Annales Univ. Sci. Budapest., Sect. Comp. 45 (2016) 101–110

ITERATES OF THE SUM OF THE UNITARY DIVISORS OF AN INTEGER

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Dedicated to Professor Pavel D. Varbanets
on the occasion of his 80-th birthday

Communicated by Karl-Heinz Indlekofer

(Received March 6, 2016; accepted July 29, 2016)

Abstract. Given an integer $k \geq 0$, let $\sigma_k^*(n)$ stand for the k-fold iterate of $\sigma^*(n)$, the sum of the unitary divisors of n. We show that $\frac{\sigma_2^*(p+1)}{\sigma_1^*(p+1)}$ tends to 1 for almost all primes p.

1. Introduction and notation

Let $\sigma^*(n)$ be the sum of the unitary divisors of n, that is,

$$\sigma^*(n) := \sum_{\substack{d|n \\ (d,n/d)=1}} d.$$

Given an integer $k \geq 0$, let $\sigma_k^*(n)$ stand for the k-fold iterate of $\sigma^*(n)$, that is, $\sigma_0^*(n) = n$, $\sigma_1^*(n) = \sigma^*(n)$, $\sigma_2^*(n) = \sigma^*(\sigma_1^*(n))$, and so on. The function $\sigma^*(n)$ is easily checked to be multiplicative with $\sigma^*(p^\alpha) = p^\alpha + 1$ for each prime p and integer $\alpha \geq 1$.

Key words and phrases: Sum of unitary divisors, shifted primes.
2010 Mathematics Subject Classification: 11N37.
[1]Research supported in part by a grant from NSERC.

In 1971, Erdős and Subbarao [3] proved that

$$\frac{\sigma_2^*(n)}{\sigma_1^*(n)} \to 1 \qquad \text{for almost all } n\ . \tag{1.1}$$

This is quite a contrast with the easily proven estimate

$$\frac{\sigma_2(n)}{\sigma_1(n)} \to \infty \qquad \text{for almost all } n\ .$$

In 1991, Kátai and Wisjmuller [6] proved that

$$\frac{\sigma_3^*(n)}{\sigma_2^*(n)} \to 1 \qquad \text{for almost all } n$$

and conjectured that, given an arbitrary integer $k \geq 0$,

$$\frac{\sigma_{k+1}^*(n)}{\sigma_k^*(n)} \to 1 \qquad \text{for almost all } n\ .$$

This remains unproven.

Here, we consider similar quotients, namely those where the arguments of the functions σ_{k+1}^* and σ_k^* are running over shifted primes.

2. Main result

Theorem 1. *We have*

$$\frac{\sigma_2^*(p+1)}{\sigma_1^*(p+1)} \to 1 \qquad \textit{for almost all primes } p\,.$$

Of course, the above statement is equivalent to the following one.

Given any $\varepsilon > 0$, we have

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \# \left\{ p \leq x : \frac{\sigma_2^*(p+1)}{\sigma_1^*(p+1)} > 1 + \varepsilon \right\} = 0.$$

In the following, we denote by $p(n)$ and $P(n)$ the smallest and largest prime factors of n, respectively. We let $\mu(n)$ stand for the Moebius function and $\phi(n)$ for the Euler totient function. For each integer $n \geq 2$, we let $\omega(n)$ stand for the number of distinct prime factors of n and set $\omega(1) = 0$. The letters p, q, π and

Q, with or without subscript, will stand exclusively for primes. On the other hand, the letters c and C, with or without subscript, will stand for absolute constants but not necessarily the same at each occurrence. We will let $\wp$ stand for the set of all primes. Moreover, we shall at times use the abbreviations $x_1 = \log x$, $x_2 = \log\log x$, and so on. We denote the logarithmic integral $\displaystyle\int_2^x \frac{dt}{\log t}$ by $\mathrm{li}(x)$. Finally, we let $\pi(x)$ stand for the number of primes $p \le x$ and we write $\pi(x; k, \ell)$ for $\#\{p \le x : p \equiv \ell \pmod{k}\}$.

3. Preliminary results

For the proof of our main result, we will need the following lemmas.

Our first lemma is a classical result.

Lemma 1. (Brun-Titchmarsh Theorem) *For every positive integer $k < x$, we have*

$$\pi(x; k, \ell) \le \frac{2x}{\phi(k)\log(x/k)}.$$

A proof of the following result follows from Theorem 3.12 in the book of Halberstam and Richert [4].

Lemma 2. *There exists a positive constant C_1 such that*

$$\#\{p, q \in \wp : p + 1 = aq \le x\} < C_1 \frac{x}{\phi(a)\log^2(x/a)}.$$

The following can be obtained from Theorem 4.2 in the book of Halberstam and Richert [4].

Lemma 3. *Given an arbitrary positive number $\delta < 1$, there exists an absolute constant $C_2 > 0$ such that*

$$\lim_{x\to\infty} \#\{p \le x : P(p+1) < x^\delta \text{ or } P(p+1) > x^{1-\delta}\} < C_2\, \delta\, li(x).$$

It was proved by the first author [5] that the distribution of the numbers

$$\frac{\omega(\sigma(p+1)) - \frac{1}{2}(\log\log p)^2}{\frac{1}{\sqrt{3}}(\log\log p)^{3/2}}$$

as p runs through the primes obeys the Gaussian Law. It is easy to show that the same statement holds if $\sigma(p+1)$ is replaced by $\sigma^*(p+1)$. The following result is an immediate consequence of this observation.

Lemma 4. *We have*

$$\frac{1}{\pi(x)}\#\{p \le x : \omega(\sigma^*(p+1)) > x_2^2\} \to 0 \qquad \text{as } x \to \infty.$$

4. Proof of the main result

First let

$$t(n) := \sum_{q^\alpha \,\|\, \sigma^*(n)} \frac{1}{q^\alpha}.$$

It is clear that, for each integer $n \ge 2$,

$$\frac{\sigma^*(n)}{n} = \prod_{q^\alpha \,\|\, n}\left(1+\frac{1}{q^\alpha}\right) = \exp\left\{\sum_{q^\alpha \,\|\, n}\log\left(1+\frac{1}{q^\alpha}\right)\right\} \le \exp\left\{\sum_{q^\alpha \,\|\, n}\frac{1}{q^\alpha}\right\}$$

and therefore that

$$\frac{\sigma_2^*(n)}{\sigma_1^*(n)} = \frac{\sigma^*(\sigma^*(n))}{\sigma^*(n)} \le \exp\left\{\sum_{q^\alpha \,\|\, \sigma^*(n)}\frac{1}{q^\alpha}\right\} = e^{t(n)}.$$

From this observation, it follows that the claim in Theorem 1 is equivalent to the assertion that

$$(4.1) \qquad t(p+1) \to 0 \qquad \text{for almost all primes } p.$$

Let $\delta > 0$ be any small number, let $\wp_x := \{p \in \wp : p \le x\}$ and consider the set

$$\wp_x^{(1)} := \{p \le x : P(p+1) < x^\delta \text{ or } P(p+1) > x^{1-\delta}\}.$$

In light of Lemma 3,

$$(4.2) \qquad \#\wp_x^{(1)} < C_2\,\delta\,\mathrm{li}(x).$$

This is why we only need to work with the set

$$\wp_x^{(2)} := \wp_x \setminus \wp_x^{(1)}.$$

So, let us assume that $p \in \wp_x^{(2)}$, let T be a large integer and consider the set $\mathcal{D}_T$ made up of all those primes p such that $\pi^2 \mid p+1$ for some prime $\pi > T$.

Using Lemma 1, for some constant $C_3 > 0$, we then have

$$\begin{aligned}
\#\{p \le x : p \in \mathcal{D}_T\} &\le \sum_{T<\pi<\sqrt{x}} \sum_{\substack{p\le x \\ p+1\equiv 0 \pmod{\pi^2}}} 1 = \sum_{T<\pi<\sqrt{x}} \pi(x;\pi^2,-1) \le \\
&\le C_3 \mathrm{li}(x) \sum_{T<\pi<\sqrt{x}} \frac{1}{\phi(\pi^2)} = C_3 \mathrm{li}(x) \sum_{T<\pi<\sqrt{x}} \frac{1}{\pi(\pi-1)} \ll \\
&\ll \frac{\mathrm{li}(x)}{T\log T} + O\big(x^{3/4}\big). \qquad (4.3)
\end{aligned}$$

Hence, in light of (4.3), we may now discard those primes $p \le x$ for which $p \in \mathcal{D}_T$, since their number is $O(\mathrm{li}(x)/(T\log T))$. This is why, for each prime number q, we now focus our attention on the set

$$E_q(x) := \{p \in \wp_x^{(2)} : p \notin \mathcal{D}_T \text{ and } q^T \nmid \sigma^*(p+1)\} \qquad (4.4)$$

and the sum

$$S_T(x) := \sum_{q\le T} \#E_q(x).$$

Moreover, for each prime q, we let $\mathcal{B}_q$ be the semigroup generated by those primes Q such that $q \nmid Q+1$.

Let us now consider a fixed prime $q \le T$ and those integers

$$\begin{aligned}
&K = \pi_1^{\alpha_1}\cdots\pi_r^{\alpha_r} \ge 2 \text{ for which } q \mid \pi_j^{\alpha_j}+1 \\
&\text{for } j = 1,\dots,r \text{ with } \alpha_j = 1 \text{ if } \pi_j > T.
\end{aligned} \qquad (4.5)$$

In order to estimate $\#E_q(x)$, we first introduce the set

$$\begin{aligned}
H_{K,R} :=\{p : p+1 = KRmP, P(R) \le T,\ p(mP) > T,\\
\mu^2(m) = 1,\ (m,\mathcal{B}_q) = 1,\ P(p+1) = P\}.
\end{aligned}$$

Writing each prime $p \in H_{K,R}$ as $p+1 = KRmP = aP$, then, since P was assumed to be such that $P > x^\delta$, it follows that $a < x^{1-\delta}$, and therefore, using

Lemma 2, that

$$
\begin{aligned}
\#H_{K,R} &\leq C_2\,\delta\,C_1 \frac{x}{\log^2 x} \sum_{m\leq x} \frac{\mu^2(m)}{\phi(KRm)} \leq \\
&\leq C_1 C_2\,\delta \frac{x}{\log^2 x} \frac{1}{\phi(KR)} \sum_{\substack{m\leq x/KR\\ (m,\mathcal{B}_q)=1}} \frac{\mu^2(m)}{\phi(m)} \leq \\
&\leq C_1 C_2\,\delta \frac{x}{\log^2 x} \frac{1}{\phi(KR)} \cdot \prod_{\substack{T<\pi\leq x\\ \pi+1\not\equiv 0 \pmod q}} \left(1+\frac{1}{\pi-1}\right) \leq \\
&\leq C_1 C_2\,\delta \frac{x}{\log^2 x} \frac{1}{\phi(KR)} \exp\left\{\frac{q-2}{q-1} x_2\right\} = \\
&= C_1 C_2\,\delta \frac{x}{\log^2 x} \frac{1}{\phi(KR)} \cdot \log x \cdot \exp\left\{-\frac{x_2}{q-1}\right\} \leq \\
&\leq C_1 C_2\,\delta \frac{1}{\phi(KR)} \mathrm{li}(x) \cdot \exp\left\{-\frac{x_2}{q-1}\right\}. \qquad (4.6)
\end{aligned}
$$

On the one hand, observe that

$$
\begin{aligned}
\sum_{\substack{R\\ P(R)\leq T}} \frac{1}{\phi(R)} &\leq \prod_{p\leq T}\left(1+\frac{1}{\phi(p)}+\frac{1}{\phi(p^2)}+\cdots\right) = \\
&= \prod_{p\leq T}\left(1+\frac{1}{p-1}+\frac{1}{p(p-1)}+\cdots\right) = \\
&= \prod_{p\leq T}\left(1+\frac{1}{p}+O\left(\frac{1}{p^2}\right)\right) = \\
&= \exp\left\{\sum_{p\leq T}\frac{1}{p}+O(1)\right\} \leq C_4 \log T \qquad (4.7)
\end{aligned}
$$

for some positive constant C_4.

On the other hand, writing each K as $K=K_1K_2$, where $P(K_1)\leq T$ and $p(K_2)>T$. Then, by the nature of K (see (4.5)), it is clear that K_2 is squarefree and that $\omega(K_2)\leq r\leq T-1$. From this, it follows that, for some constant $C_5>0$,

(4.8)

$$
\sum_K \frac{1}{\phi(K)} \leq \sum_{K_2} \frac{1}{\phi(K_2)} \leq \sum_{j=0}^{T-1} \frac{1}{j!} \left(\sum_{\substack{\pi\leq x\\ \pi+1\equiv 0 \pmod q}} \frac{1}{\pi-1} \right)^j < C_5 \frac{x_2^{T-1}}{(T-1)!}.
$$

Recalling that for all $K, R \in \mathbb{N}$, we have $\phi(KR) \geq \phi(K)\phi(R)$, and combining (4.7) and (4.8) in (4.6), it follows that

$$\#E_q(x) \leq \sum_{K,R} H_{K,R} \leq C_6\, \delta \mathrm{li}(x) \frac{x_2^{T-1}}{(T-1)!} \cdot \exp\left\{-\frac{x_2}{q-1}\right\} \cdot \log T \tag{4.9}$$

for some constant $C_6 > 0$, from which it follows that

$$S_T(x) = o(\mathrm{li}(x)) \qquad (x \to \infty). \tag{4.10}$$

We now move to estimate $t(p+1)$ when the primes q such that $q^\alpha \mid \sigma^*(p+1)$ satisfy $q > T$.

We first consider the case where $\alpha = 1$, and for this we introduce the sum

$$s(p+1) := \sum_{\substack{T<q\leq x_2^{1-\varepsilon} \\ q\|\sigma^*(p+1)}} \frac{1}{q},$$

where $\varepsilon > 0$ is an arbitrarily small number. We then have, using Lemmas 1, 2 and 3,

$$\begin{aligned}
\sum_{p\in\wp_x^{(2)}} s(p+1) = \sum_{T<q\leq x_2^{1-\varepsilon}} & \frac{1}{q} \#\{p \in \wp_2^{(2)} : p+1 = QHmP, P(H) \leq T, p(m) > T, \\
& P > x^\delta, Q+1 \equiv 0 \pmod q, (m, \mathcal{B}_q) = 1\} + \\
& + \sum_{T<q\leq x_2^{1-\varepsilon}} \frac{1}{q} \#\{p \in \wp_2^{(2)} : p+1 = HmP, P(H) \leq T, p(m) > T, \\
& P > x^\delta, P+1 \equiv 0 \pmod q\} \leq \\
& \leq C_2\, \delta \frac{x}{x_1^2} \sum_{T<q\leq x_2^{1-\varepsilon}} \frac{1}{q} \sum_{\substack{Q\equiv -1 \pmod q \\ (m,\mathcal{B}_q)=1 \\ P(H)\leq T,\ p(m)>T}} \frac{1}{\phi(HmQ)} \leq \\
& \leq C_2\, \delta \frac{x}{x_1^2} \sum_{T<q\leq x_2^{1-\varepsilon}} \frac{1}{q} \sum_{Q\equiv -1 \pmod q} \frac{1}{Q} \sum_{P(H)\leq T,\ p(m)>T} \frac{1}{\phi(H)} \cdot \\
& \cdot \prod_{\pi \not\equiv -1 \pmod q} \left(1 + \frac{1}{\pi-1} + \frac{1}{\pi(\pi-1)} + \cdots\right) \leq \\
& \leq C_2\, \delta \frac{x}{x_1^2} \sum_{T<q\leq x_2^{1-\varepsilon}} \frac{1}{q^2} \cdot 2\log T \cdot x_2 \cdot \exp\left\{-\frac{x_2}{q-1}\right\},
\end{aligned}$$

from which it follows that

$$\sum_{p\in\wp_x^{(2)}} s(p+1) \ll \delta \frac{x}{x_1^2} \frac{\log T}{T} x_2 = o(\mathrm{li}(x)) \qquad (x \to \infty). \tag{4.11}$$

To account for those $q \in [x_2^{1+\varepsilon}, x_2^{2+\varepsilon}]$, we will consider the sum

$$r(p+1) := \sum_{\substack{x_2^{1+\varepsilon} \le q \le x_2^{2+\varepsilon} \\ q \| \sigma^*(p+1)}} \frac{1}{q}.$$

Proceeding as above, we obtain that

$$\begin{aligned}
\sum_{p \in \wp_x^{(2)}} r(p+1) \quad &\le \quad \sum_{x_2^{1+\varepsilon} \le q \le x_2^{2+\varepsilon}} \frac{1}{q} \sum_{\substack{Q \le x \\ Q+1 \equiv 0 \pmod q}} \pi(x; Q, -1) \le \\
&\le \quad C_2\, \delta \mathrm{li}(x) \sum_{x_2^{1+\varepsilon} \le q \le x_2^{2+\varepsilon}} \frac{1}{q} \sum_{\substack{Q \le x \\ Q+1 \equiv 0 \pmod q}} \frac{1}{Q-1} \le \\
&\le \quad C_2\, \delta \mathrm{li}(x) \sum_{x_2^{1+\varepsilon} \le q \le x_2^{2+\varepsilon}} \frac{1}{q} \cdot C_6 \frac{x_2}{q} \le \\
&\le \quad C_2\, \delta\, C_6 \mathrm{li}(x)\, x_2 \sum_{x_2^{1+\varepsilon} \le q \le x_2^{2+\varepsilon}} \frac{1}{q^2} \le \\
&\le \quad C_2\, \delta\, C_6 \mathrm{li}(x)\, x_2 \frac{1}{x_2^{1+\varepsilon} x_3},
\end{aligned}$$

from which it clearly follows that

$$\sum_{p \in \wp_x^{(2)}} r(p+1) = o(\mathrm{li}(x)). \tag{4.12}$$

We now split $t(p+1)$ into five sums as follows.

$$t(p+1) = t_1(p+1) + t_2(p+1) + t_3(p+1) + t_4(p+1) + t_5(p+1), \tag{4.13}$$

where

$$t_j(p+1) = \sum_{\substack{q^\alpha \| \sigma^*(p+1) \\ q \in I_j}} \frac{1}{q^\alpha} \qquad (j = 1, \ldots, 5),$$

with the five intervals I_j being defined as

$$\begin{gathered}
I_1 = [2, T], \quad I_2 = (T, x_2^{1-\varepsilon}], \quad I_3 = (x_2^{1-\varepsilon}, x_2^{1+\varepsilon}), \\
I_4 = [x_2^{1+\varepsilon}, x_2^{2+\varepsilon}], \quad I_5 = (x_2^{2+\varepsilon}, \infty).
\end{gathered}$$

We will show that the next five inequalities hold for almost all primes $p \in \wp_x^{(2)}$.

First of all, in light of (4.10), we have

$$t_1(p+1) \ll \frac{1}{2^T} \qquad (p \to \infty), \tag{4.14}$$

with the exception of at most $O\left(\frac{\mathrm{li}(x)}{2^T}\right)$ primes $p \in \wp_x^{(2)}$.

In light of (4.11), we have that

$$t_2(p+1) \le s(p+1) + \sum_{T<q\le x_2^{1-\varepsilon}} \frac{1}{q^2} \le o(1) + \frac{1}{T\log T} \qquad (p \to \infty). \tag{4.15}$$

Clearly,

$$t_3(p+1) \le \sum_{x_2^{1-\varepsilon}<q<x_2^{1+\varepsilon}} \frac{1}{q} \le \log\left(\frac{1+\varepsilon}{1-\varepsilon}\right) < 2\varepsilon. \tag{4.16}$$

In light of (4.12), we have

$$t_4(p+1) \le r(p+1) + \sum_{q>x_2^{1+\varepsilon}} \frac{1}{q^2} \le o(1) + O\left(\frac{1}{x_2^{1+\varepsilon}x_3}\right) \qquad (p \to \infty). \tag{4.17}$$

Finally, using Lemma 4, it follows that

$$t_5(p+1) \le x_2^{-\varepsilon} \qquad (p \to \infty). \tag{4.18}$$

Gathering inequalities (4.14) through (4.18) in (4.13), we have thus established that

$$\frac{1}{\pi(x)}\#\left\{p \le x : t(p+1) > \frac{2}{T}\right\} \le \delta$$

and since this is true for every $\delta > 0$ and for every large number T, our claim (4.1) is established and the proof of Theorem 1 is complete. ∎

5. Final remark

The unitary analog of $\phi(n)$ denoted by $\phi^*(n)$ and introduced by Cohen [1] is defined by

$$\phi^*(n) := \prod_{p^\alpha \| n} (p^\alpha - 1).$$

Denote by $\phi_k^*(n)$ the k-fold iterate of $\phi^*(n)$. Erdős and Subbarao [3] claimed that
$$\frac{\phi_2^*(n)}{\phi_1^*(n)} \to 1 \qquad (n \to \infty)$$
except on a set of integers n of zero density. In fact, they mentioned that they could prove this result by using the same methods as those used to prove (1.1).

With the approach we used to prove Theorem 1, we can also prove the following.

Theorem 2. *Given any $\varepsilon > 0$,*
$$\frac{1}{\pi(x)} \# \left\{ \frac{\phi_2^*(p+1)}{\phi_1^*(p+1)} < 1 - \varepsilon \right\} \to 0 \qquad (x \to \infty).$$

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