Annales Univ. Sci. Budapest., Sect. Comp. 45 (2016) 57–74

A SIMPLE ALGORITHM FOR SCHWARZ WAVEFORM RELAXATION METHODS

Minh-Phuong Tran and Thanh-Nhan Nguyen
(Ho Chi Minh City, Vietnam)

Communicated by Ferenc Schipp

(Received January 18, 2016; accepted August 26, 2016)

Abstract. In this paper, we propose a new algorithm to the non-overlapping Schwarz domain decomposition methods for solving one dimensional advection reaction diffusion problem. More precisely, our method provides a good estimation of parameters which are used in Schwarz waveform relaxation methods. The advantage of our algorithm is that it is simple, efficient and easy to extend to more general partial differential equations. Numerical experiments are provided to show the efficiency of the new method and to compare with the optimized Schwarz waveform relaxation methods in [4].

1. Introduction

Domain decomposition methods are very important techniques in solving boundary value problems for partial differential equations. The main idea consists of decoupling the original problem into several sub-problems and solving the discrete problems on parallel computing. The first domain decomposition method has been known as the classical Schwarz method which was proposed by H. Schwarz in [17], to study the case of a domain that is the union of subdomains. The convergence properties of the classical Schwarz methods are well studied in several books [16], [18] and references therein. In [11], P. L. Lions developed the non-overlapping variant of the Schwarz method to obtain a convergent algorithm, since without overlap the classical Schwarz method

Key words and phrases: Domain decomposition, Schwarz method, reaction diffusion equation.
2010 Mathematics Subject Classification: 65M55, 65M12, 35K57.

does not converge. Following the pioneering work of P. L. Lions, a new domain decomposition method for solving evolution problems in parallel in space-time has been developed by M. J. Gander et al., so called Schwarz waveform relaxation method, see [2], [3], [8], [9]. The convergence rates of these methods come from the different transmission conditions which are differential type in both time and space, and depend on several coefficients. These coefficients are determined by the optimization of convergence factor. The best optimal approximation of transmission conditions was studied as the optimized Schwarz waveform relaxation (SWR) method by M. J. Gander and L. Halpern to solve the one dimensional advection diffusion equation in [4]. The optimized Schwarz waveform relaxation method is very efficient, which gives idea to study theoretical results for non-overlapping domain decomposition method. However, this method may be difficult to apply in solving some other complicated problems. One thinks about a simpler domain decomposition method that also gives good convergence results. For this purpose, in this paper we propose a simple algorithm and show that the convenient framework is efficient, flexible and robust. In addition, in this work we also point out that this new method is inherently adaptive for parallel computing. The rest of this paper is organized as follows. In the next section, we start with a brief mathematical description of the domain decomposition methods, in which the non-overlapping optimized SWR algorithm is reviewed. We present the theoretical analysis, such as the existence, uniqueness and regularity of the method. Then we analyze the convergence behavior of the proposed algorithm. In Section 3, we describe two steps of our new optimized domain decomposition methods. Our algorithm includes two steps: 1) the first step is to estimate the initial parameters by minimizing quadrature functions which are obtained by the simplicity of the min-max problem; 2) the second step is to obtain the optimized parameters using the backtracking-Armijo line search method. Finally, in the last section we perform some numerical experiments to illustrate our developed algorithms and make comparisons with the optimal method in [4].

2. The Schwarz waveform relaxation method

2.1. Setting of the problem

For the easier understanding, let us describe the main idea of our method in a simple case. Let the domain $\Omega = \mathbb{R}$, we consider the guiding example is the one dimensional advection reaction diffusion equation as follows:

$$\partial_t u + Au = f, \quad \text{in} \quad \Omega \times (0,T), \tag{2.1}$$

with the initial condition

$$u(x,0) \;=\; u_0(x), \quad \text{in} \quad \Omega, \tag{2.2}$$

where $A = -a\partial^2_{xx} + b\partial_x + c$ and a is a positive constant. The spatial domain Ω is split into two non-overlapping sub-domains $\Omega_1 = (-\infty, 0)$ and $\Omega_2 = (0, +\infty)$. We introduce the optimized non-overlapping SWR algorithm, consists of solving iteratively two sub-problems on $\Omega_1 \times (0,T)$ and $\Omega_2 \times (0,T)$ using some coupling transmission conditions at the interfaces $\Gamma = \{0\}$. More precisely, we define a sequence (u_1^k, u_2^k), $k \in \mathbb{N}$ as a solution of the equations corresponding to transmission conditions, for $i = 1, 2$:

$$\begin{cases} \partial_t u_i^{k+1} + A u_i^{k+1} \;=\; f, & \text{in} \quad \Omega_i \times (0,T), \\ u_i^{k+1}(x,0) \;=\; u_0(x), & \text{in} \quad \Omega_i, \end{cases} \tag{2.3}$$

$$\begin{cases} (\partial_x + S_2) u_1^{k+1}(0,t) \;=\; (\partial_x + S_2) u_2^k(0,t), & \text{in } (0,T), \\ (-\partial_x + S_1) u_2^{k+1}(0,t) \;=\; (-\partial_x + S_1) u_1^k(0,t), & \text{in } (0,T), \end{cases} \tag{2.4}$$

where S_1 and S_2 are linear operators, possibly pseudo-differential. Note that we need to provide some initial guesses $u_1^0(0,t)$, $u_2^0(0,t)$, for $t \in (0,T)$ as solving each sub-problem approximately. The sequence (u_1^k, u_2^k) must converge to the exact solution $\left(u_{|\Omega_1}, u_{|\Omega_2}\right)$ of (2.1) as k tends to infinity. This convergence property depends on the choice of the operators S_1 and S_2. Furthermore, in order to reduce the complexity and the computing time, the number of iterations for reaching the convergence must be as small as possible.

In this paper, we see below that if S_i are chosen as the Dirichlet to Neumann operators $\tilde{S}_i$, for $i = 1, 2$, then the SWR algorithm converges in two iterations. However, the operators $\tilde{S}_i$ are non-local operators in time, and it is very expensive to approximate them directly in sub-problems. The simple approach of the optimized SWR algorithm has been afterwards proposed, that the operators S_i are chosen as the approximation of $\tilde{S}_i$ in the form $S_i := p_i I_d$ or $S_i := p_i I_d + q_i \partial_t$ for $i = 1, 2$, where the real parameters p_i, q_i are given in order to optimize the convergence rate.

The goal of the optimized Schwarz methods is to choose the positive parameters p_i and q_i to obtain good performance of convergence. Many variants of this approach have been proposed and investigated in the recent literature [1], [4], [5], [6], [7], [10], [14], [15] and references therein. In particular, Gander, Martin J. et al. in [1], [4], the authors determine analytically the best parameter p_i, $q_i > 0$ for the advection reaction diffusion equation. For other problems with varying coefficients, it may be very difficult to find the optimal parameters analytically. From all that reasons, this paper studies a new simple method for computing numerically efficient transmission conditions.

2.2. Existence, uniqueness and regularity

In this section, we will explain the mathematical formulation and algorithmic composition of the domain decomposition method. Let us consider the advection diffusion equation (2.1). Without loss of generality, we may assume that the reaction coefficient c is non-negative because of a change of variables $v = u^{-\lambda t}$ with $\lambda + c > 0$ will lead to the equation with a positive reaction coefficient. A weak solution to (2.1) is defined as a function $u \in \mathcal{C}(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$, such that for any v in $H^1(\Omega)$, we have

$$\partial_t(u,v) + a(\partial_x u, \partial_x v) + \frac{b}{2}((\partial_x u, v) - (\partial_x v)) + c(u,v) = (f,v), \quad \text{in} \quad \mathcal{D}'(0,T),$$

where $(.,.)$ denotes the inner product in $L^2(\Omega)$. For the transmission conditions, we first introduce the anisotropic Sobolev spaces as $H^{r,s}(\Omega \times (0,T)) = L^2(0,T;H^r(\Omega)) \cap H^s(0,T;L^2(\Omega))$. We refer the readers to [12], [13] for the proof of the existence, uniqueness and regularity results.

Theorem 2.1. *Assume that $u_0 \in L^2(\Omega)$ and $f \in L^2(0,T;L^2(\Omega))$. Then there exists a unique weak solution u of the system (2.1), (2.2). Moreover, if $u_0 \in H^1(\Omega)$ and $f \in L^2(0,T;L^2(\Omega))$, then the weak solution $u \in H^{2,1}(\Omega \times (0,T))$.*

One need to analyze under which condition our sub-domain problems of the algorithm with Robin transmission conditions is well posed. Without loss of generality, we only study the well-posedness of the sub-domain problem in Ω_1

$$\begin{cases} \partial_t u + Au = f, & \text{in } \Omega_1 \times (0,T), \\ u(.,0) = u_0, & \text{in } \Omega_1, \\ \partial_x u + S_2 u(0,.) = g, & \text{in } (0,T). \end{cases} \tag{2.5}$$

First, we consider the case $S_i = p_i I_d$, for $i = 1,2$, which is justified by the following lemma:

Lemma 2.1. *If $u_0 \in H^1(\Omega_1)$ and $g \in H^{1/4}(0,T)$, then there exists an extension v in $H^{2,1}(\Omega_1 \times (0,T))$, such that $v(.,0) = u_0$ in Ω_1 and $(\partial_x v + S_2 v)(0,.) = g$ on $(0,T)$.*

Proof. Let $\tilde{g}$ be in $H^{3/4}(0,T)$ such that $\tilde{g}(0) = u_0(0)$. By the continuous extension theorem, there exists a function v_1 in $H^{2,1}(\Omega_1 \times (0,T))$ such that

$$v_1(.,0) = u_0, \ v_1(0,.) = \tilde{g}, \ \partial_x v_1(0,.) = 0,$$

and a function v_2 in $H^{2,1}(\Omega_1 \times (0,T))$ such that

$$v_2(.,0) = 0,\ v_2(0,.) = 0,\ \partial_x v_2(0,.) = g - S_2\tilde{g}.$$

Now the sum $v = v_1 + v_2$ is the desired extension in $H^{2,1}(\Omega_1 \times (0,T))$.

So it is sufficient to analyze the well-posedness of the problem with homogeneous initial and boundary conditions

$$\begin{cases} \partial_t \tilde{u} + A\tilde{u} = F, & \text{in } \Omega_1 \times (0,T), \\ \tilde{u}(.,0) = 0, & \text{in } \Omega_1, \\ (\partial_x \tilde{u} + S_2\tilde{u})(0,.) = 0, & \text{in } (0,T), \end{cases} \tag{2.6}$$

where $\tilde{u} = u - v$ and the right hand side function $F = f - (\partial_t v + Av)$ is in $L^2\left(0,T,L^2(\Omega_1)\right)$ if f is in $L^2\left(0,T,L^2(\Omega_1)\right)$. We start with the weak formulation: for any φ in $H^1(\Omega_1)$, we multiply the equation by φ, integrate, and use Greens formula and the boundary condition, to obtain in $\mathcal{D}'(0,T)$ as follows

$$\partial_t(\tilde{u},\varphi) + a(\partial_x\tilde{u},\partial_x\varphi) + \frac{b}{2}\left((\partial_x\tilde{u},\varphi) - (\partial_x\varphi,\tilde{u})\right) + c(\tilde{u},\varphi) + \left(ap_2 + \frac{b}{2}\right)\tilde{u}(0)\varphi(0) = (F,\varphi).$$

The following theorem gives the existence, uniqueness and regularity of the weak solution, see [14] for the proof. ■

Theorem 2.2. *If $F \in L^2\left(0,T,L^2(\Omega_1)\right)$, then (2.6) has a unique weak solution $\tilde{v} \in H^{2,1}(\Omega_1 \times (0,T))$, for any p_2.*

Theorem 2.3. *If $f \in L^2\left(0,T,L^2(\Omega_1)\right)$ and $u_0 \in H^1(\Omega_1)$, $g \in H^{1/4}(0,T)$, then the problem (2.5) has a unique solution $u \in H^{2,1}(\Omega_1 \times (0,T))$, for any p_2.*

Proof. Using Lemma 2.1 and Theorem 2.2, we get the proof. ■

This theorem also holds on the sub-domain Ω_2 by changing x to $-x$, b to $-b$.

Theorem 2.4. *Let g be in $H^{1/4}(0,T)$, and the transmission conditions of (2.3) are $(\partial_x u_1^1 + S_2 u_1^1)(0,.) = g$ and $(-\partial_x u_2^1 + S_1 u_2^1)(0,.) = (-\partial_x u_1^1 + S_1 u_1^1)(0,.)$. Then (2.3) defines a sequence of solutions (u_1^k, u_2^k) in $H^{2,1}(\Omega_1 \times (0,T)) \times \times H^{2,1}(\Omega_1 \times (0,T))$ for any p_1, $p_2 > 0$.*

Proof. For $k = 1$, the equation (2.3) defines a solution (u_1^1, u_2^1), where u_1^1 is in $H^{2,1}(\Omega_1 \times (0,T))$. As $(-\partial_x u_1^1 + S_1 u_1^1)(0,.)$ is in $H^{\frac{1}{4}}(0,T)$, then u_2^1 is also in $H^{2,1}(\Omega_1 \times (0,T))$. Assume that

$$(u_1^k, u_2^k) \in H^{2,1}(\Omega_1 \times (0,T)) \times H^{2,1}(\Omega_1 \times (0,T)).$$

Similar to the case $k = 1$, we also have $(\partial_x u_2^k + S_2 u_2^1)(0,.)$ and $(-\partial_x u_1^k + +S_1 u_1^k)(0,.)$ are in $H^{\frac{1}{4}}(0,T)$, and thus by Theorem 2.3, (u_1^{k+1}, u_2^{k+1}) must be in $H^{2,1}(\Omega_1 \times (0,T)) \times H^{2,1}(\Omega_1 \times (0,T))$. The proof is complete. ■

We also have the well-posed of the sub-domain problems for two parameters $S_i = p_i I_d + q_i \partial_t$. The following theorem is well proved in [1].

Theorem 2.5. *For $p_2 > -\dfrac{b}{2a}$ and $q_2 > 0$, if $f \in L^2(0,T;L^2(\Omega_1))$, $u_0 \in$ $\in H^1(\Omega_1)$ and $g \in L^2(0,T)$, then the problem (2.5) has a unique solution in $H^{2,1}(\Omega_1 \times (0,T))$.*

2.3. Convergence of the SWR algorithm

In this section, we consider the convergence of the SWR algorithm with assumption that all hypotheses for the well-posedness are satisfied. Let us define e_i^k as the errors between the solution u and u_i^k in the SWR algorithm for $i = 1, 2$. The errors e_i^k satisfy a homogeneous equation with the initial condition and the transmission conditions as follows:

$$\begin{cases} \partial_t e_i^{k+1} + A e_i^{k+1} = 0, & \text{in} \quad \Omega_i \times (0,T), \\ e_i^{k+1}(.,0) = 0, & \text{in} \quad \Omega_i, \end{cases} \quad \text{and} \tag{2.7}$$

$$\begin{cases} (\partial_x + S_2) e_1^{k+1}(0,.) = (\partial_x + S_2) e_2^k(0,.), \\ (-\partial_x + S_1) e_2^{k+1}(0,.) = (-\partial_x + S_1) e_1^k(0,.). \end{cases} \tag{2.8}$$

Definition 2.1. (Dirichlet to Neumann operators) The Dirichlet to Neumann operators

$$\tilde{S}_1 g = \partial_x u_1(0,.), \quad \tilde{S}_2 g = -\partial_x u_2(0,.)$$

satisfy that

$$\begin{cases} \partial_t u_i + A u_i = 0, & \text{in } \Omega_i \times (0,T), \\ u_i(.,0) = 0, & \text{in } \Omega_i, \\ u_i(0,.) = g, & \text{in } (0,T), \end{cases} \quad \text{for } i = 1, 2. \tag{2.9}$$

We note that the SWR algorithm converges in two iterations if the operators S_i are given by the Dirichlet to Neumann operators. The following lemma will be used in the rest of the paper.

Lemma 2.2. *Let $\tilde{S}_i$ be the Dirichlet and Neumann operators defined by* (2.9)*, for $i = 1, 2$. If we choose $S_i = \tilde{S}_i$ then the SWR algorithm converges in two steps.*

Proof. The error e_i^{k+1} satisfies

$$(\partial_x + S_2)e_1^{k+1}(0,.) \;=\; (\partial_x + S_2)e_2^{k}(0,.).$$

Since $S_i = \tilde{S}_i$, we got that

$$(\partial_x + S_2)e_1^{k+1}(0,.) = (-\tilde{S}_2 + S_2)e_2^{k}(0,.) = 0.$$

Therefore one obtains $e_1^2 = 0$. ■

It can be seen that the transmission operators $\tilde{S}_i$ are non local operators in time, for $i = 1, 2$. Nevertheless, it is expensive to calculate them directly. Hence, a natural idea is to choose S_i as an approximation of $\tilde{S}_i$ in terms of $S_i = p_i I_d$ or $S_i = p_i I_d + q_i \partial_t$. Let us describe the behavior of the error afterwards.

Theorem 2.6. *Assume that $p_1 > \dfrac{b}{2a}$, $p_2 > -\dfrac{b}{2a}$, $q_1 > 0$ and $q_2 > 0$. Then both SWR algorithms corresponding to $S_i = p_i I_d$ and $S_i = p_i I_d + q_i \partial_t$, for $i = 1, 2$ converge.*

Proof. First, taking the Fourier transform the equations (2.7) in time, we obtain that the solutions on the sub-domains are in the form

$$\hat{e}_1^{k+1} = \lambda_1^{k+1} e^{r^+ x}, \quad \hat{e}_2^{k+1} = \lambda_2^{k+1} e^{r^- x},$$

where r^+, r^- are respectively the roots of characteristic equation

$$-ar^2 + br + c + i\omega = 0.$$

By the transmission conditions, we get that

$$\begin{aligned}(r^+ + \sigma_2)\lambda_1^{k+1} &= (r^- + \sigma_2)\lambda_2^{k},\\ (-r^- + \sigma_1)\lambda_2^{k+1} &= (-r^+ + \sigma_1)\lambda_1^{k}.\end{aligned}$$

with σ_1 and σ_2 are the symbols of S_1 and S_2 respectively. So, we obtain

$$\hat{e}_1^{2k} = [R(\omega)]^k \, \hat{e}_1^0, \quad \hat{e}_2^{2k+1} = [R(\omega)]^k \, \hat{e}_2^1,$$

where

$$R(\omega) = \frac{r^- + \sigma_2}{r^+ + \sigma_2}.\frac{-r^+ + \sigma_1}{-r^- + \sigma_1} =: R_2(\omega).R_1(\omega). \tag{2.10}$$

It can be seen that the convergence of the algorithm is obtained if $|R(\omega)| < 1$. So, we will prove this fact for both algorithms. In the first case, we have $S_i = p_i I_d$. This implies $\sigma_i = p_i$. Then,

$$|R_2(\omega)|^2 = \frac{(b + 2ap_2 - \mathrm{Re}(\sqrt{\mathrm{d}}))^2 + (\mathrm{Im}(\sqrt{\mathrm{d}}))^2}{(b + 2ap_2 + \mathrm{Re}(\sqrt{\mathrm{d}}))^2 + (\mathrm{Im}(\sqrt{\mathrm{d}}))^2},$$

where $d = b^2 + 4a(c + i\omega)$. Thus, for any $p_2 > -\dfrac{b}{2a}$, it follows that $\mathrm{Re}(\sqrt{\mathrm{d}}) > 0$, so we have $|R_2(\omega)| < 1$. Similarly, we also obtain $|R_1(\omega)| < 1$. Then, we deduce the convergence of the SWR algorithm in the first case.

In the second case, we have $\sigma_i = p_i + iq_i\omega$. So the convergence factor in this case becomes

$$R(\omega) = \frac{r^- + p_2 + iq_2}{r^+ + p_2 + iq_2}.\frac{-r^+ + p_1 + iq_1}{-r^- + p_1 + iq_1} =: R_2(\omega).R_1(\omega).$$

By simple computations, it gives that

$$|R_2(\omega)|^2 = \frac{(b + 2ap_2 - x)^2 + (y - 2aq_2\omega)^2}{(b + 2ap_2 + x)^2 + (y + 2aq_2\omega)^2},$$

where $x = \mathrm{Re}(\sqrt{\mathrm{d}}) > 0$, $y = \mathrm{Im}(\sqrt{\mathrm{d}})$. Next, we show that $y\omega \geq 0$. Indeed, since

$$d = b^2 + 4a(c + i\omega) = r^2 e^{2i\varphi}, \quad \varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right],$$

we get that $\sqrt{d} = re^{i\varphi} = r(\cos\varphi + i\sin\varphi)$. Hence,

$$y\omega = r\sin\varphi.\omega = r\sin\varphi.\frac{r^2\sin 2\varphi}{4a} = \frac{2r^3}{4a}\sin^2\varphi\cos\varphi \geq 0.$$

Therefore, using $p_2 > -\dfrac{b}{2a}, q_2 > 0$, we obtain $|R_2(\omega)|^2 < 1$ for all ω.

The second factor can be obtained similarly. Finally, we get $|R(\omega)| < 1$ and then the SWR algorithm also converges in the second case. ■

Remark 2.1. For simplicity, we present the previous results for $\Omega = \mathbb{R}$ to skip the boundary conditions. However, they also hold in a bounded domain Ω with the corresponding boundary conditions.

3. A new optimized Schwarz method

In this section, we present a new simple optimized Schwarz method follows the theoretical Schwarz decomposition method in [4]. In the previous paper

[4], the authors showed that the optimal transmission conditions are obtained for S_1, S_2 are $\sigma_1 = r^+$, $\sigma_2 = -r^-$, where r^+, r^- are introduced in the proof of Theorem 2.6. Since the operators σ_1 and σ_2 are not polynomials in $i\omega$, hence the optimal corresponding transmission operators S_1, S_2 are non-local operators in time. In order to overcome this difficulty, σ_1 and σ_2 are approximated corresponding to the optimal transmission operators by constants, that means

$$S_1 := \frac{b+p}{2a}, \qquad S_2 := \frac{-b+p}{2a}.$$

The main goal of optimized Schwarz methods is to choose the free parameter $p > 0$ to obtain a good performance. The choice of the parameter p is restricted by the requirement that the sub-domain problems need to be well-posed, leads to a good convergence algorithm. The best performance is obtained by minimizing the convergence rate. For this reason, the parameter p in [4] was chosen such that the convergence factor $R(\omega, p)$ is minimized over all $\omega_{\min} \leq \omega \leq \omega_{\max}$, where $\omega_{min} = 0$ and $\omega_{max} = \dfrac{\pi}{\Delta t}$. Hence the optimal choice of the parameter p for the SWR algorithm is the solution of the following min-max problem:

$$\min_{p>0} \left\{ \max_{\omega_{\min} \leq \omega \leq \omega_{\max}} R(\omega, p) \right\}. \tag{3.1}$$

In order to solve (3.1), the authors approximated the optimized parameter in [4] as

$$p := p^* = (8x_0^2 \pi a)^{\frac{1}{4}} \Delta t^{-\frac{1}{4}} + O(\Delta t^{\frac{1}{4}}),$$

where $x_0 = \sqrt{a^2 + 4bc}$. On the other hand, the transmission operators with two parameters are defined by

$$S_1 := \frac{b+p}{2a} + 2q\partial_t, \qquad S_2 := \frac{-b+p}{2a} + 2q\partial_t.$$

In this case, the optimized parameters are obtained as follows

$$p := p^* = (2a\pi x_0^6)^{\frac{1}{8}} \Delta t^{-\frac{1}{8}},$$
$$q := q^* = (\pi^3 x_0^2)^{-\frac{1}{8}} (2a)^{\frac{5}{8}} \Delta t^{\frac{3}{8}}.$$

The main idea of the optimized Schwarz waveform relaxation method is to solve the min-max problem (3.1) analytically. In other words, they optimized the convergence factor $R(\omega, p)$. However, the method the authors used in [4] is very complicated. So we believe that it may be very tricky to extend to another problems such as elliptic problems with varying coefficients or systems. Therefore, in this section, we establish a new optimized methods to obtain good parameters for the Schwarz domain decomposition method. We remind that

in Lemma 2.2, we prove that the best transmission conditions corresponds to Dirichlet to Neumann operators $\tilde{S}_i$. Our method is proposed to approximate Dirichlet to Neumann operators $\tilde{S}_i$ numerically. It includes two steps. The first step is to estimate the initial parameters by minimizing quadrature functions. These functions are obtained by the simplicity of the min-max problem. The second step is to optimize these parameters using the backtracking-Armijo line search method. The proposed algorithm is quite simple and efficient. Moreover, it can be easily extended to more general problems because of the simplicity of the method.

3.1. The first step

In the first case, we approximate $\tilde{S}_i$ in the form of $S_i = p_i I_d$, where p_i are chosen as minimizers of the following min-max problems:

$$\min_{p_i} \left(\max_{\|g\| \leq 1} \left\| \tilde{S}_i g - p_i g \right\| \right), \quad \text{for } i = 1, 2.$$

These problems are difficult even when we only solve them numerically. However, the problems are simpler if we assume that the problems $\max\limits_{\|g\| \leq 1} \left\| \tilde{S}_i g - p_i g \right\|$ are well solved. We then study the first step of our method is to minimize the functions

$$Q_i(p_i) = \left\| \tilde{S}_i g - p_i g \right\|^2_{L^2(0\times(0,T))}, \quad \text{for} \quad i = 1, 2. \tag{3.2}$$

We note that in these functions, g is already known. However, instead of solving the difficult problems $\max\limits_{\|g\| \leq 1} \left\| \tilde{S}_i g - p_i g \right\|$, we choose g be a random functions such that $\|g\| = 1$. In numerical experiment, this fact shows that for random function g, we always obtain a good enough parameters p_i.

Theorem 3.1. *Let p_i^{st} be the minimizers of the functions* (3.2) *respectively. Then we have*

$$p_i^{st} = \frac{\left\langle \tilde{S}_i g, g \right\rangle}{\|g\|^2}, \quad \textit{for } i = 1, 2.$$

Moreover, the SWR method using these parameters converges, that means

$$p_1^{st} > \frac{b}{2a} \quad \textit{and} \quad p_2^{st} > -\frac{b}{2a}.$$

Proof. The first state of this theorem is obtained by the fact that p_i minimizes

the quadratic function

$$Q_i(p_i) = \left\langle \tilde{S}_i g - p_i g, \tilde{S}_i g - p_i g \right\rangle = \\ = \|g\|^2 p_i^2 - 2p_i \left\langle \tilde{S}_i g, g \right\rangle + \left\| \tilde{S}_i \right\|^2.$$

Taking the Fourier transform of equation (2.9) in time, we obtain that $\widehat{u_1}(x,.) = e^{r^+ x}\hat{g}$ and $\widehat{u_2}(x,.) = e^{r^- x}\hat{g}$. We therefore propose the approximation as $\hat{\tilde{S}}_i g = r_i \hat{g}$, where $r_1 = r^+, r_2 = -r^-$. Then,

$$Q_i(p_i) = \left\| \tilde{S}_i g - p_i g \right\|^2 = \left\| \hat{\tilde{S}}_i g - p_i \hat{g} \right\|^2 = \|(r_i - p_i)\hat{g}\|^2.$$

Set $x_\omega = \mathrm{Re}(\sqrt{\mathrm{d}})$ and $y_\omega = \mathrm{Im}(\sqrt{\mathrm{d}})$, we get that

$$Q_i(p_i) \;=\; \int_{\mathbb{R}} \left[\left(\frac{(-1)^{i+1} b + x_\omega}{2a} - p_i \right)^2 + \left(\frac{y_\omega}{2a} \right)^2 \right] |\hat{g}(\omega)|^2 \, d\omega,$$

and

$$\frac{d}{dp_i} Q_i(p_i) \;=\; -2 \int_{\mathbb{R}} \left(\frac{x_\omega}{2a} + (-1)^{i+1} \frac{b}{2a} - p_i \right) |\hat{g}(\omega)|^2 \, d\omega, \qquad \text{for } i = 1, 2.$$

Note that $x_\omega > 0$, so if $p_1 \le \dfrac{b}{2a}$ then $\dfrac{d}{dp_1} Q_1(p_1) < 0$.

Moreover, since p_1^{st} minimizes the function $Q_1(p_1)$, it implied $\dfrac{d}{dp_1} Q_1$ vanishes at p_1^{st}. So we obtain $p_1^{st} > \dfrac{b}{2a}$. Similarly, we also have $p_2^{st} > -\dfrac{b}{2a}$. ■

Remark 3.1. In the proof of Theorem 3.1, it can be seen that

$$-p_1^{st} + \frac{b}{2a} = -p_2^{st} - \frac{b}{2a}.$$

Therefore, it implies $p_1^{st} - p_2^{st} = \dfrac{b}{a}$. So we only need to calculate one parameter p_1^{st} instead of both p_1^{st} and p_2^{st}.

In the second case, we approximate $\tilde{S}_i$ in the form $S_i = p_i I_d + q_i \partial_t$ by minimizing the function

$$Q_i(p_i, q_i) = \left\| \tilde{S}_i g - (p_i + q_i \partial_t) g \right\|^2_{L^2(0 \times (0,T))}. \tag{3.3}$$

Theorem 3.2. *The minimizers* (p_i^{nd}, q_i^{nd}) *of the functions* (3.3) *satisfy the following system*

$$\begin{cases} \|g\|^2 p_i + \langle g, \partial_t g\rangle q_i = \left\langle \tilde{S}_i g, g \right\rangle, \\ \\ \langle g, \partial_t g\rangle p_i + \|\partial_t g\|^2 q_i = \left\langle \partial_t g, \tilde{S}_i g \right\rangle. \end{cases} \tag{3.4}$$

Moreover, we even have

$$p_1^{nd} > \frac{b}{2a},\ p_2^{nd} > -\frac{b}{2a},\ p_1^{nd} - p_2^{nd} = \frac{b}{a},\ q_1^{nd} = q_2^{nd} > 0.$$

Proof. For every $i = 1, 2$, we have that

$$\begin{aligned} Q_i(p_i, q_i) = \|g\|^2 p_i^2 + \|\partial_t g\|^2 q_i^2 + 2\langle g, \partial_t g\rangle p_i q_i - \\ - 2\left\langle \tilde{S}_i g, g \right\rangle p_i - 2\left\langle \partial_t g, \tilde{S}_i g \right\rangle q_i + \left\| \tilde{S}_i \right\|^2. \end{aligned}$$

Since p_i^{nd} and q_i^{nd} minimize the functions $Q_i(p_i, q_i)$, so $\nabla Q_i(p_i^{nd}, q_i^{nd}) = 0$. We imply that p_i^{nd} and q_i^{nd} satisfy the system (3.4). Similar to the proof of Theorem 3.1, after taking a Fourier transform, we also obtain

$$Q_i(p_i, q_i) = \|(r_i - p_i - i\omega q_i)\hat{g}\|^2.$$

Let x_ω, y_ω be the same notations as in the proof of Theorem 3.1, we get

$$\begin{aligned} Q_1 &= \int_{\mathbb{R}} \left[\left(\frac{b + x_\omega}{2a} - p_1 \right)^2 + \left(\frac{y_\omega}{2a} - \omega q_1 \right)^2 \right] |\hat{g}(\omega)|^2 \, d\omega, \\ \frac{\partial Q_1}{\partial p_1}(p_1, q_1) &= -2 \int_{\mathbb{R}} \left(\frac{x_\omega}{2a} + \frac{b}{2a} - p_1 \right) |\hat{g}(\omega)|^2 \, d\omega, \\ \frac{\partial Q_1}{\partial q_1}(p_1, q_1) &= -2 \int_{\mathbb{R}} \left(\frac{y_\omega \omega}{2a} - \omega^2 q_1 \right) |\hat{g}(\omega)|^2 \, d\omega. \end{aligned}$$

It is noticed that $x_\omega > 0$ and followed proof of Theorem 2.6, we get $y_\omega \omega > > 0$. Therefore, two above estimations show that if $p_1 \le \dfrac{b}{2a}$ and $q_1 \le 0$ then $\dfrac{\partial Q_1}{\partial p_1}(p_1, q_1) < 0$ and $\dfrac{\partial Q_1}{\partial q_1}(p_1, q_1) < 0$. However, since the parameters (p_1^{nd}, q_1^{nd}) minimize the function $Q_1(p_1, q_1)$. So it implies that $p_1^{nd} > \dfrac{b}{2a},\ q_1^{nd} > 0$.

Similarly, we can also prove that $p_2^{nd} > -\dfrac{b}{2a},\ q_2^{nd} > 0$.

Moreover, let us take a changing of variables by

$$t_1 = p_1 - \frac{b}{2a} \quad \text{and} \quad t_2 = p_2 + \frac{b}{2a},$$

we then obtain

$$Q_i(p_i, q_i) \;=\; T(t_i, q_i) \;:=\; \int\limits_{\mathbb{R}} \left[\left(\frac{x_\omega}{2a} - t_i \right)^2 + \left(\frac{y_\omega}{2a} - \omega q_i \right)^2 \right] |\hat{g}(\omega)|^2 \, d\omega.$$

Since $\arg\min\limits_{q>0} T(t,q)$ does not depend on the variables, so we obtain that $t_1^{nd} = t_2^{nd}$ and $q_1^{nd} = q_2^{nd}$. ■

3.2. The second step

The purpose of the second step is to optimize the parameters using the backtracking-Armijo line search method. Let e_1, e_2 be the solutions of the following equations

$$\begin{cases} \partial_t e_i + A e_i \;=\; 0, & \text{in} \quad \Omega_i \times (0,T), \\ e_i(.,0) \;=\; 0, & \text{in} \quad \Omega_i, \end{cases}$$

$$\begin{cases} (\partial_x + S_2) e_1(0,.) \;=\; g, & \text{in } (0,T), \\ (-\partial_x + S_1) e_2(0,.) \;=\; (-\partial_x + S_1) e_1(0,.), & \text{in } (0,T). \end{cases}$$

Then, let us consider the function

$$F = \|(\partial_x + S_2) e_2\|^2_{L^2(\{0\}\times(0,T))} = \|g_2\|^2_{L^2(0,T)},$$

where $g_2 := (\partial_x + S_2) e_2(0,.)$.
The number of variables of the function F depends on the number of parameters in S_i, where $S_i = p_i I_d$ or $S_i = p_i I_d + q_i \partial_t$. In fact, we can estimate G_2 by setting $g_1 = (-\partial_x + S_1) e_1(0,.)$, then we have

$$g_1 = -(\partial_x + S_2) e_1(0,.) + (S_1 + S_2) e_1(0,.) \;=\; -g + (S_1 + S_2) e_1(0,.).$$
$$g_2 := (\partial_x + S_2) e_2(0,.) \;=\; -g_1 + (S_1 + S_2) e_2(0,.).$$

Our method bases on Lemma 2.2 which states that the function F will be vanish if we choose $S_i = \tilde{S}_i$. So we will approximate the operators S_i by minimizing the function F numerically. In this article, we use the backtracking-Armijo line search method to solve this problem. For the initial guess, we choose the parameters given by the first step in Section 3.1. For simplicity, we describe the algorithm in the case of one parameter $S_i = p_i I_d$. Let us set $p^k = (p_1^k, p_2^k)$ and the gradient of F at p^k is $g^k = \nabla F(p^k)$, for all $k = 1, 2, ...$ The backtracking-Armijo algorithm has two steps:

1. Pick an initial iterate $p^0 = (p_1^{st}, p_2^{st})$, set $k = 0$.
2. Until p^k has converged:
   (i) Choose a search direction $d^k = -g^k/\|g^k\|$.
   (ii) Calculate a suitable step-length $\lambda^k > 0$ so that

$$F(p^k + \lambda^k d^k) \leq F(p^k) + \lambda^k \beta \left[g^k\right]^T d^k,$$

   for some fixed $\beta \in (0,1)$ (e.g., $\beta = 0.1$).
   (iii) Set $p^{k+1} = p^k + \lambda^k d^k$.

In the step ii), we will compute λ^k as following:

- Given $\lambda_{init} > 0$ (e.g., $\lambda_{init} = 1$), let $\lambda^{(0)} = \lambda_{init}$ and $l = 0$.
- Set $\lambda^{(l+1)} = \frac{1}{2}\lambda^{(l)}$ until

$$F(p^k + \lambda^{(l)} d^k) \leq F(p^k) + \lambda^{(l)} \beta \left[g^k\right]^T d^k.$$

It is similar for the second case of two parameters $S_i = p_i I_d + q_i \partial_t$, the differences are $p^k = (p_1^k, p_2^k, q_1^k, q_2^k)$ and the initial guess $p^0 = (p_1^{nd}, p_2^{nd}, q_1^{nd}, q_2^{nd})$.

4. Numerical experiments

In this section, we perform some numerical experiments to present the efficiency of our method in a comparison with the optimized SWR method in [4]. We consider the parabolic model problem (2.1) on the domain $\Omega = (-3, 3)$. We chose for the problem parameters

$$a = 0.2,\ b = 1,\ c = 0.$$

We imposed homogeneous boundary condition

$$u(-3, t) = u(3, t) = 0,\ t \in (0, T),$$

with time interval $T = 1$, and the initial condition

$$u(x, 0) = e^{-3(0.2-x)^2},\ x \in \Omega.$$

We use a decomposition the domain Ω into two sub-domains $\Omega_1 = (-3, 0)$ and $\Omega_2 = (0, 3)$. Then, the problem (2.1) is discretized with a finite element method

$\mathbb{P}_1$ in space with spatial step $\Delta x = 0.01$, and the backward Euler discretization in time with time steps are $\Delta t = 0.01$ and $\Delta t = 0.005$ respectively. To compare the optimized Schwarz waveform relaxation method and our method, we compute the relative error defined by

$$err \;=\; \frac{\|u - u_{\text{ref}}\|}{\|u_{\text{ref}}\|},$$

where u is numerical solution, u_{ref} is the "reference solution" and $\|.\|$ denotes the discrete L_2-norm. Here, we use the "reference solution" instead of the exact solution. The "reference solution" is obtained by solving the equation using a finite element method $\mathbb{P}_1$ with very small time step $\Delta t = 10^{-4}$ and spatial step $\Delta x = 10^{-4}$.

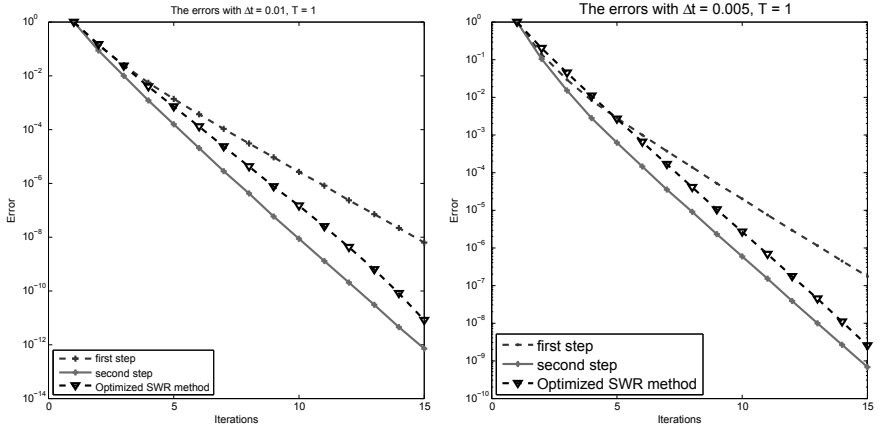

Figure 1. Convergence of our method compared to SWR method using one parameter.

In Figure 1 and 2, one can see the performance of the optimized Schwarz waveform relaxation method and the new algorithm in two steps. These methods are performed in two cases with one and two parameters respectively. We observe that the convergence behaviors of the algorithms in two methods are quite close. In more detail, the optimized parameters p_i^{opt}, q_i^{opt} from the second step always lead to a better performance than the parameters p_i^{st} and p_i^{nd}, q_i^{nd} from the first one.

In particular, in the case of one parameter transmission conditions (Figure 1), the algorithm with the optimized parameters p_i^{opt} even converges faster than the one with the optimized parameters p^* in [4]. However, it does not hold in the case of a two parameter transmission conditions (Figure 2). The

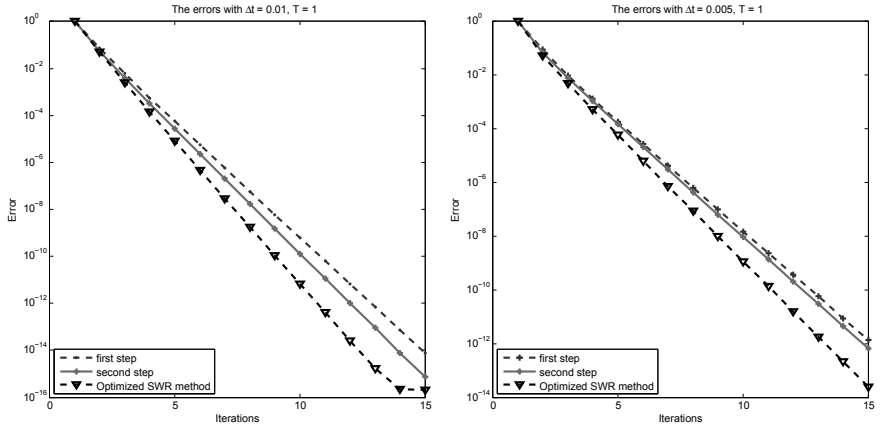

Figure 2. Convergence of our method compared to SWR method using two parameters.

reason comes from the fact that our optimization algorithm for the function F is not efficient enough.

In order to compare the computational cost, we compute the number of iterations as a function of time step Δt, until the errors are smaller than $\varepsilon = 10^{-6}$ in Figure 3. It can be seen that the behaviors of number of iterations for two methods are quite the same.

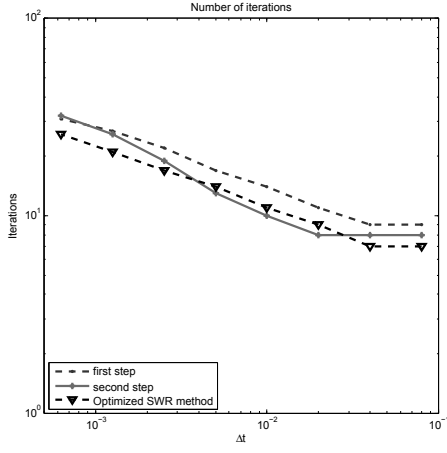

Figure 3. The number of iterations until the errors are smaller than 10^{-6} in one parameter case.

As a conclusion, we have described our method to estimate the optimized parameters for efficient transmission conditions. The first step is very simple. But its convergence is not as good as the convergence of the analytical method given in [1]. The second step numerically improves the parameters of the first one. However, we have performed the numerical experiments to show that the convergence of our method is quite close to the optimized SWR method [4].

References

[1] **Bennequin, D., M.J. Gander and L. Halpern,** A homographic best approximation problem with application to optimized Schwarz waveform relaxation, *Math. Comp.*, **78** (2009), 185–223.

[2] **Gander, M.J.,** Overlapping Schwarz for parabolic problems, *Ninth international conference on Domain Decomposition Methods*, 1997, 97–104.

[3] **Gander, M.J.,** Optimized Schwarz methods, *SIAM J. Numer. Anal.*, **44** (2006), 699–731.

[4] **Gander, M.J. and L. Halpern,** Optimized Schwarz waveform relaxation methods for advection reaction diffusion problems, *SIAM J. Numer. Anal.*, **45** (2007), 666–697.

[5] **Gander, M.J., L. Halpern and F. Nataf,** Optimized Schwarz methods, *Domain decomposition methods in sciences and engineering*, 2001, 15–27.

[6] **Gander, M.J., L. Halpern and F. Nataf,** Optimal convergence for overlapping and non-overlapping Schwarz waveform relaxation, *Eleventh International Conference on Domain Decomposition Methods*, London, 1999, 27–36.

[7] **Gander, M.J., F. Magoulès and F. Nataf,** Optimized Schwarz methods without overlap for the Helmholtz equation, *SIAM J. Sci. Comput.* **24** (2002), 38–60.

[8] **Gander, M.J. and A. M. Stuart,** Space-time continuous analysis of waveform relaxation for the heat equation, *SIAM J. Sci. Comput.* **19** (1998), 2014–2031.

[9] **Giladi, E. and H. B. Keller,** Space-time domain decomposition for parabolic problems, *Numer. Math.*, **93** (2002), 279–313.

[10] **Lelarasmee, E., A. E. Ruehli and A. L. Sangiovanni-Vincentelli,** The waveform relaxation methods for time domain analysis of large scale integrated circuits, *IEEE Trans. on CAD of IC and Systems*, **1** (1982), 131–145.

[11] **Lions, P.-L.,** On the Schwarz alternating method. III. A variant for nonoverlapping subdomains, *Third International Symposium on Domain Decomposition Methods for Partial Differential Equations*, Houston, TX, 1990, 202–223.

[12] **Lions, J.-L. and E. Magenes,** *Problèmes aux Limites non Homogènes et Applications*, Vol. 1, Travaux et Recherches Mathématiques, Paris, 1968.

[13] **Lions, J.-L. and E. Magenes,** *Problèmes aux Limites non Homogènes et Applications*, Vol. 2, Travaux et Recherches Mathématiques, Paris, 1968.

[14] **Martin, V.,** An optimized Schwarz waveform relaxation method for the unsteady convection diffusion equation in two dimensions, *Appl. Numer. Math.* **52** (2005), 401–428.

[15] **Martin, V.,** Schwarz waveform relaxation method for the viscous shallow water equations, *Lect. Notes Comput. Sci. Eng.*, **40** (2005), 653–660.

[16] **Quarteroni, A. and A. Valli,** *Domain Decomposition Methods for Partial Differential Equations*, The Clarendon Press, Oxford University Press, New York, 1999.

[17] **Schwarz, H.A.,** Ueber einige Abbildungsaufgaben., *Journal für die reine und angewandte Mathematik*, **70** 1869, 105–120.

[18] **Smith, B.F., P.E. Bjørstad and W.D. Gropp,** *Domain Decomposition*, Cambridge University Press, Cambridge, 1996.

Minh-Phuong Tran
Faculty of Mathematics and Statistics
Ton Duc Thang University
Ho Chi Minh City
Vietnam
tranminhphuong@tdt.edu.vn

Thanh-Nhan Nguyen
Department of Mathematics
Ho Chi Minh City University of Education
Ho Chi Minh City
Vietnam
nguyenthnhan@hcmup.edu.vn
nguyenthnhan@gmail.com