1 + 12000f},$$

$$f_2 := \frac{6000f - 1}{2 + 24000f}.$$

This way, we get that our system is stable only if $f < 1.3353 \cdot 10^{-3}$, so $g(1) = 1.3353 \cdot 10^{-3}$ for every (D_p, D_r, D_t) value.

3. Numerical results

For the numerical calculations, we will use the bisection method. In this case, we choose one point (f_0) to be very small (in the algorithm, we use 0) and the other (f_1) to be very large (we choose it to be 10^7). If the system is stable in f_0 and unstable in f_1, then the border is between these two, so we check it in the point $\frac{f_0 + f_1}{2}$. If it is stable, then the border is above this point, and if it is unstable, then it is under this point. Then we continue this iteration while the distance of f_0 and f_1 will be small (we usually use 0.0001), and then we say that the border is the mean of the two endpoints of the last interval. This way we search for every N for the only value of f where the system changes its stability. We get the graph on Figure 2 for the (D_p, D_r, D_t) values $(0.1, 0.003, 0.003)$, where the points above the graph are unstable points, and the ones below it are stable.

So we got the function $g(N)$ with the convergence property proved before. We can also see that we got case (B) mentioned before, because $g(2) = g(N)$ for every even N.

For (D_p, D_r, D_t) values $(0.1, 0.03, 0.03)$ we get the graph on Figure 3.

In this case, we got a constant function $g(N)$, so we got case (A).

Note that we also got the value of $g(1)$ calculated before.

During our examinations, we tested several values, but found only cases (A) and (B). Because of this, we state the following conjecture:

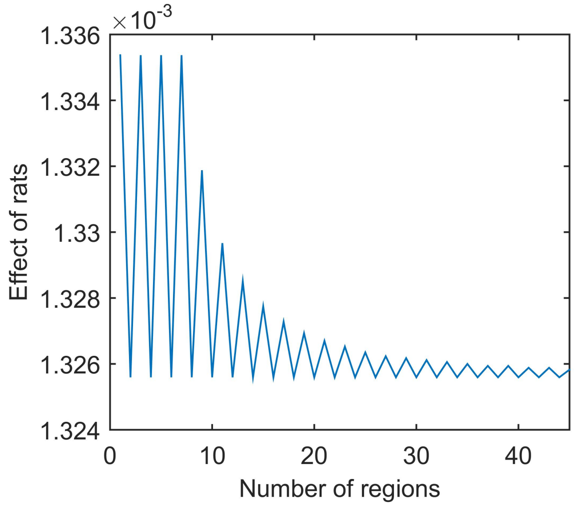

Figure 2. The border of stability (and also the image of the function $g(N)$) for $(0.1, 0.003, 0.003)$.

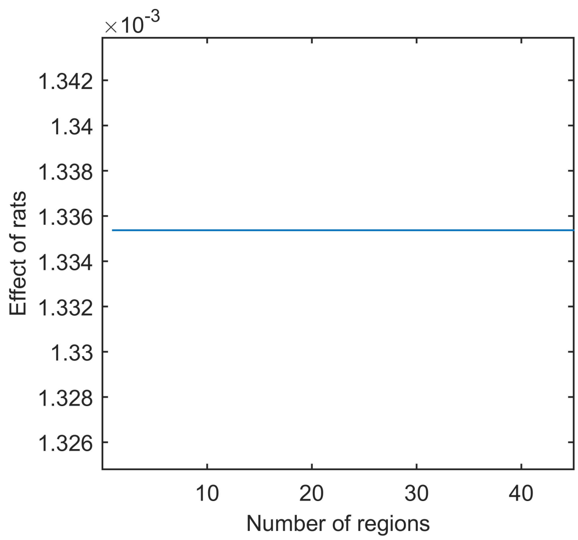

Figure 3. The border of stability (and also the image of the $g(N)$ function) for $(0.1, 0.03, 0.03)$.

Conjecture 3.1. *For every* (D_p, D_r, D_t) *triplet*

$$\min_{n\in\mathbb{N}} g(n) = g(2).$$

This way, Proposition 2.1 has the following form:

$$g(N) \longrightarrow g(2) \text{ as } N \to \infty.$$

Note that in this case the previous case (A) is just a special version of case (B).

4. Conclusions

In the previous pages we studied the stability region on the (f, N) plane for fixed (D_p, D_r, D_t) values. It turned out that the border of this region described by the function $g(n)$ can have three different shapes, depending on the (D_p, D_r, D_t) triplet. We also stated a conjecture, from which we get the following property for the $g(n)$ function:

$$g(2) \le g(n) \le g(1).$$

Without the conjecture, we only have an upper bound for this function, which means that for every $f > g(1)$, the system is unstable.

From all this we can conclude that the choice $f = 0.001$ in [3] is under the value $g(1)$, so the system can change stability there for some (D_p, D_r, D_t) values, but not necessarily (see the $(0.1, 0.03, 0.03)$ case above).

However, we can make a better choice than $N = 10$ using the previous propositions: if our conjecture is true, it is enough to examine our system for $N = 2$. If the conjecture is false, then we can also use the $N = 2$ value (it is not worse than $N = 10$), or we have to calculate $\min_{n\in\mathbb{N}} g(n)$ for every (D_p, D_r, D_t) triplet.

This way, the results of this paper give motivation on how to fix the parameters (f, N) in this analysis.

In our paper we assumed that $g(n)$ is well-defined. The proof of this statement and the conjecture may be the subject of further research. Also, we only examined our system for fixed (D_p, D_r, D_t) values: another method would be to fix the (f, N) values and only change the parameters of diffusion (like in [3]). This method will be used in [4].

5. Appendix

Our system (described by (1.6)) is stable if and only if all the eigenvalues of its characteristic polynomial have negative real parts. The characteristic polynomial is the determinant of the matrix $|I\lambda - A|$, which is

$$\lambda^3 + F_1\lambda^2 + F_2\lambda + F_3,$$

where

$$F_1 = C_rD_t - f_2 + 0.03 + C_rD_p + 10 + C_rD_r,$$
$$F_2 = (10 + C_rD_r)(C_rD_t - f_2) + (0.03 + C_rD_p)(C_rD_t - f_2)+ \\ +(10 + C_rD_r)(0.03 + C_rD_p) - 10f_1 + 0.0075,$$
$$F_3 = (0.03{+}C_rD_p)(10{+}C_rD_r)(C_rD_t{-}f_2){-}10f_1(0.03{+}C_rD_p){+}0.0075(10{+}C_rD_r).$$

It is quite difficult to compute the real parts of the roots of this function. Because of this, we will use the following theorem:

Theorem 5.1 (Routh - Hurwitz criteria for stability, [7], [8])**.** *Let us consider the following polynomial:* $p(x) = x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + a_0$. *Then all of its roots will have a negative real part if and only if all the minors of the following matrix are positive definite:*

$$\begin{pmatrix} a_{n-1} & 1 & 0 & \dots & \dots & 0 \\ a_{n-3} & a_{n-2} & a_{n-1} & 1 & 0 & \dots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & a_0 & a_1 & a_2 \\ 0 & \dots & \dots & \dots & 0 & a_0 \end{pmatrix}.$$

If $n = 3$, the previous theorem can be stated the following way:

Theorem 5.2 (Routh - Hurwitz criteria for three dimensional systems)**.** *Let us consider the following polynomial:* $p(x) = x^3 + a_2x^2 + a_1x + a_0$. *Then all of its roots will have a negative real part if and only if* $a_2 > 0$, $a_1 > 0$, $a_0 > 0$ *and* $a_2a_1 - a_0 > 0$.

It is also easy to see that in this case a_1 is the opposite of the trace of the matrix, a_2 is the sum of all of its minors, and a_3 is the determinant.

Using Theorem 5.1, we get four functions which must be positive if our system is stable, which are as follows. The first one is

$$A(C_r) := C_r(D_p + D_r + D_t) + 10.03 - f_2,$$

which is always positive. The second one is

$$B(C_r) := B_1 C_r^3 + B_2 C_r^2 + B_3 C_r + B_4,$$

where

$$B_1 = D_p D_r D_t,$$
$$B_2 = 0.03 D_r D_t + 10 D_p D_t - f_2 D_p D_r,$$
$$B_3 = -10 f_2 D_p + 0.3 D_t - 0.03 f_2 D_r - 10 f_1 D_p + 0.0075 D_r,$$
$$B_4 = -0.3 f_2 - 0.3 f_1 + 0.075,$$

which is a first concave, then convex (as C_r increases) cubic function.

For the third one, we define three other terms:

$$G_1 = D_r D_t + D_p D_t + D_r D_p,$$
$$G_2 = 10 D_t - f_2 D_r + 0.03 D_t - f_2 D_p + 0.03 D_r + 10 D_p,$$
$$G_3 = -10.03 f_2 - 10 f_1 + 0.3075;$$

with these, we can define function C the following way:

$$C(C_r) := H_1 C_r^3 + H_2 C_r^2 + H_3 C_r + H_4,$$

where

$$H_1 = G_1(D_p + D_r + D_t) - B_1,$$
$$H_2 = G_2(D_p + D_r + D_t) + G_1(10.03 - f_2) - B_2,$$
$$H_3 = G_2(10.03 - f_2) + G_3(D_p + D_r + D_t) - B_3,$$
$$H_4 = G_3(10.03 - f_2) - B_4,$$

which has the same shape as B. The fourth one is an upward parabola:

$$D(C_r) := L_1 C_r^2 + L_2 C_r + L_3,$$

where

$$L_1 = D_r D_t + D_p D_t + D_r D_p,$$
$$L_2 = 10.03 D_t + (10 - f_2) D_p + (0.03 - f_2) D_r,$$
$$L_3 = -10 f_1 - 10.03 f_2 + 0.375.$$

With these, we can state the following:

$$\text{The system is stable} \iff B(C_r) > 0, C(C_r) > 0, D(C_r) > 0 \text{ for } \forall C_r \in [0, 4].$$

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