ITERATES OF THE SUM OF THE UNITARY DIVISORS OF AN INTEGER

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Dedicated to Professor Pavel D. Varbanets
on the occasion of his 80-th birthday

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Abstract. Given an integer \( k \geq 0 \), let \( \sigma_k^*(n) \) stand for the \( k \)-fold iterate of \( \sigma^*(n) \), the sum of the unitary divisors of \( n \). We show that \( \frac{\sigma_k^*(p+1)}{\sigma_1^*(p+1)} \) tends to 1 for almost all primes \( p \).

1. Introduction and notation

Let \( \sigma^*(n) \) be the sum of the unitary divisors of \( n \), that is,

\[ \sigma^*(n) := \sum_{d|n \text{ gcd}(d,n/d) = 1} d. \]

Given an integer \( k \geq 0 \), let \( \sigma_k^*(n) \) stand for the \( k \)-fold iterate of \( \sigma^*(n) \), that is, \( \sigma_0^*(n) = n, \sigma_1^*(n) = \sigma^*(n), \sigma_2^*(n) = \sigma^*(\sigma_1^*(n)) \), and so on. The function \( \sigma^*(n) \) is easily checked to be multiplicative with \( \sigma^*(p^\alpha) = p^\alpha + 1 \) for each prime \( p \) and integer \( \alpha \geq 1 \).

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In 1971, Erdős and Subbarao proved that

\[ \frac{\sigma_2(n)}{\sigma_1(n)} \rightarrow 1 \quad \text{for almost all } n. \]

This is quite a contrast with the easily proven estimate

\[ \frac{\sigma_2(n)}{\sigma_1(n)} \rightarrow \infty \quad \text{for almost all } n. \]

In 1991, Kátai and Wisjmueller proved that

\[ \frac{\sigma_3(n)}{\sigma_2(n)} \rightarrow 1 \quad \text{for almost all } n \]

and conjectured that, given an arbitrary integer \( k \geq 0 \),

\[ \frac{\sigma_{k+1}(n)}{\sigma_k(n)} \rightarrow 1 \quad \text{for almost all } n. \]

This remains unproven.

Here, we consider similar quotients, namely those where the arguments of the functions \( \sigma_{k+1} \) and \( \sigma_k \) are running over shifted primes.

2. Main result

**Theorem 1.** We have

\[ \frac{\sigma_2(p+1)}{\sigma_1(p+1)} \rightarrow 1 \quad \text{for almost all primes } p. \]

Of course, the above statement is equivalent to the following one.

Given any \( \varepsilon > 0 \), we have

\[ \lim_{x \to \infty} \frac{1}{\pi(x)\#\left\{p \leq x : \frac{\sigma_2(p+1)}{\sigma_1(p+1)} > 1 + \varepsilon\right\}} = 0. \]

In the following, we denote by \( p(n) \) and \( P(n) \) the smallest and largest prime factors of \( n \), respectively. We let \( \mu(n) \) stand for the Moebius function and \( \phi(n) \) for the Euler totient function. For each integer \( n \geq 2 \), we let \( \omega(n) \) stand for the number of distinct prime factors of \( n \) and set \( \omega(1) = 0 \). The letters \( p, q, \pi \) and
3. Preliminary results

For the proof of our main result, we will need the following lemmas.

Our first lemma is a classical result.

Lemma 1. (Brun-Titchmarsh Theorem) For every positive integer \( k < x \), we have

\[
\pi(x; k, \ell) \leq \frac{2x}{\phi(k) \log(x/k)}.
\]

A proof of the following result follows from Theorem 3.12 in the book of Halberstam and Richert [4].

Lemma 2. There exists a positive constant \( C_1 > 0 \) such that

\[
\#\{p, q \in \mathbb{P} : p + 1 = aq \leq x\} < C_1 \frac{x}{\phi(a) \log^2(x/a)}.
\]

The following can be obtained from Theorem 4.2 in the book of Halberstam and Richert [4].

Lemma 3. Given an arbitrary positive number \( \delta < 1 \), there exists an absolute constant \( C_2 > 0 \) such that

\[
\lim_{x \to \infty} \#\{p \leq x : P(p + 1) < x^\delta \text{ or } P(p + 1) > x^{1-\delta}\} < C_2 \delta \text{ li}(x).
\]

It was proved by the first author [5] that the distribution of the numbers

\[
\frac{\omega(\sigma(p + 1)) - \frac{1}{2}(\log \log p)^2}{\frac{1}{\sqrt{3}}(\log \log p)^{3/2}}
\]

as \( p \) runs through the primes obeys the Gaussian Law. It is easy to show that the same statement holds if \( \sigma(p + 1) \) is replaced by \( \sigma^*(p + 1) \). The following result is an immediate consequence of this observation.
Lemma 4. We have
\[
\frac{1}{\pi(x)} \# \{p \leq x : \omega(\sigma^*(p+1)) > x^2 \} \to 0 \quad \text{as } x \to \infty.
\]

4. Proof of the main result

First let
\[
t(n) := \sum_{q^\alpha \| n} \frac{1}{q^\alpha}.
\]
It is clear that, for each integer \( n \geq 2 \),
\[
\frac{\sigma^*(n)}{n} = \prod_{q^\alpha \| n} \left( 1 + \frac{1}{q^\alpha} \right) = \exp \left\{ \sum_{q^\alpha \| n} \log \left( 1 + \frac{1}{q^\alpha} \right) \right\} \leq \exp \left\{ \sum_{q^\alpha \| n} \frac{1}{q^{\alpha}} \right\}
\]
and therefore that
\[
\frac{\sigma^+_2(n)}{\sigma^*_1(n)} = \frac{\sigma^*(\sigma^*(n))}{\sigma^*(n)} \leq \exp \left\{ \sum_{q^\alpha \| n} \frac{1}{q^{\alpha}} \right\} = e^{t(n)}.
\]

From this observation, it follows that the claim in Theorem 1 is equivalent to
the assertion that
\[
(4.1) \quad t(p+1) \to 0 \quad \text{for almost all primes } p.
\]

Let \( \delta > 0 \) be any small number, let \( \wp_x := \{ p \in \wp : p \leq x \} \) and consider the set
\[
\wp_x^{(1)} := \{ p \leq x : P(p+1) < x^\delta \text{ or } P(p+1) > x^{1-\delta} \}.
\]
In light of Lemma 3,
\[
(4.2) \quad \# \wp_x^{(1)} < C_2 \delta \text{li}(x).
\]

This is why we only need to work with the set
\[
\wp_x^{(2)} := \wp_x \setminus \wp_x^{(1)}.
\]

So, let us assume that \( p \in \wp_x^{(2)} \), let \( T \) be a large integer and consider the set \( \mathcal{D}_T \) made up of all those primes \( p \) such that \( \pi^2 \mid p+1 \) for some prime \( \pi > T \).
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Using Lemma 1 for some constant $C_3 > 0$, we then have

$$\#\{p \leq x : p \in \mathcal{D}_T\} \leq \sum_{T < \pi < \sqrt{x}} \sum_{p \equiv 0 \pmod{\pi^2}} 1 = \sum_{T < \pi < \sqrt{x}} \pi(x; \pi^2, -1) \leq C_3 \text{li}(x) \sum_{T < \pi < \sqrt{x}} \frac{1}{\phi(\pi^2)} = C_3 \text{li}(x) \sum_{T < \pi < \sqrt{x}} \frac{1}{\pi(\pi - 1)} \ll \frac{\text{li}(x)}{T \log T} + O(x^{3/4}).$$

Hence, in light of (4.3), we may now discard those primes $p \leq x$ for which $p \in \mathcal{D}_T$, since their number is $O(\text{li}(x)/(T \log T))$. This is why, for each prime number $q$, we now focus our attention on the set

(4.4) $E_q(x) := \{p \in \psi^{(2)}_x : p \notin \mathcal{D}_T \text{ and } q \nmid \sigma^*(p + 1)\}$

and the sum

$$S_T(x) := \sum_{q \leq T} \#E_q(x).$$

Moreover, for each prime $q$, we let $\mathcal{B}_q$ be the semigroup generated by those primes $Q$ such that $q \nmid Q + 1$.

Let us now consider a fixed prime $q \leq T$ and those integers

$$K = \pi_1^{\alpha_1} \cdots \pi_r^{\alpha_r} \geq 2 \text{ for which } q \mid \pi_j^{\alpha_j} + 1$$

for $j = 1, \ldots, r$ with $\alpha_j = 1$ if $\pi_j > T$.

In order to estimate $\#E_q(x)$, we first introduce the set

$$H_{K,R} := \{p : p + 1 = KRmP, P(R) \leq T, p(mp) > T, \mu^2(m) = 1, (m, B_q) = 1, P(p + 1) = P\}.$$

Writing each prime $p \in H_{K,R}$ as $p + 1 = KRmP = aP$, then, since $P$ was assumed to be such that $P > x^\delta$, it follows that $a < x^{1-\delta}$, and therefore, using
Lemma that

\[ \#H_{K,R} \leq C_2 \delta C_1 \frac{x}{\log^2 x} \sum_{m \leq x} \frac{\mu^2(m)}{\phi(KRm)} \leq \]

\[ \leq C_1 C_2 \delta \frac{x}{\log^2 x} \frac{1}{\phi(KR)} \sum_{m \leq x/KR \atop (m,B_q)=1} \frac{\mu^2(m)}{\phi(m)} \leq \]

\[ \leq C_1 C_2 \delta \frac{x}{\log^2 x} \frac{1}{\phi(KR)} \cdot \prod_{\tau \leq \pi \leq x \atop \pi \equiv 0 (\text{mod } q)} \left( 1 + \frac{1}{\pi - 1} \right) \leq \]

\[ \leq C_1 C_2 \delta \frac{x}{\log^2 x} \frac{1}{\phi(KR)} \cdot \log x \cdot \exp \left\{ -\frac{x}{q - 1} \right\} \leq \]

\[ \leq C_1 C_2 \delta \frac{1}{\phi(KR) \log(x)} \cdot \exp \left\{ -\frac{x}{q - 1} \right\}. \tag{4.6} \]

On the one hand, observe that

\[ \sum_{R \leq T} \frac{1}{\phi(R)} \leq \prod_{p \leq T} \left( 1 + \frac{1}{\phi(p)} + \frac{1}{\phi(p^2)} + \cdots \right) = \]

\[ = \prod_{p \leq T} \left( 1 + \frac{1}{p - 1} + \frac{1}{p(p - 1)} + \cdots \right) = \]

\[ = \prod_{p \leq T} \left( 1 + \frac{1}{p} + O \left( \frac{1}{p^2} \right) \right) = \]

\[ = \exp \left\{ \sum_{p \leq T} \frac{1}{p} + O(1) \right\} \leq C_4 \log T \tag{4.7} \]

for some positive constant \( C_4 \).

On the other hand, writing each \( K \) as \( K = K_1 K_2 \), where \( P(K_1) \leq T \) and \( p(K_2) > T \). Then, by the nature of \( K \) (see (4.5)), it is clear that \( K_2 \) is square-free and that \( \omega(K_2) \leq r \leq T - 1 \). From this, it follows that, for some constant \( C_5 > 0 \),

\[ \sum_{K} \frac{1}{\phi(K)} \leq \sum_{K_2} \frac{1}{\phi(K_2)} \leq \sum_{j=0}^{T-1} \frac{1}{j!} \left( \sum_{\pi \leq x \atop \pi \equiv 0 (\text{mod } q)} \frac{1}{\pi - 1} \right)^j < C_5 \frac{x^{T-1}}{(T-1)!}. \tag{4.8} \]
Recalling that for all $K, R \in \mathbb{N}$, we have $\phi(KR) \geq \phi(K)\phi(R)$, and combining (4.7) and (4.8) in (4.6), it follows that

$$
\#E_q(x) \leq \sum_{K,R} H_{K,R} \leq C_6 \delta \text{li}(x) \frac{x^{2T-1}}{(T-1)!} \cdot \exp \left\{ -\frac{x_2}{q-1} \right\} \cdot \log T
$$

for some constant $C_6 > 0$, from which it follows that

$$
S_T(x) = o(\text{li}(x)) \quad (x \to \infty).
$$

We now move to estimate $t(p+1)$ when the primes $q$ such that $q^\alpha | \sigma^*(p+1)$ satisfy $q > T$.

We first consider the case where $\alpha = 1$, and for this we introduce the sum

$$
s(p+1) := \sum_{T < q \leq x^{1-\varepsilon}_2} \frac{1}{q},
$$

where $\varepsilon > 0$ is an arbitrarily small number. We then have, using Lemmas 1, 2 and 3

$$
\sum_{p \in \wp_2^{(2)}} s(p+1) = \sum_{T < q \leq x^{1-\varepsilon}_2} \frac{1}{q} \sum_{\substack{p \in \wp_2^{(2)} : p+1 = QHmP, P(H) \leq T, p(m) > T, \cr P > x^\delta, Q + 1 \equiv 0 \pmod{q}, (m, B_q) = 1 \}} + \\
\sum_{T < q \leq x^{1-\varepsilon}_2} \frac{1}{q} \sum_{\substack{p \in \wp_2^{(2)} : p+1 = HmP, P(H) \leq T, p(m) > T, \cr P > x^\delta, P + 1 \equiv 0 \pmod{q} \}} \leq \\
C_2^2 \delta \frac{x}{x_2} \sum_{T < q \leq x^{1-\varepsilon}_2} \frac{1}{q} \sum_{\substack{Q \equiv -1 \pmod{q}, (m, B_q) = 1 \cr P(H) \leq T, p(m) > T}} \frac{1}{\phi(HmQ)} \leq \\
C_2^2 \delta \frac{x}{x_2} \sum_{T < q \leq x^{1-\varepsilon}_2} \frac{1}{q} \sum_{\substack{Q \equiv -1 \pmod{q} \cr P(H) \leq T, p(m) > T}} \frac{1}{\phi(H)} \cdot \\
\prod_{\pi \equiv -1 \pmod{q}} \left( 1 + \frac{1}{\pi - 1} + \frac{1}{\pi(\pi - 1)} + \ldots \right) \leq \\
C_2^2 \delta \frac{x}{x_2} \sum_{T < q \leq x^{1-\varepsilon}_2} \frac{1}{q^2} \cdot 2 \log T \cdot x_2 \cdot \exp \left\{ -\frac{x_2}{q-1} \right\},
$$

from which it follows that

$$
\sum_{p \in \wp_2^{(2)}} s(p+1) \ll \frac{x}{x_1} \frac{\log T}{T} x_2 = o(\text{li}(x)) \quad (x \to \infty).
$$
To account for those \( q \in [x_2^{1+\varepsilon}, x_2^{2+\varepsilon}] \), we will consider the sum
\[
r(p + 1) := \sum_{x_2^{1+\varepsilon} \leq q \leq x_2^{2+\varepsilon}, q \mid \sigma^*(p+1)} \frac{1}{q}.
\]
Proceeding as above, we obtain that
\[
\sum_{p \in \wp^{(2)}} r(p + 1) \leq \sum_{x_2^{1+\varepsilon} \leq q \leq x_2^{2+\varepsilon}} \frac{1}{q} \sum_{Q \leq x} \pi(x; Q, -1) \leq C_2 \delta \log(x) \sum_{x_2^{1+\varepsilon} \leq q \leq x_2^{2+\varepsilon}} \frac{1}{q} \leq C_2 \delta \log(x) \sum_{x_2^{1+\varepsilon} \leq q \leq x_2^{2+\varepsilon}} \frac{1}{q} \leq C_2 \delta \log(x) x_2 \sum_{x_2^{1+\varepsilon} \leq q \leq x_2^{2+\varepsilon}} \frac{1}{q^3},
\]
from which it clearly follows that
\[
(4.12) \quad \sum_{p \in \wp^{(2)}} r(p + 1) = o(\log(x)).
\]

We now split \( t(p + 1) \) into five sums as follows.
\[
(4.13) \quad t(p + 1) = t_1(p + 1) + t_2(p + 1) + t_3(p + 1) + t_4(p + 1) + t_5(p + 1),
\]
where
\[
t_j(p + 1) = \sum_{q^a \mid \sigma^*(p+j)} \frac{1}{q^a} \quad (j = 1, \ldots, 5),
\]
with the five intervals \( I_j \) being defined as
\[
I_1 = [2, T], \quad I_2 = (T, x_2^{1-\varepsilon}], \quad I_3 = (x_2^{1-\varepsilon}, x_2^{1+\varepsilon}), \quad I_4 = [x_2^{1+\varepsilon}, x_2^{2+\varepsilon}], \quad I_5 = (x_2^{2+\varepsilon}, \infty).
\]

We will show that the next five inequalities hold for almost all primes \( p \in \wp^{(2)} \).
First of all, in light of (4.10), we have
\[(4.14)\quad t_1(p + 1) \leq \frac{1}{2^T} \quad (p \to \infty),\]
with the exception of at most \(O \left( \frac{\text{li}(x)}{2^T} \right) \) primes \( p \in \varphi_2^2 \).

In light of (4.11), we have that
\[(4.15)\quad t_2(p + 1) \leq s(p + 1) + \sum_{T < q \leq x^2_2} \frac{1}{q^2} \leq o(1) + \frac{1}{T \log T} \quad (p \to \infty).\]

Clearly,
\[(4.16)\quad t_3(p + 1) \leq \sum_{x^{1-\varepsilon}_2 < q < x^{1+\varepsilon}_2} \frac{1}{q} \leq \log \left( \frac{1 + \varepsilon}{1 - \varepsilon} \right) < 2\varepsilon.\]

In light of (4.12), we have
\[(4.17)\quad t_4(p + 1) \leq r(p + 1) + \sum_{q > x^{1+\varepsilon}_2} \frac{1}{q^2} \leq o(1) + O \left( \frac{1}{x^{1+\varepsilon}_2 x^2_3} \right) \quad (p \to \infty).\]

Finally, using Lemma 4, it follows that
\[(4.18)\quad t_5(p + 1) \leq x^{-\varepsilon}_2 \quad (p \to \infty).\]

Gathering inequalities (4.14) through (4.18) in (4.13), we have thus established that
\[
\frac{1}{\pi(x)} \# \left\{ p \leq x : t(p + 1) > \frac{2}{T} \right\} \leq \delta
\]
and since this is true for every \( \delta > 0 \) and for every large number \( T \), our claim (4.1) is established and the proof of Theorem 1 is complete.

5. Final remark

The unitary analog of \( \phi(n) \) denoted by \( \phi^*(n) \) and introduced by Cohen [1] is defined by
\[
\phi^*(n) := \prod_{p^a || n} (p^a - 1).
\]
Denote by $\phi_k^*(n)$ the $k$-fold iterate of $\phi^*(n)$. Erdős and Subbarao [3] claimed that

$$\frac{\phi_2^*(n)}{\phi_1^*(n)} \rightarrow 1 \quad (n \rightarrow \infty)$$

except on a set of integers $n$ of zero density. In fact, they mentioned that they could prove this result by using the same methods as those used to prove [1,1].

With the approach we used to prove Theorem 1, we can also prove the following.

**Theorem 2.** Given any $\varepsilon > 0$,

$$\frac{1}{\pi(x)} \# \left\{ \frac{\phi_2^*(p+1)}{\phi_1^*(p+1)} < 1 - \varepsilon \right\} \rightarrow 0 \quad (x \rightarrow \infty).$$

References


