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SPECTRAL SYNTHESIS ON COMMUTATIVE HYPERGROUPS

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Abstract. The problem of spectral synthesis is formulated on commutative hypergroups and is solved for finite dimensional varieties.

1. Introduction

The study of spectral analysis and spectral synthesis problems is based on the concept of exponential monomials on Abelian groups. Until the appropriate definitions were not available on general hypergroups these problems could be studied on special types of hypergroups only. In the papers [4] and [5] we introduced the corresponding concept on polynomial hypergroups in one variable and in several variables, respectively. Using these concepts we were able to prove spectral analysis and spectral synthesis on those types of hypergroups. Based on the generally accepted term *exponential* on hypergroups in [6] we proved spectral analysis for finite dimensional varieties on commutative hypergroups (see also [7]).

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As in the papers [8, 9] the first author introduced a definition for exponential monomials on general (commutative) hypergroups, it is possible to formulate and to study the problems of spectral synthesis on those structures. In this paper we prove that spectral synthesis holds for finite dimensional varieties on any commutative hypergroup.

In this paper K will denote a commutative hypergroup. The nonzero continuous function $m : K \to \mathbb{C}$ is an *exponential* if

$$m(x)m(y) = m(x * y) = \int_K m(t)\, d(\delta_x * \delta_y)(t) \tag{1}$$

holds for any x, y in K. Here, as usual, δ_x denotes the point mass with support at the singleton $\{x\}$.

If y is in K and $f : K \to \mathbb{C}$ is a continuous function, then the function $\tau_y f$ defined by

$$\tau_y f(x) = f(x * y) = \int_K f(t)\, d(\delta_x * \delta_y)(t)$$

for each x in K is called the *translate* of f by y. Let $\mathcal{C}(K)$ denote the space of all continuous complex valued functions on K equipped with the pointwise operations and the topology of uniform convergence on compact sets. A closed subspace V of $\mathcal{C}(K)$ is called a *variety*, if it is translation invariant, that is, if f belongs to V then all translates of f belongs to V, too. Given a continuous function f the variety *generated by* f is the intersection of all varieties including f and it is denoted by $\tau(f)$.

The problem of *spectral analysis* for a given variety means that we are looking for exponentials in the variety. If there is an exponential in the variety then we say that spectral analysis holds for the variety. If spectral analysis holds for each nonzero variety, then we say that spectral analysis holds on the hypergroup K. In [6] it has been proved that spectral analysis holds for every finite dimensional variety on any commutative hypergroup.

For each exponential on K and for every y in K we define the *modified difference* $\Delta_{m;y}$ as the the linear operator on $\mathcal{C}(K)$ by the equation

$$\Delta_{m;y} = \tau_y - m(y)\tau_e,$$

that is, we have for each x, y in K and f in $\mathcal{C}(K)$

$$\Delta_{m;y} f(x) = f(x * y) - m(y)f(x).$$

Iterates of modified difference operators are defined by successive composition and we use the notation

$$\Delta_{m;y_1,y_2,\ldots,y_{n+1}} = \Delta_{m;y_1,y_2,\ldots,y_n} \circ \Delta_{m;y_{n+1}}$$

for each $y_1, y_2, \ldots, y_{n+1}$ in K.

The continuous function $\varphi : K \to \mathbb{C}$ is called a *generalized exponential monomial*, if there exists an exponential m and a natural number n such that

$$\Delta_{m;y_1,y_2,\dots,y_{n+1}} f(x) = 0$$

holds for each $x, y_1, y_2, \dots, y_{n+1}$ in K. We note that if $\varphi \neq 0$, then m is unique, and the smallest n with this property is called the *degree* of φ. In this case we say that φ *corresponds* to the exponential m. The generalized exponential monomial φ is simply called an *exponential monomial*, if $\tau(\varphi)$ is finite dimensional (see [8, 9]). Linear combinations of exponential monomials are called *exponential polynomials.*

We say that the variety V in $\mathcal{C}(K)$ is *synthesizable* if all exponential monomials in V span a dense subspace. We say that *spectral synthesis holds for* V if every subvariety of V is synthesizable. If every variety in $\mathcal{C}(K)$ is synthesizable, then we say that *spectral synthesis holds on* K. In [4] and in [5] we proved that spectral synthesis holds on every polynomial hypergroup. It is known that spectral synthesis holds for every finite dimensional variety on any commutative topological group (see e.g. [3, 10]). In the subsequent paragraphs we prove the same result for commutative hypergroups.

2. A matrix equation

In this section first we shall consider the matrix equation

$$L(x * y) = L(x)L(y) \tag{2}$$

on the commutative hypergroup K, where $L : K \to \mathcal{L}(\mathbb{C}^n)$ is a continuous mapping and the equation is supposed to hold for each x, y in K. Here $\mathcal{L}(\mathbb{C}^n)$ denotes the space of all linear operators on $\mathbb{C}^n$, which is identified with the space of all $n \times n$ complex matrices. This equation has been studied on Abelian groups, even on commutative semigroups (see e.g. [3], and further references given therein). We shall apply our results for the Levi–Civitá functional equation and for spectral synthesis on finite dimensional varieties.

We shall use the following result (see [1], [2]).

Theorem 1. *Let $\mathcal{S}$ be a family of commuting linear operators in $\mathcal{L}(\mathbb{C}^n)$. Then $\mathbb{C}^n$ decomposes into a direct sum of linear subspaces X_j such that each X_j is a minimal invariant subspace under the operators in $\mathcal{S}$. Further, $\mathbb{C}^n$ has a basis in which every operator in $\mathcal{S}$ is represented by an upper triangular matrix.*

In other words, there exist positive integers $k, n_1, n_2, \ldots, n_k$ with the property $n_1+n_2+\cdots+n_k = n$, and there exists a regular matrix C such that every matrix L in $\mathcal{S}$ has the form

$$L = S^{-1}\mathrm{diag}\,\{L_1, L_2, \ldots, L_k\}S$$

where L_j is upper triangular for $j = 1, 2, \ldots, k$. Here $\mathrm{diag}\,\{L_1, L_2, \ldots, L_k\}$ denotes the block matrix with blocks $L_1, L_2, \ldots, L_k$ along the main diagonal, and all diagonal elements of the block L_j are the same. As a consequence the following theorem holds true.

Theorem 2. *Let K be a hypergroup and let $L : K \to \mathcal{L}(\mathbb{C}^n)$ be a continuous mapping satisfying (2) for each x, y in K. Then there exist positive integers $k, n_1, n_2, \ldots, n_k$ with the property $n_1 + n_2 + \cdots + n_k = n$, and there exists a regular matrix S such that*

$$L(x) = S^{-1}\mathrm{diag}\,\{L_1(x), L_2(x), \ldots, L_k(x)\}S \tag{3}$$

for each x in K, where $L_j(x)$ is an upper triangular $n_j \times n_j$ matrix in which all diagonal elements are equal, and it satisfies (2) for each x, y in K and for every $j = 1, 2, \ldots, l$.

3. Spectral synthesis for finite dimensional varieties

Theorem 3. *Spectral synthesis holds for finite dimensional varieties on every commutative hypergroup.*

Proof. Suppose that K is a commutative hypergroup and $V \neq \{0\}$ is a finite dimensional variety in $\mathcal{C}(K)$. We show that V consists of exponential polynomials. Let $f_1, f_2, \ldots, f_n$ be a basis of V, then there exist complex valued functions $\lambda_{i,j}$ for $i, j = 1, 2, \ldots, n$ such that

$$f_i(x * y) = \sum_{j=1}^{n} \lambda_{i,j}(y) f_j(x) \tag{4}$$

holds for every x, y in K and $i = 1, 2, \ldots, n$. As the functions $f_1, f_2, \ldots, f_n$ are linearly independent, hence there are elements x_k for $k = 1, 2, \ldots, n$ in K such that the matrix $\big(f_j(x_k)\big)_{j,k=1}^{n}$ is regular. We have

$$f_i(x_k * y) = \sum_{j=1}^{n} \lambda_{i,j}(y) f_j(x_k)$$

for each y in K and $k = 1, 2, \ldots, n$.

Using the associativity of convolution we infer that

$$\sum_{j=1}^{n} \lambda_{i,j}(z) f_j(x * y) = \sum_{j=1}^{n} \lambda_{i,j}(y * z) f_j(x)\,, \tag{5}$$

or

$$\sum_{j=1}^{n} \lambda_{i,j}(z) \sum_{l=1}^{n} \lambda_{j,l}(y) f_l(x) = \sum_{j=1}^{n} \lambda_{i,j}(y * z) f_j(x) \tag{6}$$

holds for each x, y, z in K. This is equivalent to

$$\sum_{k=1}^{n} \sum_{j=1}^{n} \lambda_{i,j}(z) \lambda_{j,k}(y) f_k(x) = \sum_{k=1}^{n} \lambda_{i,k}(y * z) f_k(x)\,. \tag{7}$$

By the linear independence of the f_k's we have

$$\sum_{j=1}^{n} \lambda_{i,j}(z) \lambda_{j,k}(y) = \lambda_{i,k}(y * z) = \lambda_{i,k}(z * y) \tag{8}$$

for each y, z in K. Let $L(x)$ be the matrix $\big(\lambda_{i,j}(x)\big)_{i,j=1}^{n}$, then from (8) it follows

$$L(x * y) = L(x) \cdot L(y) \tag{9}$$

for each x, y in K. In particular, the matrices $L(x)$ are commuting for different x's. By Theorem 2, $L(x)$ has the form given in Theorem 2. We show that the elements of each L_j are exponential polynomials for $j = 1, 2, \ldots, l$. For the sake of simplicity we suppose that $l = 1$ and $L = L_1$. Then L is upper triangular: $\lambda_{i,j} = 0$ for $i > j$, it satisfies equation (2), and all diagonal elements in L are the same: $\lambda_{i,i} = \lambda_{j,j}$ for $i, j = 1, 2, \ldots, n$. Then we have

$$\lambda_{i,j}(x * y) = \sum_{k=i}^{j} \lambda_{i,k}(x) \cdot \lambda_{k,j}(y)$$

holds for $i = 1, 2, \ldots, j$ and for each x, y in K. If we put $j = i$ we get

$$\lambda_{i,i}(x * y) = \lambda_{i,i}(x) \cdot \lambda_{i,i}(y) \tag{10}$$

for $i = 1, 2, \ldots, n$ and for each x, y in K, which means that the functions $\lambda_{i,i}$ ($i = 1, 2, \ldots, n$) are exponentials. We denote $m(x) = \lambda_{i,i}(x)$ for x in K and $i = 1, 2, \ldots, n$. We show by induction on $j - i$ that $\lambda_{i,j}$ is an exponential monomial of order $j-i$ corresponding to the exponential m. As $\lambda_{i,j}$ is in V for each i,j and V is a variety, hence it is enough to show that

$$\Delta_{m;y_1,y_2,\dots,y_{j-i+1}}\lambda_{i,j}(x)=0.$$

Clearly, the statement holds for $j-i=0$. Suppose that we have proved it for $j-i\leq l$ and let $j=i+l+1$. Then we have

$$\Delta_{m;y_1,y_2,\dots,y_{l+1},y_{l+2}}\lambda_{i,i+l+1}(x)=$$

$$\Delta_{m;y_1,y_2,\dots,y_{l+1}}[\lambda_{i,i+l+1}(x*y_{l+2})-m(y_{l+2})\lambda_{i,i+l+1}(x)]=$$

$$\Delta_{m;y_1,\dots,y_{l+1}}\Big[\sum_{k=i}^{i+l+1}\lambda_{i,k}(x)\lambda_{k,i+l+1}(y_{l+2})\Big]-m(y_{l+2})\Delta_{m;y_1,\dots,y_{l+1}}\lambda_{i,i+l+1}(x)=$$

$$\Delta_{m;y_1,\dots,y_{l+1}}\Big[\lambda_{i,i+l+1}(x)m(y_{l+2})\Big]-m(y_{l+2})\Delta_{m;y_1,\dots,y_{l+1}}\lambda_{i,i+l+1}(x)=0.$$

This shows that the functions $\lambda_{i,j}$ are all generalized exponential monomials. As $\tau(\lambda_{i,j})$ is in V, hence, in fact, the functions $\lambda_{i,j}$ are exponential monomials. By (4), substitution $x=e$ gives that f_i is an exponential polynomial for each i, hence the variety V consists of exponential polynomials. The proof is complete. ■

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