Annales Univ. Sci. Budapest., Sect. Comp. 45 (2016) 45–55

COMPETING RISKS WEIBULL MODEL: PARAMETER ESTIMATES AND THEIR ACCURACY

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Communicated by Imre Kátai

(Received January 7, 2016; accepted January 28, 2016)

Abstract. Competing risks Weibull model arises as the distribution of the minimum of two independent (two parameter) Weibull random variables having distinct Weibull exponents. This model is commonly used in the strength testing of certain brittle fibers, but it also appears in modeling the life of a series system determined by the shortest of its components. We implemented the EM algorithm to estimate the bi-Weibull model parameters (when the failure mode is unknown) and we explicitly derived the observed information matrix to address the precision of these estimates.

1. Introduction

The simple two parameter Weibull model is often used in the statistical analysis of certain brittle fiber strength. Since this model does not always adequately fit the experimental data [4, 11, 12], several other models, such as the competing risks Weibull model, have been used to describe the failure of fibers. This competing risks Weibull model arises in the strength testing of certain brittle fibers in which failure is known to arise from two competing causes: surface defects and internal defects [3]. It is common to assume that

Key words and phrases: Competing risks, EM algorithm, missing data, observed information matrix, Weibull, bi-Weibull.
2010 Mathematics Subject Classification: 62-P30.

each type of failure follows a two parameter Weibull distribution, leading to a four parameter competing risks model for fiber failure. In practice, it may or may not be the case that the cause of failure is known for any particular specimen [7]. Some models are further complicated by the testing of fibers at several distinct lengths and a fundamental objective is the estimation of fiber strength at extremely short lengths from data taken at longer lengths [3]. Competing risks Weibull model also arises when the life of a series system is determined by the shortest of two component lives, where the life distribution of each component is Weibull [5].

We introduced using the EM algorithm to derive the maximum likelihood estimates (MLE) of the model parameters when the cause of failure is unknown for any particular specimen [6]. The advantage of the EM technique over other techniques, such as Quasi-Newton procedure, is that the maximization reduces to two separate single variable numerical maximizations. Because the EM procedure avoids maximization in four parameters simultaneously, it also avoids the need for estimating the inverse Hessian matrix. There is no need for artful reparameterizations to speed convergence as is suggested in [5] for the Quasi-Newton method. It is not merely enough to obtain the maximum likelihood estimates of the parameters, as they are of little value unless one has some idea of their accuracy. To address this, we derive an explicit formula for the observed information matrix, which we use to give a large sample approximation to the variance-covariance matrix for the maximum likelihood estimates.

2. Competing risks model

A random variable X follows the two parameter Weibull distribution if its cumulative distribution function $P(X \leq x) = W(x;\rho,\beta)$ takes the form

$$W(x;\rho,\beta) = 1 - \exp\left\{-\left(\frac{x}{\beta}\right)^{\rho}\right\} \qquad x \geq 0,$$

where $\beta > 0$ is the scale parameter, and $\rho > 0$ is the Weibull exponent or shape parameter.

Let $X^{(0)}$ and $X^{(1)}$ be independent Weibull distributed random variables having distinct Weibull exponents ρ_0 and ρ_1, and having cumulative distribution function $F_i(x)$, $i = 0, 1$. The minimum $X = \min\{X^{(0)}, X^{(1)}\}$ is said to have a bi-Weibull distribution, whose cumulative distribution function is

$$W(x;\rho_0,\rho_1,\beta_0,\beta_1) = 1 - \exp\left\{-\left[\left(\frac{x}{\beta_0}\right)^{\rho_0} + \left(\frac{x}{\beta_1}\right)^{\rho_1}\right]\right\} \qquad x \geq 0. \tag{2.1}$$

Let $S_i(x) = 1 - F_i(x)$ be the survival function, and $f_i(x)$ be the density function of the simple Weibull random variable $X^{(i)}$, $i = 0, 1$. Denoting the hazard rate of $X^{(i)}$ by $r_i(x)$, we obtain the following relation

$$f_i(x) = r_i(x)S_i(x) = \frac{f_i(x)}{1 - F_i(x)} S_i(x).$$

We will say that a "type 0 defect caused the failure" if $X^{(0)} \leq X^{(1)}$, and conversely, a "type 1 defect caused the failure" if $X^{(0)} > X^{(1)}$. Denote the defect type by

$$I = \begin{cases} 0 & if X^{(0)} \leq X^{(1)} \\ 1 & if X^{(0)} > X^{(1)}. \end{cases}$$

Let X be the minimum of $X^{(0)}$ and $X^{(1)}$. We now express the conditional probability that X arises from a type 0 defect, given that $X = x$. We first find the probability that $x < X \leq x + dx$, or with a short notation, $X \in dx$. From

$$P(X \in dx, I = i) = [1 - F_i(x)]f_i(x)dx = r_i(x)S_0(x)S_1(x)dx, \qquad i = 0, 1,$$

we have

$$P(X \in dx) = (r_0(x) + r_1(x))S_0(x)S_1(x)dx.$$

Therefore,

$$\Pi_i(x) := P(I = i | X \in dx) = \frac{r_i(x)}{r_0(x) + r_1(x)}, \qquad i = 0, 1.$$

Thus the joint density function of X and I (with respect to the product of the Lebesgue measure for X, and the counting measure for I) is

$$f(x, i) = r_0(x)^{1-i} r_1(x)^i S_0(x)S_1(x), \qquad i = 0, 1.$$

Let $X_1, X_2, \ldots, X_n$ be independent identically distributed random variables and let I_j be the type of defect that caused the failure in X_j. Each X_j arises as a minimum of two independent random variables $X_j^{(0)}$ and $X_j^{(1)}$, $j = 1, 2, \ldots, n$. Then the likelihood function of $X = (X_1, X_2, \ldots, X_n)$ and $I = (I_1, I_2, \ldots, I_n)$ is

$$l(\mathbf{x}, \mathbf{i}) = \prod_{j=1}^{n} f(x_j, i_j) = \prod_{j=1}^{n} r_0(x_j)^{1-i_j} r_1(x_j)^{i_j} S_0(x_j)S_1(x_j).$$

Therefore the log likelihood function is

$$\begin{aligned} \log l(\mathbf{x}, \mathbf{i}) = & \textstyle\sum_{j=1}^{n} (1 - i_j) \log r_0(x_j) + \sum_{j=1}^{n} i_j \log r_1(x_j) + \\ & + \textstyle\sum_{j=1}^{n} \log S_0(x_j) + \sum_{j=1}^{n} \log S_1(x_j). \end{aligned} \tag{2.2}$$

The above equations clearly hold regardless of the distribution of $X^{(i)}$, $i = 0, 1$.

3. Maximum likelihood estimation

If the type of defect that caused the failure is known, the parameter estimation may simply be done by maximizing the likelihood function. If this fracture information is unknown - as in most applications - maximizing the likelihood function involves maximization in four parameters simultaneously. According to Ishioka and Nonaka [5] this may be achieved using the Quasi-Newton method. We implemented the EM algorithm to derive the maximum likelihood estimates (MLE) of the model parameters when the cause of failure is unknown for any particular specimen [6]. This widely applicable algorithm for computing the maximum likelihood when some observations are missing is presented by Dempster, Laird and Rubin [1]. Each iteration of the algorithm involves two steps, an expectation (*E-step*) and a maximization (*M-step*), and can be applied to estimate the four parameters of the bi-Weibull distribution by viewing the defect type as missing data.

In order to apply the algorithm, we introduce the parameters Then the *E-step* calculates $Q(\theta^{'}|\theta) = E(\log(l(\mathbf{x}, \mathbf{I}; \theta^{'}))|\mathbf{x}, \theta)$, where θ corresponds to the parameters in the model. Suppose $\theta^{(p)}$ denotes the current value of θ after p cycles of the algorithm. Then the EM iteration $\theta^{(p)} \to \theta^{(p+1)}$ is as follows:

E-step : Compute $Q(\theta|\theta^{(p)})$:

$$\begin{aligned} Q(\theta|\theta^{(p)}) &= \sum_{j=1}^{n} \Pi_0^{(p)}(x_j)\log(\alpha_0\rho_0 x_j^{\rho_0-1}) + \sum_{j=1}^{n} \Pi_1^{(p)}(x_j)\log(\alpha_1\rho_1 x_j^{\rho_1-1}) \\ &\quad -\alpha_0\sum_{j=1}^{n} x_j^{\rho_0} - \alpha_1\sum_{j=1}^{n} x_j^{\rho_1} \end{aligned} \tag{3.1}$$

where

$$\Pi_0^{(p)}(x_j) = \frac{\alpha_0{}^{(p)}\rho_0^{(p)}x_j^{\rho_0^{(p)}-1}}{\alpha_0{}^{(p)}\rho_0^{(p)}x_j^{\rho_0^{(p)}-1} + \alpha_1{}^{(p)}\rho_1^{(p)}x_j^{\rho_1^{(p)}-1}},$$

and

$$\Pi_1^{(p)}(x_j) = 1 - \Pi_0^{(p)}(x_j).$$

Since Equation (3.1) can be written as a sum of two functions, so that each function depends on the parameters α_0, ρ_0 and α_1, ρ_1 respectively, the maximization in the *M-step* can be split into two parts:

M-step : Choose $\theta^{(p+1)}$ to be a value of θ which maximizes $Q(\theta|\theta^{(p)})$:

choose $\alpha_i^{(p+1)}$ and $\rho_i^{(p+1)}$ which maximizes (3.1) as a function of α_i and ρ_i $i = 0, 1$:

$$\begin{aligned}\sum_{j=1}^{n} \Pi_i^{(p)}(x_j) \log(r_i(x_j)) + \sum_{j=1}^{n} \log(S_i(x_j)) = \\ = \sum_{j=1}^{n} \Pi_i^{(p)}(x_j) \log(\alpha_i \rho_i x_j^{\rho_i - 1}) - \alpha_i \sum_{j=1}^{n} x_j^{\rho_i}.\end{aligned} \tag{3.2}$$

The choice of $\alpha_i^{(p+1)}$ $i = 0, 1$ is clear as follows:

$$\alpha_i^{(p+1)} = \frac{\sum_{j=1}^{n} \Pi_i^{(p)}(x_j)}{\sum_{j=1}^{n} x_j^{\rho_i^{(p+1)}}}, \qquad i = 0, 1. \tag{3.3}$$

Then by substituting (3.3) into (3.2) and simplifying, choose $\rho_i^{(p+1)}$, $i = 0, 1$, which maximizes

$$\sum_{j=1}^{n} \Pi_i^{(p)}(x_j) \log \left(\frac{\rho_i x_j^{\rho_i - 1}}{\sum_{j=1}^{n} x_j^{\rho_i}} \right).$$

4. Observed information matrix

In many cases, determining the expected value in the Fisher information is complicated or impossible. Since the justification of the Fisher information lies in its asymptotic properties, any function of the data that is asymptotically equivalent to the Fisher information can be equally justified. One very important such equivalent is the so-called observed information matrix:

$$I(\theta) = (I(\theta_r, \theta_s))_{r,s} \tag{4.1}$$

where

$$I(\theta_r, \theta_s) := -\frac{\partial^2 \log l(\mathbf{X}; \theta)}{\partial \theta_r \partial \theta_s} = -\sum_{i=1}^{n} \frac{\partial^2 \log f(X_i; \theta)}{\partial \theta_r \partial \theta_s}$$

and the covariance matrix of $\hat{\theta}$ is now estimated by the matrix inverse $I^{-1}(\hat{\theta})$. Efron [2] gave justification for preferring to estimate the variance by the inverse of the observed information matrix rather than the inverse Fisher information matrix. There are procedures to estimate the observed information matrix when the EM algorithm is used to find the MLE in incomplete data problems [8, 9]. In the bi-Weibull case it is possible to calculate explicitly the observed information matrix.

Let $\mathbf{Y} = (Y_1, \dots, Y_{2n})$ denote the complete data with associated density $f(\mathbf{Y}|\theta)$, where $\theta = (\theta_1, \dots, \theta_d)$ is the unknown parameter. Let us write $\mathbf{Y} = (\mathbf{Y}_{obs}, \mathbf{Y}_{mis})$, where $\mathbf{Y}_{obs}$ represents the observed part $\mathbf{X}$, a bi-Weibull distributed random sample and $\mathbf{Y}_{mis} = \mathbf{I} = (I_1, \dots, I_n)$ the missing part, the type of defect that caused the failure. The EM algorithm finds $\hat{\theta}$, the MLE for θ based on $\mathbf{Y}_{obs}$ the observed data, by maximizing $f(\mathbf{Y}_{obs}|\theta)$ in θ.

The observed information matrix $I_o(\theta|\mathbf{Y}_{obs})$ is the negative second derivative of the log likelihood function of θ given $\mathbf{Y}_{obs}$,

$$I_o(\theta|\mathbf{Y}_{obs}) = -\frac{\partial^2 \log f(\mathbf{Y}_{obs}|\theta)}{\partial\theta\partial\theta}, \tag{4.2}$$

and

$$I_o(\hat{\theta}|\mathbf{Y}_{obs}) = -\frac{\partial^2 \log f(\mathbf{Y}_{obs}|\theta)}{\partial\theta\partial\theta}\bigg|_{\theta=\hat{\theta}}. \tag{4.3}$$

This can be more complicated to evaluate than the observed information using the complete data $\mathbf{Y}$. Denote

$$I_o(\theta|\mathbf{Y}) = -\frac{\partial^2 \log f(\mathbf{Y}|\theta)}{\partial\theta\partial\theta}, \tag{4.4}$$

and

$$I_{oc} = -\frac{\partial^2 \log f(\mathbf{Y}|\theta)}{\partial\theta\partial\theta}\bigg|_{\theta=\hat{\theta}}.$$

Notice that $f(\mathbf{Y}|\theta)$ can be factorized as

$$f(\mathbf{Y}|\theta) = f(\mathbf{Y}_{obs}|\theta) f(\mathbf{Y}_{mis}|\mathbf{Y}_{obs}, \theta).$$

This expression can be used to evaluate (4.4):

$$I_o(\theta|\mathbf{Y}) = -\frac{\partial^2 \log f(\mathbf{Y}|\theta)}{\partial\theta\partial\theta} = -\frac{\partial^2 \log f(\mathbf{Y}_{obs}|\theta)}{\partial\theta\partial\theta} - \frac{\partial^2 \log f(\mathbf{Y}_{mis}|\mathbf{Y}_{obs}, \theta)}{\partial\theta\partial\theta}.$$

By writing

$$I_{om} = -\frac{\partial^2 \log f(\mathbf{Y}_{mis}|\mathbf{Y}_{obs}, \theta)}{\partial\theta\partial\theta}\bigg|_{\theta=\hat{\theta}},$$

and evaluating (4.2) and (4.3) at $\hat{\theta}$, we get

$$I_o(\hat{\theta}|\mathbf{Y}_{obs}) = I_{oc} - I_{om}.$$

For the bi-Weibull distribution (2.1), it is possible to evaluate I_{oc} and I_{om}, and thus $I_o(\hat{\theta}|\mathbf{Y}_{obs})$, whose inverse approximates the asymptotic variance-covariance matrix of $\hat{\theta}$.

Calculating the complete information matrix, I_{oc} is straightforward as follows. The 'complete likelihood function' is

$$f(\mathbf{Y}|\theta) = l(\mathbf{x}, \mathbf{i}) =$$

$$= \prod_{j=1}^{n} \left(\frac{\rho_0}{\beta_0} \left(\frac{x_j}{\beta_0} \right)^{\rho_0 - 1} \right)^{1-i_j} \left(\frac{\rho_1}{\beta_1} \left(\frac{x_j}{\beta_1} \right)^{\rho_1 - 1} \right)^{i_j} \exp \left\{ - \left[\left(\frac{x_j}{\beta_0} \right)^{\rho_0} + \left(\frac{x_j}{\beta_1} \right)^{\rho_1} \right] \right\}.$$

Taking the negative second derivatives of the complete log likelihood function with respect to the parameters, we get

$$\begin{aligned}
\frac{\partial^2}{\partial \beta_k^2} \log l(\mathbf{x}, \mathbf{i}) &= \sum_{j=1}^{n} (1 - i_j) \frac{\rho_k}{\beta_k^2} - \rho_k \left(\rho_k + 1 \right) \beta_k^{-\rho_k - 2} \\
\frac{\partial^2}{\partial \rho_k^2} \log l(\mathbf{x}, \mathbf{i}) &= \sum_{j=1}^{n} - \frac{(1 - i_j)}{\rho_k} + \left(\frac{x_j}{\beta_k} \right)^{\rho_k} \log^2 \left(\frac{x_j}{\beta_k} \right) \\
\frac{\partial^2}{\partial \beta_k \partial \rho_k} \log l(\mathbf{x}, \mathbf{i}) &= \sum_{j=1}^{n} - \frac{(1 - i_j)}{\beta_k} + \frac{1}{\beta_k} \left(\frac{x_j}{\beta_k} \right)^{\rho_k} \left(1 + \rho_k \log \left(\frac{x_j}{\beta_k} \right) \right).
\end{aligned}$$

for $k = 0$ and 1. The cross derivatives,

$$\frac{\partial^2}{\partial \beta_0 \partial \beta_1} \log l(\mathbf{x}, \mathbf{i}), \quad \frac{\partial^2}{\partial \rho_0 \partial \rho_1} \log l(\mathbf{x}, \mathbf{i}), \quad \frac{\partial^2}{\partial \beta_0 \partial \rho_1} \log l(\mathbf{x}, \mathbf{i}), \quad \frac{\partial^2}{\partial \rho_0 \partial \beta_1} \log l(\mathbf{x}, \mathbf{i})$$

all vanish. Taking the expected value, using $E(1 - I_j | X_j = x_j) = \Pi_0(x_j)$, and $E(I_j | X_j = x_j) = \Pi_1(x_j)$ results in the complete information matrix as follows:

$$I_{oc}(\beta_k, \beta_k) = -\frac{\rho_k}{\beta_k^2} \sum_{j=1}^{n} \Pi_k(x_j) + \frac{\rho_k(\rho_k + 1)}{\beta_k^2} \sum_{j=1}^{n} \left(\frac{x_j}{\beta_k} \right)^{\rho_k} \tag{4.5}$$

$$I_{oc}(\beta_k, \rho_k) = \frac{1}{\beta_k} \sum_{j=1}^{n} \Pi_k(x_j) - \frac{1}{\beta_k} \sum_{j=1}^{n} \left(\frac{x_j}{\beta_k} \right)^{\rho_k} \left(1 + \rho_k \log \left(\frac{x_j}{\beta_k} \right) \right) \tag{4.6}$$

$$I_{oc}(\rho_k, \rho_k) = \frac{1}{\rho_k^2} \sum_{j=1}^{n} \Pi_k(x_j) + \sum_{j=1}^{n} \left(\frac{x_j}{\beta_k} \right)^{\rho_k} \log^2 \left(\frac{x_j}{\beta_k} \right). \tag{4.7}$$

for $k = 0, 1$. When we evaluate (4.5) and (4.6) at the MLE, they simplify to:

$$\begin{aligned} I_{oc}(\beta_k, \beta_k) &= \frac{\rho_k^2}{\beta_k^2} \sum_{j=1}^{n} \Pi_k(x_j) \\ I_{oc}(\beta_k, \rho_k) &= -\frac{1}{\beta_k} \sum_{j=1}^{n} \Pi_k(x_j) \left(1 + \rho_k \log\left(\frac{x_j}{\beta_k}\right)\right). \end{aligned}$$

In matrix form

$$I_{oc} = \begin{pmatrix} I_{oc}(\beta_0, \beta_0) & I_{oc}(\beta_0, \rho_0) & \multicolumn{2}{c}{\mathbf{0}} \\ I_{oc}(\beta_0, \rho_0) & I_{oc}(\rho_0, \rho_0) & & \\ \multicolumn{2}{c}{\mathbf{0}} & I_{oc}(\beta_1, \beta_1) & I_{oc}(\beta_1, \rho_1) \\ & & I_{oc}(\beta_1, \rho_1) & I_{oc}(\rho_1, \rho_1) \end{pmatrix}.$$

The evaluation of the missing information matrix is more complicated, and involves long calculations. The 'missing likelihood function' is

$$f(\mathbf{Y}_{mis} | \mathbf{Y}_{obs}, \theta) = \prod_{j=1}^{n} \Pi_0(x_j)^{1-i_j} \Pi_1(x_j)^{i_j}$$

where $\Pi_0(x_j)$ and $\Pi_1(x_j)$ are defined in (2.2). After much calculation and simplification the missing information matrix is

$$I_{om} = \sum_{j=1}^{n} \Pi_0(x_j) \Pi_1(x_j) \times$$

$$\begin{pmatrix} \frac{\rho_0^2}{\beta_0^2} & -\frac{1+\rho_0 \log(\frac{x_j}{\beta_0})}{\beta_0} & -\frac{\rho_0 \rho_1}{\beta_0 \beta_1} & \frac{\rho_0 (1+\rho_1 \log(\frac{x_j}{\beta_1}))}{\beta_0 \rho_1} \\ . & \frac{(1+\rho_0 \log(\frac{x_j}{\beta_0}))^2}{\rho_0^2} & \frac{\rho_1 (1+\rho_0 \log(\frac{x_j}{\beta_0}))}{\beta_1 \rho_0} & -\frac{(1+\rho_0 \log(\frac{x_j}{\beta_0}))(1+\rho_1 \log(\frac{x_j}{\beta_1}))}{\rho_0 \rho_1} \\ . & . & \frac{\rho_1^2}{\beta_1^2} & -\frac{1+\rho_1 \log(\frac{x_j}{\beta_1})}{\beta_1} \\ . & . & . & \frac{(1+\rho_1 \log(\frac{x_j}{\beta_1}))^2}{\rho_1^2} \end{pmatrix}.$$

Since this matrix is symmetric, the values in the lower triangle of the matrix are the same as the corresponding values in the upper triangle.

As the MLE satisfies the $\frac{\partial}{\partial \theta} \log \prod_{i=1}^{n} f(\mathbf{x}, \mathbf{i}) = 0$ log likelihood equations, the difference of I_{oc} and I_{om} for (β, β) and (β, ρ) can be reduced as follows,

$$
\begin{aligned}
I_o(\beta_k, \beta_k) &= \frac{\rho_k^2}{\beta_k^2} \sum_{j=1}^{n} \Pi_k^2(x_j) \\
I_o(\beta_k, \rho_k) &= -\frac{1}{\beta_k} \sum_{j=1}^{n} \left(1 + \beta_k \log\left(\frac{x_j}{\beta_k}\right) \Pi_k^2(x_j)\right)
\end{aligned}
$$

for $k = 0$ and 1.

5. Example

A numerical example is now given to illustrate the above method. We first simulate data from two simple Weibull distributions with parameters $\rho_0 = 6$, $\beta_0 = 3$, and $\rho_1 = 2$, $\beta_1 = 4$. Then generate the minimum of these two series of data sets. 100 simulation runs were performed on each of the simple Weibull random variables. The results show that in 37 cases of the 100, the first random variable was smaller, i.e. a type 0 defect caused the failure. The above EM-procedure was applied to estimate the parameters of the bi-Weibull distribution (2.1).

The observed information matrix

$$
I_o = \begin{pmatrix} 130.22 & -4.44 & 25.31 & -4.78 \\ -4.44 & 1.17 & 1.13 & 0.81 \\ 25.31 & 1.13 & 15.76 & 8.63 \\ -4.78 & 0.81 & 8.63 & 13.90 \end{pmatrix},
$$

while its inverse is

$$
I_o^{-1} = \begin{pmatrix} 0.05 & 0.27 & -0.15 & 0.10 \\ 0.27 & 2.38 & -0.88 & 0.50 \\ -0.15 & -0.88 & 0.56 & -0.35 \\ 0.10 & 0.50 & -0.35 & 0.29 \end{pmatrix}.
$$

The estimates of the standard deviation for the parameter estimates are shown in Table 1.

Parameters	ρ_0	β_0	ρ_1	β_1
Real	6	3	2	4
Estimated (standard deviation)	6.35 (1.54)	3.08 (0.22)	2.25 (0.54)	3.29 (0.75)

Table 1: Standard deviations of the parameter estimates of the bi-Weibull model

6. Conclusion

We implemented the EM algorithm to estimate the bi-Weibull model parameters (when the failure mode is unknown) and we explicitly derived the observed information matrix, which we used to give a large sample approximation to the variance-covariance matrix for the maximum likelihood estimates. This allows one to provide confidence intervals for the parameters, as well as for the distribution mean or its functions of interest.

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