

ON CONVOLUTED SUMS

Jean-Marie De Koninck¹ (Québec, Canada)

Imre Kátai (Budapest, Hungary)

Dedicated to the memory of C.A. Corrádi

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Abstract. Given a complex valued multiplicative function f such that $|f(n)| = 1$ for each $n \in \mathbb{N}$, let $h_f(n) := \sum_{\nu=1}^{n-1} f(\nu)f(n-\nu)$. We investigate under which conditions we have $h_f(n) = o(n)$ for almost all positive integers n as $n \rightarrow \infty$.

1. Introduction and notation

Let \mathcal{M}_1 stand for the set of those multiplicative functions f which are such that $|f(n)| = 1$ for each $n \in \mathbb{N}$. Then, given $f \in \mathcal{M}_1$, consider the corresponding convoluted sum

$$h_f(n) := \sum_{\nu=1}^{n-1} f(\nu)f(n-\nu).$$

This function was studied by Corrádi and Kátai [2] in 1969 in the case where the function f takes the values ± 1 only, and more recently by De Koninck, Germán and Kátai [4] in the case where $|f(n)| \leq 1$ for each $n \in \mathbb{N}$. Here, we are interested in establishing under which conditions we have that

$$\frac{h_f(n)}{n} \rightarrow 0 \quad \text{as } n \rightarrow \infty \quad \text{for almost all } n.$$

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Letting f be as above and $\alpha \in \mathbb{R}$, consider the exponential sum

$$S_f(N, \alpha) := \sum_{n=1}^{N-1} f(n)e(n\alpha),$$

where we used the classical notation $e(y) = e^{2\pi iy}$. Then it is clear that

$$S_f^2(N, \alpha) = \sum_{n=1}^{2N-2} h_{f,N}(n)e(n\alpha),$$

where

$$h_{f,N}(n) = \begin{cases} h_f(n) & \text{if } n \leq N, \\ \sum_{\max(\nu, n-\nu) \leq N-1} f(\nu)f(n-\nu) & \text{if } n > N. \end{cases}$$

From this, it follows that

$$\begin{aligned} \sum_{n=1}^N |h_f(n)|^2 &\leq \sum_{n \leq 2N} |h_{f,N}(n)|^2 = \int_{-1/2}^{1/2} |S_f(N, \alpha)|^4 d\alpha \leq \\ &\leq \max_{\alpha \in [0,1]} |S_f(N, \alpha)|^2 \cdot \int_{-1/2}^{1/2} |S_f(N, \alpha)|^2 d\alpha, \end{aligned}$$

from which it follows, in light of the fact that this very last integral is equal to N , that

$$(1.1) \quad \sum_{N/2 \leq n \leq N} \frac{|h_f(n)|^2}{n^2} \leq \max_{\alpha \in [0,1]} \left| \frac{S_f(N, \alpha)}{N} \right|^2.$$

Hence, using (1.1), it follows that if

$$(A) \quad \max_{\alpha \in [0,1]} \left| \frac{S_f(N, \alpha)}{N} \right| \rightarrow 0 \quad (N \rightarrow \infty),$$

then

$$(B) \quad \frac{h_f(n)}{n} \rightarrow 0 \quad (n \rightarrow \infty) \quad \text{for almost all } n.$$

This raises the following question: ‘‘Under which condition does (A) hold?’’

Now, an obvious necessary condition for (A) to hold is that for every integers $q \geq 1$ and $\ell \geq 0$, we have that

$$(C) \quad \frac{1}{N} \sum_{\substack{n \leq N \\ n \equiv \ell \pmod{q}}} f(n) \rightarrow 0 \quad (N \rightarrow \infty).$$

Indeed, this follows from the fact that

$$\sum_{\substack{n \leq N \\ n \equiv \ell \pmod{q}}} f(n) = \sum_{n \leq N} f(n) \cdot \frac{1}{q} \sum_{a=0}^{q-1} e\left(\frac{a(n-\ell)}{q}\right) = \frac{1}{q} \sum_{a=0}^{q-1} e\left(-\frac{a\ell}{q}\right) S_f(N, a/q).$$

Here, we investigate the particular cases when f is a completely multiplicative function and when f is q -multiplicative.

2. Main results

Let \mathcal{M}_1^* stand for those functions $f \in \mathcal{M}_1$ which are completely multiplicative.

Theorem 1. *Let $f \in \mathcal{M}_1^*$. Statement (A) holds if and only if for every $q \in \mathbb{N}$ and every corresponding Dirichlet character χ_q and every $\tau \in \mathbb{R}$ we have*

$$\sum_p \frac{\Re(1 - \chi_q(p)p^{i\tau} f(p))}{p} = \infty$$

and

$$\sum_p \frac{\Re(1 - p^{i\tau} f(p))}{p} = \infty.$$

Let $\mathcal{M}_q^{(1)}$ stand for the set of all q -multiplicative functions $f : \mathbb{N}_0 \rightarrow \mathbb{C}$ satisfying $f(0) = 1$ and $|f(n)| = 1$ for each positive integer n . To each $f \in \mathcal{M}_q^{(1)}$, we associate the sum

$$(2.1) \quad S_x(z) := \sum_{n < x} f(n)z^n.$$

Theorem 2. *Let $f \in \mathcal{M}_q^{(1)}$ with corresponding sum $S_x(z)$ defined in (2.1). Further set*

$$G_m(z) = \sum_{a=0}^{q-1} f(aq^m)z^a \quad (m = 1, 2, \dots)$$

and assume that

$$(2.2) \quad \sum_{m=1}^{\infty} \left(1 - \max_{|z|=1} \left| \frac{G_m(z)}{q} \right|^2\right) = \infty.$$

Then,

$$(2.3) \quad \lim_{x \rightarrow \infty} \max_{|z|=1} \frac{|S_x(z)|}{x} = 0$$

and

$$(2.4) \quad \frac{h_f(n)}{n} \rightarrow 0 \quad \text{as } n \rightarrow \infty \quad \text{for almost all } n.$$

3. Preliminary results

In 1974, Daboussi [3] proved that given $\alpha \in [0, 1)$ and assuming that $\left| \alpha - \frac{a}{q} \right| \leq \frac{1}{q^2}$, where a and q are positive integers satisfying $(a, q) = 1$ and $3 \leq q \leq \sqrt{N/\log N}$, there exists an absolute constant $c_1 > 0$ such that

$$\max_{f \in \mathcal{M}_1^*} |S_N(f, \alpha)| \leq \frac{c_1 N}{\sqrt{\log \log q}}.$$

In 1977, Montgomery and Vaughan [7] proved that letting a and q be two positive integers such that $(a, q) = 1$ and $q \leq N$, there exists an absolute constant $c_2 > 0$ such that

$$\max_{f \in \mathcal{M}_1^*} |S_N(f, a/q)| \leq \frac{c_2 N}{\sqrt{\log 2N}} + \frac{c_2}{\phi(q)} + c_2 \left(\frac{q}{N} \right)^{1/2} \left(\log \frac{2N}{q} \right)^{3/2},$$

where ϕ stands for the Euler totient function.

As an immediate consequence of Montgomery and Vaughan's inequality, we have the following result.

Let $\alpha \in [0, 1)$ and assume that $\left| \alpha - \frac{a}{q} \right| \leq \frac{1}{q^2}$, where a , q and R are positive integers satisfying $(a, q) = 1$ and $2 \leq R \leq q \leq N/R$. Then, there exists an absolute constant $c_3 > 0$ such that

$$(3.1) \quad \max_{f \in \mathcal{M}_1^*} |S_N(f, \alpha)| \leq \frac{c_3}{\log N} + c_3 \frac{(\log R)^{3/2}}{\sqrt{R}}.$$

The following is a variant of a theorem of Kátai (see Kátai [6]).

Lemma 1. *Let \wp_N be an arbitrary subset of the primes, each element of which does not exceed N . Further set $A_N := \sum_{p \in \wp_N} 1/p$ and, given any arbitrary real number $\alpha \in [0, 1)$, let*

$$(3.2) \quad h_N(\alpha) := \sum_{\substack{p, q \in \wp_N \\ p \neq q}} \frac{1}{\|\alpha(p - q)\|},$$

where $\|y\|$ stands for the distance from y to the nearest integer. Then there exist absolute constants $C_4 > 0$ and $C_5 > 0$ such that

$$(3.3) \quad \max_{f \in \mathcal{M}_1^*} \frac{|S_N(f, \alpha)|}{N} \leq \frac{C_4}{\sqrt{A_N}} + \frac{C_5}{A_N} \sqrt{\frac{h_N(\alpha)}{N}}.$$

Proof. Let $\omega_{\wp_N}(n) := \sum_{\substack{p|n \\ p \in \wp_N}} 1$. Then, by using the Turán-Kubilius inequality,

it is clear that there exists an absolute constant $C_1 > 0$ such that

$$\sum_{n \leq N} |\omega_{\wp_N}(n) - A_N|^2 \leq C_1 N A_N$$

and therefore that, for some absolute constant $C_2 > 0$, we have

$$(3.4) \quad \sum_{n \leq N} |\omega_{\wp_N}(n) - A_N| \leq C_2 N \sqrt{A_N}.$$

Let

$$U_N(f, \alpha) := \sum_{n \leq N} f(n) e(n\alpha) \omega_{\wp_N}(n).$$

Since

$$U_N(f, \alpha) = \sum_{\substack{pm \leq N \\ p \in \wp_N}} f(p) f(m) e(pm\alpha),$$

it follows from (3.4) that

$$(3.5) \quad |S_N(f, \alpha)| A_N \leq c_2 N \sqrt{A_N} + |U_N(f, \alpha)|.$$

Now, observe that

$$\begin{aligned}
|U_N(f, \alpha)|^2 &\leq \left\{ \sum_{m \leq N} 1 \right\} \left\{ \sum_{m \leq N} \left| \sum_{\substack{p \leq N/m \\ p \in \wp_N}} f(p) e(pm\alpha) \right|^2 \right\} \leq \\
&\leq N \left\{ \sum_{\substack{pm \leq N \\ p \in \wp_N}} 1 + \sum_{\substack{p_1, p_2 \in \wp_N \\ p_1 \neq p_2}} \sum_{m \leq \min(N/p_1, N/p_2)} e((p_1 - p_2)m\alpha) \right\} \leq \\
&\leq N \left\{ c_3 N A_N + \sum_{\substack{p_1, p_2 \in \wp_N \\ p_1 \neq p_2}} \min \left(\frac{1}{\|\alpha(p_1 - p_2)\|}, \frac{N}{p_1}, \frac{N}{p_2} \right) \right\},
\end{aligned}$$

thereby implying that

$$\left| \frac{1}{N} U_N(f, \alpha) \right|^2 \leq C_3 A_N + \frac{h_N(\alpha)}{N},$$

that is,

$$\left| \frac{1}{N} U_N(f, \alpha) \right| \leq C_4 \sqrt{A_N} + \frac{C_5}{\sqrt{N}} \sqrt{h_N(\alpha)},$$

which, with (3.5), proves (3.3), thereby completing the proof of Lemma 1. ■

As an immediate consequence of Lemma 1, we have the following.

Lemma 2. *Let $f \in \mathcal{M}_1^*$. Let $\varepsilon > 0$ be an arbitrarily small number. Let $p_1 < \dots < p_k$ be a sequence of prime numbers satisfying $\sum_{j=1}^k 1/p_j > 1/\varepsilon$. Then, provided $N > p_k$, we have*

$$\frac{|S_N(f, \alpha)|}{N} \leq C_4 \sqrt{\varepsilon} + C_5 \varepsilon \sqrt{\frac{h_N(\alpha)}{N}}.$$

The following is a consequence of a theorem of G. Halász [5].

Lemma 3. *If statement (A) holds for some $f \in \mathcal{M}_1^*$, then for every positive integer q and every corresponding Dirichlet character χ_q , we have for each $\tau \in \mathbb{R}$,*

$$(3.6) \quad \sum_p \frac{\Re(1 - \chi_q(p) p^{i\tau} f(p))}{p} = \infty$$

and

$$(3.7) \quad \sum_p \frac{\Re(1 - p^{i\tau} f(p))}{p} = \infty.$$

Lemma 4. *Let $f \in \mathcal{M}_1^*$. If estimate (3.6) holds for every q and χ_q and if relation (3.7) holds as well, then*

$$(3.8) \quad \frac{S_N(f, a/q)}{N} \rightarrow 0 \quad (N \rightarrow \infty)$$

for every congruence class $a \pmod{q}$, $a = 0, 1, \dots, q-1$.

Proof. Assume first that $(a, q) = 1$ and let

$$\ell(n) := \begin{cases} 1 & \text{if } n \equiv a \pmod{q}, \\ 0 & \text{otherwise.} \end{cases}$$

It is known that

$$\ell(n) = \sum_{\chi} d_{\chi} \cdot \chi(n),$$

where χ runs over the characters mod q and d_{χ} are suitable constants. Hence, it follows that

$$\sum_{\substack{n \leq N \\ n \equiv a \pmod{q}}} f(n) = \sum_{\chi} d_{\chi} \cdot S_N(f, \chi_q) = o(N) \quad (N \rightarrow \infty),$$

so that

$$\frac{1}{N} S_N(f, a/q) = \frac{1}{N} \sum_{n \leq N} f(n) e\left(\frac{an}{q}\right) \rightarrow 0 \quad (N \rightarrow \infty),$$

thus completing the proof of Lemma 4 in the case where $(a, q) = 1$.

In the more general case, that is when $(a, q) = \Delta \geq 1$, let $a_1 = a/\Delta$ and $q_1 = q/\Delta$. We then have that

$$S_N(f, a/q) = S_N(f, a_1/q_1),$$

allowing one to easily establish that (3.8) holds in the general case as well. ■

Let $f \in \mathcal{M}_q^{(1)}$ with corresponding sum $S_x(z)$ defined in (2.1). We then have

$$(3.9) \quad S_{q^N}(z) = \prod_{j=0}^{N-1} \left\{ \sum_{a=0}^{q-1} f(aq^j) z^{aq^j} \right\}.$$

The following result will be used in the proof of Theorem 2.

Lemma 5. *Let $f \in \mathcal{M}_q^{(1)}$ with corresponding function $S_x(z)$ defined in (2.1). If*

$$\max_{|z|=1} \frac{|S_{q^N}(z)|}{q^N} \rightarrow 0 \quad (N \rightarrow \infty),$$

then

$$\max_{|z|=1} \frac{|S_x(z)|}{x} \rightarrow 0 \quad (x \rightarrow \infty).$$

Proof. Let

$$\Lambda_N := \sup_{|z|=1} \frac{|S_{q^N}(z)|}{q^N}$$

and write successively

$$\begin{aligned} x &= b_0 q^N + x_1, \\ \text{where } x_1 &= b_1 q^{N-1} + x_2, \\ \text{where } x_2 &= b_2 q^{N-2} + x_3, \\ &\vdots \end{aligned}$$

We then have

$$S_x(z) = \left(\sum_{a=0}^{b_0} f(aq^N) z^{aq^N} \right) S_{q^N}(z) + f(b_0 q^N) z^{b_0 q^N} S_{x_1}(z),$$

where

$$S_{x_1}(z) = \left(\sum_{a=0}^{b_1} f(aq^{N-1}) z^{aq^{N-1}} \right) S_{q^{N-1}}(z) + f(b_1 q^{N-1}) z^{b_1 q^{N-1}} S_{x_2}(z),$$

and so on, at each step introducing successively the definitions of $S_{x_2}(z)$, $S_{x_3}(z)$,
...

It follows from this representation of $S_x(z)$ that

$$\begin{aligned} \frac{|S_x(z)|}{x} &\leq (b_0 + 1) \frac{q^N \Lambda_N}{x} + \frac{(b_1 + 1) q^{N-1} \Lambda_{N-1}}{x} + \dots \leq \\ &\leq q \left(\Lambda_N + \frac{\Lambda_{N-1}}{q} + \frac{\Lambda_{N-2}}{q^2} + \dots \right), \end{aligned}$$

which clearly implies (3.9), thus completing the proof of Lemma 5. ■

4. The proof of Theorem 1

It is sufficient to prove that if (3.6) and (3.7) hold, then (A) holds. To do so, we first observe that Lemmas 3 and 4 imply that

$$\frac{1}{N} S_N(f, a/q) \rightarrow 0 \quad \text{as } N \rightarrow \infty \text{ for every } q \text{ and every } a \pmod{q}.$$

Consequently, there exists a suitable sequence $(K_N)_{N \geq 1}$ which tends to infinity with N such that

$$(4.1) \quad \max_{\substack{q \leq K_N \\ a=0,1,\dots,q-1}} \frac{1}{N} |S_N(f, a/q)| \rightarrow 0 \quad (N \rightarrow \infty).$$

Now, let $\delta > 0$ be an arbitrarily small number. Then, for every pair of positive integers $N_1 < N_2$ such that $N_2(1 - \delta) \geq N_1$, we have

$$(4.2) \quad \max_{\substack{q \leq K_{N_1} \\ a=0,1,\dots,q-1}} \frac{1}{N_2 - N_1} \sum_{N_1 \leq n \leq N_2} f(n) e\left(\frac{an}{q}\right) \rightarrow 0 \quad (N_1 \rightarrow \infty).$$

Assume that $|\alpha| \leq \varepsilon/N$. We then have

$$(4.3) \quad |S_N(f, \alpha) - S_N(f, 0)| \leq \varepsilon N,$$

so that

$$(4.4) \quad \max_{|\alpha| \leq \varepsilon/N} \frac{1}{N} |S_N(f, \alpha)| \leq \varepsilon + \frac{1}{N} |S_N(f, 0)|.$$

If, on the other hand, we have

$$0 \leq \frac{\varepsilon}{N} < \alpha \leq \frac{K}{N},$$

we then create the sequence

$$(4.5) \quad N_0 = \delta N, \quad N_j = (j+1)\delta N \quad \text{for } j = 1, 2, \dots,$$

so that

$$(4.6) \quad S_{N_{j+1}}(f, \alpha) - S_{N_j}(f, \alpha) = e(N_j \alpha) \sum_{\ell=0}^{N_{j+1}-N_j} f(N_j + \ell) e(\ell \alpha).$$

Using the fact that $|e(\ell \alpha) - 1| \leq \ell K/N$, it follows from relations (4.3) to (4.6) that

$$\begin{aligned} |S_{N_{j+1}}(f, \alpha) - S_{N_j}(f, \alpha)| &\leq |S_{N_{j+1}}(f, 0) - S_{N_j}(f, 0)| + \frac{K}{N} \sum_{\ell=0}^{\delta N} \ell = \\ &= O(1)\delta N + K\delta^2 N. \end{aligned}$$

Summing the above inequality for all $j \leq 1/\delta$, we obtain that

$$|S_N(f, \alpha)| \leq o(1) \frac{\delta}{\delta} N + K\delta N \quad (N \rightarrow \infty)$$

and therefore that

$$(4.7) \quad \left| \frac{1}{N} S_N(f, \alpha) \right| \leq o(1) + K\delta \quad (N \rightarrow \infty).$$

Let us now assume that we have chosen $\delta = \varepsilon$. It then follows from (4.7) that

$$(4.8) \quad \max_{|\alpha| \leq K/N} \frac{1}{N} |S_N(f, \alpha)| \leq K\varepsilon + o(1) \quad (N \rightarrow \infty).$$

Assume that $p_1 < \dots < p_k$ are fixed primes and further assume that $\sum_{j=1}^k 1/p_j > 1/\varepsilon$ and that $N > p_k$. Moreover, let T be a large number and let \mathcal{B}_N be the set of those $\alpha \in [0, 1)$ for which the inequality

$$(4.9) \quad \|(p_i - p_j)\alpha\| > T/N$$

holds for every prime pair p_i, p_j with $1 \leq j < i \leq k$. In this case, we obtain that

$$\left| \alpha - \frac{R}{p_i - p_j} \right| > \frac{T}{N(p_i - p_j)} \quad (R \in \mathbb{Z}).$$

Using this and the representation of $h_N(\alpha)$ provided by (3.2), it follows that

$$h_N(\alpha) \leq \frac{Np_k k^2}{T}.$$

Choosing $T = p_k k^2$, it follows, using Lemma 2, that

$$\sup_{\alpha} \frac{1}{N} |S_N(f, \alpha)| \leq \frac{C_6}{\sqrt{A_N}} < C_6 \sqrt{\varepsilon}.$$

On the other hand, if (4.9) does not hold, then

$$\left| \alpha - \frac{R}{p_i - p_j} \right| \leq \frac{T}{N(p_i - p_j)} \quad (R \in \mathbb{Z}),$$

we may first write that

$$\alpha = \frac{R}{p_i - p_j} + \beta, \quad \text{where } |\beta| < \frac{T}{N(p_i - p_j)}.$$

Repeating the argument used above, namely by first defining the sequence $(N_j)_{j \geq 0}$ as in (4.5), we obtain that

$$\begin{aligned} S_{N_{j+1}}(f, \alpha) - S_{N_j}(f, \alpha) &= \\ &= e(N_j \alpha) \sum_{\ell=0}^{N_{j+1}-N_j} f(N_j + \ell) e\left(\frac{R}{p_i - p_j}(N_j - \ell)\right) \cdot e(\beta \ell) = \\ &= O(\beta \delta^2 N) + e(N_j \beta) \left(S_{N_{j+1}}\left(f, \frac{R}{p_i - p_j}\right) - S_{N_j}\left(f, \frac{R}{p_i - p_j}\right) \right). \end{aligned}$$

Observing that

$$S_{N_{j+1}}\left(f, \frac{R}{p_i - p_j}\right) - S_{N_j}\left(f, \frac{R}{p_i - p_j}\right) = o(N_{j+1} - N_j)$$

for every j uniformly as $j \leq 1/\delta$, it follows that

$$\sup_{\alpha \in \bar{B}_N} \frac{1}{N} |S_N(f, \alpha)| \rightarrow 0 \quad (N \rightarrow \infty),$$

thus completing the proof of Theorem 1. ■

5. The proof of Theorem 2

Let us separate the real and imaginary parts of $G_m(z) - 1$ by writing

$$G_m(z) - 1 = \sum_{a=1}^{q-1} g(aq^m)z^a = U + iV, \quad U, V \in \mathbb{R},$$

where U and V depend on z . Since $U^2 + V^2 \leq (q-1)^2$, it follows that, for each $m \geq 1$, there exists some $\rho_m > 0$ such that

$$\max_{|z|=1} \Re(G_m(z) - q) = -\rho_m.$$

Now, $1 + U - q \leq -\rho_m$ implies that $U \leq (q-1) - \rho_m$, from which it follows that

$$\begin{aligned} |G_m(z)|^2 &= (1+U)^2 + V^2 = U^2 + V^2 + 2U + 1 \leq \\ &\leq (q-1)^2 + 1 + 2(q-1) - 2\rho_m = q^2 - 2\rho_m. \end{aligned}$$

From this, we obtain that

$$\left| \frac{G_m(z)}{q} \right|^2 \leq 1 - \frac{2\rho_m}{q^2}$$

and therefore that

$$\frac{2\rho_m}{q^2} \leq 1 - \left| \frac{G_m(z)}{q} \right|^2.$$

Using this, we get that

$$\left| \frac{S_{q^N}(z)}{q^N} \right|^2 = \prod_{m=0}^{N-1} \left| \frac{G_m(z)}{q} \right|^2 \leq \prod_{m=0}^{N-1} \left(1 - \max_{|z|=1} \left(1 - \left| \frac{G_m(z)}{q} \right|^2 \right) \right).$$

In light of hypotheses (2.2), it follows from the above inequality that

$$\max_{|z|=1} \frac{S_{q^N}(z)}{q^N} \rightarrow 0 \quad (N \rightarrow \infty),$$

thus, in light of Lemma 5, establishing (2.3). Finally, since (2.4) is an immediate consequence of (2.3), the proof of Theorem 2 is complete. ■

6. Final remarks

For the general case, that is when we do not assume that the arithmetic function f belongs to \mathcal{M}_1^* or to $\mathcal{M}_q^{(1)}$, we are unable to prove results similar to those stated in Theorems 1 or 2. However, it is interesting to observe that if the arithmetic function f is such that $|f(n)| \leq 1$, one can prove that, given any $\varepsilon > 0$,

$$(6.1) \quad S_f(N, \alpha) := \sum_{n \leq N} f(n)e(n\alpha) = O(\sqrt{N} \cdot (\log N)^{\frac{1}{2} + \varepsilon})$$

for almost all $\alpha \in \mathbb{R}$.

This can be deduced from the famous result of Carleson [1] which states that if $\sum_{k=0}^{\infty} |c_k|^2 < \infty$, then the corresponding Fourier series $\sum_{k=0}^{\infty} c_k e(k\theta)$ converges for almost all $\theta \in \mathbb{R}$. A deduction of (6.1) from the Carleson result can be found in the paper of Murty and Sankaranarayanan [8].

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J.-M. De Koninck

Dép. math. et statistique
Université Laval
Québec
Québec G1V 0A6
Canada
jmdk@mat.ulaval.ca

I. Kátai

Computer Algebra Department
Eötvös Loránd University
1117 Budapest
Pázmány Péter Sétány I/C
Hungary
katali@inf.elte.hu