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DIFFERENTIAL POLYNOMIALS AND VALUE-SHARING

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Dedicated to Professor Ha Huy Khoai
on the occasion of his 70-th birthday

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Abstract. In this paper, we give some theorems on uniqueness problem of differential polynomials of meromorphic functions. Let a, b be non-zero constants and let n, m, l, k be positive integers satisfying $n \geq 3l(k+1) + 3m + 9$ and $m \geq l(k+1) + 1$. If $f^n + af^m(f^{(k)})^l$ and $g^n + ag^m(g^{(k)})^l$ share the value b CM, then f and g are closely related. We also consider the case sharing the value IM.

1. Introduction and main results

Let $\mathbb{C}$ denote the complex plane and $f(z)$ be a non-constant meromorphic function in $\mathbb{C}$. It is assumed that the reader is familiar with the standard notion used in Nevanlinna value distribution theory such as $T(r, f), m(r, f), N(r, f), \ldots$ (see [9, 24]), and $S(r, f)$ denotes any quantity that satisfies the condition $S(r, f) = o(T(r, f))$ as $r \to \infty$, outside of a possible exceptional set of finite linear measure.

In 1959, Hayman considered the problem which was motivated by Picard exceptional values and proved the following result in [10].

Theorem A (Hayman's Theorem)**.** *For all $z \in \mathbb{C}$, each complex meromorphic function f satisfying*

$$f^n(z) + af'(z) \neq b$$

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is constant if $n \geq 5$ *and* $a, b \in \mathbb{C}, a \neq 0$. *However, if* f *is entire, this holds also for* $n \geq 3$ *and for* $n = 2, b = 0$.

As a consequence, if $n \geq 3$ then $f^n(z)f'(z)$ assumes all finite values except possibly zero and infinitely often unless f is a rational function. When f is an entire function, the remain case is only $n = 1$, which was proved later by Cluine in [4]. In 1982, Döringer has shown, Hayman's theorem remains valid for $f^n + af^m(f^{(k)})^l$ instead of $f^n(z) + af'(z)$ provided that $n \geq 3 + (k+1)l + m$ in [5]. These results are related to the value sharing problem of meromorphic functions and their derivatives. Let us first recall some basic definitions.

For f be a non-constant meromorphic function and $S \subset \mathbb{C} \cup \{\infty\}$, we define

$$E_f(S) = \bigcup_{a \in S} \{(z, m) \mid f(z) = a \text{ with multiplicity } m\},$$

$$\overline{E}_f(S) = f^{-1}(S) = \bigcup_{a \in S} \{z \mid f(z) = a\}.$$

Let $\mathcal{F}$ be a non-empty set of meromorphic functions. Two functions f and g of $\mathcal{F}$ are said to *share* S, *counting multiplicity* (share S CM), if $E_f(S) = E_g(S)$. Similarly, two functions f and g are said to *share* S, *ignoring multiplicity* (share S IM), if $\overline{E}(S) = \overline{E}_g(S)$.

In 1997, Yang-Hua studied the unicity problem for meromorphic functions and the differential monomials of the form $u^n u'$, when they share only one value, and obtained the following result in [22].

Theorem B. *Let* f *and* g *be two non-constant meromorphic functions,* $n \geqslant 11$ *be an integer and* $a \in \mathbb{C} \backslash \{0\}$. *If* $f^n f'$ *and* $g^n g'$ *share the value* a *CM, then either* $f = dg$ *for some* $(n+1)$*-th root of unity* d *or* $g(z) = c_1 e^{cz}$ *and* $f(z) = c_2 e^{-cz}$ *where* c, c_1 *and* c_2 *are constants and satisfy* $(c_1 c_2)^{n+1} c^2 = -a^2$.

Since then, several authors study the uniqueness of meromorphic functions by considering differential polynomials like $(u^n)^{(k)}, u^n(u-1)u', u^n(u-1)^2 u', \ldots$ (see [6, 7, 15, 16, 17, 18]).

In 2011, Grahl-Nevo studied the unicity problem for meromorphic functions and the differential polynomial of the form $u^n + au^{(k)}$ and obtained the following theorems in [8].

Theorem C. *Let* f *and* g *be non-constant meromorphic functions on* $\mathbb{C}, a, b \in \in \mathbb{C} \setminus \{0\}$ *and let* n *and* k *be positive integers satisfying* $n \geq 5k + 17$. *Assume that the functions*

$$\psi_f := f^n + af^{(k)} \text{ and } \psi_g := g^n + ag^{(k)} \tag{1.1}$$

share the value b CM. Then

$$\frac{\psi_f - b}{\psi_g - b} = \frac{f^n}{g^n} = \frac{af^{(k)} - b}{ag^{(k)} - b} \tag{1.2}$$

or

$$\frac{\psi_f - b}{\psi_g - b} = \frac{f^n}{ag^{(k)} - b} = \frac{af^{(k)} - b}{g^n} \tag{1.3}$$

or

$$f = g, f^{(k)} = g^{(k)} \equiv \frac{b}{a}. \tag{1.4}$$

Theorem D. *Let f and g be two non-constant entire functions on $\mathbb{C}$, $a, b \in \mathbb{C} \setminus \{0\}$ and let n, k be positive integers satisfying $n \geq 11$ and $n \geq k + 2$. Assume that the functions ψ_f and ψ_g defined as in* (1.1) *share the value b CM. Then* (1.2) *or* (1.4) *holds.*

In 2014, Zhang-Yang added an assumption that "the b-point of ψ_f are not the zeros of f and g" and proved the following theorems in [27].

Theorem E. *Let f and g be two non-constant meromorphic functions on $\mathbb{C}$, $a, b \in \mathbb{C} \setminus \{0\}$ and let n, k be positive integers satisfying $n \geq 3k + 12$. Assume that ψ_f and ψ_g defined as in* (1.1) *share the value b CM and the b-point of ψ_f are not the zeros of f and g. Then* (1.2) *or* (1.3) *holds.*

Theorem F. *Let f and g be two non-constant entire functions on $\mathbb{C}$, $a, b \in \mathbb{C} \setminus \{0\}$ and let n, k be positive integers satisfying $n \geq 8$. Assume that ψ_f and ψ_g defined as in* (1.1) *share the value b CM and the b-point of ψ_f are not the zeros of f and g. Then* (1.2) *holds.*

In this paper, we study the unicity problem for $f^n + af^m(f^{(k)})^l$, where $n, m, k, l \geq 1$, which is related to this kind of differential polynomial. Namely, we prove the following theorems.

Theorem 1.1. *Let f and g be non-constant meromorphic functions on $\mathbb{C}$, $a, b \in \mathbb{C} \setminus \{0\}$ and let n, m, k, l be positive integers satisfying $n \geq 3l(k+1)+3m+9$ and $m \geq l(k + 1) + 1$. Assume that the functions $\phi_f := f^n + af^m(f^{(k)})^l$ and $\phi_g := g^n + ag^m(g^{(k)})^l$ share the value b CM. Then*

$$\frac{\phi_f - b}{\phi_g - b} = \frac{f^n}{g^n} = \frac{af^m(f^{(k)})^l - b}{ag^m(g^{(k)})^l - b} \tag{1.5}$$

or

$$\frac{\phi_f - b}{\phi_g - b} = \frac{f^n}{ag^m(g^{(k)})^l - b} = \frac{af^m(f^{(k)})^l - b}{g^n}. \tag{1.6}$$

Theorem 1.2. *Let ϕ_f and ϕ_g be given as in Theorem 1.1, where f and g be non-constant entire functions. Assume that ϕ_f and ϕ_g share the value b CM. If $n \geq 3l + 3m + 5$ and $m \geq l + 3$, then (1.5) holds.*

Theorem 1.3. *Let ϕ_f and ϕ_g be given as in Theorem 1.1. Assume that ϕ_f and ϕ_g share the value b IM. If $n \geq 6l(k+1) + 6m + 15$ and $m \geq l(k+1) + 1$, then (1.5) or (1.6) holds.*

Theorem 1.4. *Let ϕ_f and ϕ_g be given as in Theorem 1.1, where f and g be non-constant entire functions. Assume that ϕ_f and ϕ_g share the value b IM. If $n \geq 6l + 6m + 8$ and $m \geq l + 4$, then (1.5) holds.*

2. Some basic lemmas

Let us recall a few classical lemmas.

Lemma 2.1. [9] *Let f, g be non-constant meromorphic functions on $\mathbb{C}$, $a \in \mathbb{C}$. Then*

$$T(r, f+g) \leq T(r,f) + T(r,g) + O(1),$$
$$T(r, fg) \leq T(r,f) + T(r,g) + O(1),$$
$$T(r, f-a) = T(r,f) + O(1),$$
$$T(r, \frac{1}{f}) = T(r,f) + O(1).$$

Lemma 2.2. [9] *Let f be a non-constant meromorphic function on $\mathbb{C}$ and let $P(z) \in \mathbb{C}[x]$ be a polynomial of degree q. Then*

$$T\left(r, P(z)\right) = qT(r,f) + O(1).$$

Lemma 2.3. [9] *Let f be a non-constant meromorphic function on $\mathbb{C}$. Then for any positive integer k, we have*

$$T(r, f^{(k)}) \leq T(r,f) + k\overline{N}(r,f) + S(r,f) \leq (k+1)T(r,f) + S(r,f).$$

Moreover, if f be a non-constant entire function, then

$$T(r, f^{(k)}) \leq T(r,f) + S(r,f).$$

Lemma 2.4 (Lemma of Logarithmic Derivative)**.** [9] *Let f be a non-constant meromorphic function on $\mathbb{C}$. Then for any positive integer k, we have*

$$m(r, \frac{f^{(k)}}{f}) = S(r,f).$$

Lemma 2.5 (First Main Theorem)**.** [9] *Let f be a non-constant meromorphic function on $\mathbb{C}$. Then for $a \in \mathbb{C}$, we have*

$$T(r, \frac{1}{f-a}) = T(r, f) + O(1).$$

Lemma 2.6 (Second Main Theorem)**.** [9] *Let $a_1, ..., a_n \in \mathbb{C}$ with $n \geqslant 2, n \in \mathbb{N}$, and let f be a non-constant meromorphic function on $\mathbb{C}$. Then for $r > 0$, we have*

$$(n-1)T(r, f) \leqslant \overline{N}(r, f) + \sum_{i=1}^{n} \overline{N}(r, \frac{1}{f-a_i}) + S(r, f).$$

Suppose that $f_1, \ldots, f_l$ be meromorphic functions on $\mathbb{C}$. Let n_{ij} $(0 \le i \le l$, $1 \le j \le k_i)$ be non-negative integers. We denote by

$$M[f_1, \ldots, f_l] = f_1^{n_{10}} (f_1')^{n_{11}} \cdots (f_1^{(k_1)})^{n_{1k_1}} \cdots f_l^{n_{l0}} (f_l')^{n_{l1}} \cdots (f_l^{(k_l)})^{n_{lk_l}}$$

the *differential monomial* in $f_1, \ldots, f_l$.

Let $f_1, \ldots, f_l$ be meromorphic functions on $\mathbb{C}$, $M_1[f_1, \ldots, f_l], \ldots, M_k[f_1, \ldots, f_l]$ be differential monomials in $f_1, \ldots, f_l$ and $a_1, \ldots, a_k \in \mathbb{C} \setminus \{0\}$. The summation

$$P[f_1, \ldots, f_l] = a_1 M_1[f_1, \ldots, f_l] + \cdots + a_k M_k[f_1, \ldots, f_l]$$

is said to be a *differential polynomial* in $f_1, \ldots, f_l$.

Lemma 2.7. *Let f be a meromorphic function on $\mathbb{C}$. Suppose that $f = \frac{f_1}{f_2}$, where f_1 and f_2 be entire functions that have no common zeros and let k be a positive integer number. Then there exists a differential polynomial $\omega_k[f_1, f_2]$ in f_1, f_2 such that*

$$f^{(k)} = \frac{\omega_k(f_1, f_2)}{f_2^{k+1}}.$$

Proof. We prove by induction. With $k = 1$, we have

$$f' = \frac{f_1' f_2 - f_2' f_1}{f_2^2} = \frac{\omega_1[f_1, f_2]}{f_2^2}.$$

Assume

$$f^{(k)} = \frac{\omega_k[f_1, f_2]}{f_2^{k+1}}.$$

We have

$$
\begin{aligned}
f^{(k+1)} &= \frac{f_2^{k+1}\omega_k'[f_1,f_2]-(k+1)f_2^k f_2'\omega_k[f_1,f_2]}{f_2^{2(k+1)}} = \\
&= \frac{\omega_k'[f_1,f_2]f_2-(k+1)f_2'\omega_k[f_1,f_2]}{f_2^{k+2}} = \\
&= \frac{\omega_{k+1}[f_1,f_2]}{f_2^{k+2}}.
\end{aligned}
$$

This completes the proof of Lemma 2.7. ■

Lemma 2.8. *Let f be an entire function on $\mathbb{C}$, $a, b \in \mathbb{C}\setminus\{0\}$ and m, l, k be positive integers. Suppose that $f^m(f^{(k)})^l$ is a non-constant function. Then we have*

$$T\left(r, f^m(f^{(k)})^l\right) \le N\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{af^m(f^{(k)})^l - b}\right) + T\left(r, f^{(k)}\right) + S(r,f).$$

Proof. By Lemma 2.6 and the assumption that f is a non-constant entire function, we have

$$
\begin{aligned}
T(r, f^m(f^{(k)})^l) &\le \overline{N}(r, \frac{1}{f^m(f^{(k)})^l}) + \overline{N}(r, \frac{1}{f^m(f^{(k)})^l - \frac{b}{a}}) + S(r,f) \le \\
&\le \overline{N}(r, \frac{1}{f^m}) + \overline{N}(r, \frac{1}{(f^{(k)})^l}) + \overline{N}(r, \frac{1}{af^m(f^{(k)})^l - b}) + S(r,f) \le \\
&\le \overline{N}(r, \frac{1}{f}) + \overline{N}(r, \frac{1}{f^{(k)}}) + \overline{N}(r, \frac{1}{af^m(f^{(k)})^l - b}) + S(r,f),
\end{aligned}
$$

which implies,

$$T(r, f^m(f^{(k)})^l) \le N(r, \frac{1}{f}) + \overline{N}(r, \frac{1}{af^m(f^{(k)})^l - b}) + T(r, f^{(k)}) + S(r,f).$$

Lemma 2.8 is proved. ■

Lemma 2.9 ([22], Lemma 3)**.** *Let f and g be non-constant meromorphic functions on $\mathbb{C}$. If f and g share 1 CM, then one of the following three cases holds:*

1) $T(r,f)+T(r,g) \le 2\{N_2(r,f)+N_2(r,g)+N_2(r,\frac{1}{f})+N_2(r,\frac{1}{g})\}+S(r,f)+$ $+ S(r,g)$;
2) $f \equiv g$;
3) $fg \equiv 1$.

Lemma 2.10 ([24], Theorem 1)**.** *Let f and g be non-constant meromorphic functions on $\mathbb{C}$. If f and g share 1 IM, then one of the following three cases holds:*

1) $T(r,f)+T(r,g)\le 2N_2(r,f)+3\overline{N}(r,f)+2N_2(r,g)+3\overline{N}(r,g)+$
$+2N_2(r,\frac{1}{f})+3\overline{N}(r,\frac{1}{f})+2N_2(r,\frac{1}{g})+3\overline{N}(r,\frac{1}{g})+S(r,f)+S(r,g)$;

2) $f\equiv g$;

3) $fg\equiv 1$.

3. Proof of the Theorems

Proof. [Proof of Theorem 1.1]

We claim that $af^m(f^{(k)})^l-b\not\equiv 0$. Suppose that $af^m(f^{(k)})^l-b\equiv 0$, we have $f^{(k)}\not\equiv 0$ and

$$\begin{aligned} mT(r,f) &= T(r,f^m)+O(1)= \\ &= lT(r,f^{(k)})+O(1)\le \\ &\le l(k+1)T(r,f)+S(r,f). \end{aligned}$$

Hence

$$(m-l(k+1))T(r,f)\le S(r,f),$$

which contradicts the assumption that $m\ge l(k+1)+1$. Similarly, we have $ag^m(g^{(k)})^l-b\not\equiv 0$.

Setting

$$F=\frac{-f^n}{af^m(f^{(k)})^l-b},G=\frac{-g^n}{ag^m(g^{(k)})^l-b}. \tag{3.1}$$

By Lemma 2.1 and Lemma 2.3, we have

$$\begin{aligned} nT(r,f) &= T(r,-f^n)+O(1)\le \\ &\le T(r,\frac{-f^n}{af^m(f^{(k)})^l-b})+T(r,af^m(f^{(k)})^l-b)+O(1)\le \\ &\le T(r,F)+mT(r,f)+lT(r,f^{(k)})+O(1)\le \\ &\le T(r,F)+(m+l(k+1))T(r,f)+S(r,f)\le \\ &\le T(r,F)+(m+l(k+1))T(r,f)+S(r,f). \end{aligned}$$

Hence

$$(n-m-l(k+1))T(r,f)\le T(r,F)+S(r,f).$$

From this and the assumption that $n \geq 3l(k+1)+3m+9$, we have F is non-constant. Similarly, we have G is non-constant.

Suppose that $f = \frac{f_1}{f_2}$, where f_1, f_2 are entire functions which have no common zeros and $g = \frac{g_1}{g_2}$, where g_1, g_2 are entire functions which have no common zeros. By Lemma 2.7, there exists a differential polynomial $\omega_k[f_1, f_2]$ such that $f^{(k)} = \frac{\omega_k[f_1,f_2]}{f_2^{k+1}}$ and $g^{(k)} = \frac{\omega_k[g_1,g_2]}{g_2^{k+1}}$. So

$$\phi_f - b = \frac{f_1^n + (af_1^m(\omega_k[f_1,f_2])^l - bf_2^{m+l(k+1)})f_2^{n-m-l(k+1)}}{f_2^n}$$

and

$$\phi_g - b = \frac{g_1^n + (ag_1^m(\omega_k[g_1,g_2])^l - bg_2^{m+l(k+1)})g_2^{n-m-l(k+1)}}{g_2^n}.$$

In the following we prove that the functions

$$f_1^n + (af_1^m(\omega_k[f_1,f_2])^l - bf_2^{m+l(k+1)})f_2^{n-m-l(k+1)}$$

and f_2 have no common zeros. Suppose that there exists a constant γ such that

$$(f_1(\gamma))^n + \Big(a(f_1(\gamma))^m(\omega_k[f_1,f_2](\gamma))^l - b(f_2(\gamma))^{m+l(k+1)}\Big)(f_2(\gamma))^{n-m-l(k+1)} = 0$$

and

$$f_2(\gamma) = 0.$$

This implies $f_1(\gamma) = 0$ and $f_2(\gamma) = 0$, which contradicts to the assumption that f_1 and f_2 have no common zeros. Hence $f_1^n + (af_1^m(\omega_k[f_1,f_2])^l - -bf_2^{m+l(k+1)})f_2^{n-m-l(k+1)}$ and f_2 have no common zeros. Therefore

$$E_{\phi_f}(b) = E_{f_1^n + (af_1^m(\omega_k[f_1,f_2])^l - bf_2^{m+l(k+1)})f_2^{n-m-l(k+1)}}(0). \tag{3.2}$$

Similarly, we have

$$E_{\phi_g}(b) = E_{g_1^n + (ag_1^m(\omega_k[g_1,g_2])^l - bg_2^{m+l(k+1)})g_2^{n-m-l(k+1)}}(0). \tag{3.3}$$

On the other hand, we have

$$F = \frac{f_1^n}{-(af_1^m(\omega_k(f_1,f_2))^l - bf_2^{m+l(k+1)})f_2^{n-m-l(k+1)}}. \tag{3.4}$$

Hence

$$F - 1 = \frac{f_1^n + (af_1^m(\omega_k(f_1,f_2))^l - bf_2^{m+l(k+1)})f_2^{n-m-l(k+1)}}{-(af_1^m(\omega_k(f_1,f_2))^l - bf_2^{m+l(k+1)})f_2^{n-m-l(k+1)}}.$$

We will show that the functions $f_1^n+(af_1^m(\omega_k(f_1,f_2))^l-bf_2^{m+l(k+1)})f_2^{n-m-l(k+1)}$ and $af_1^m(\omega_k(f_1,f_2))^l - bf_2^{m+l(k+1)}$ have no common zeros. Suppose that there exists a constant $\alpha \in \mathbb{C}$ such that

$$\begin{cases} (f_1(\alpha))^n + (a(f_1(\alpha))^m(\omega_k(f_1,f_2)(\alpha))^l - b(f_2(\alpha))^{m+l(k+1)})\cdot \\ \qquad \cdot (f_2(\alpha))^{n-m-l(k+1)} = 0 \\ a(f_1(\alpha))^m(\omega_k(f_1,f_2)(\alpha))^l - b(f_2(\alpha))^{m+l(k+1)} = 0. \end{cases}$$

From this and the assumption that $m \geq 1$, we have $f_1(\alpha) = f_2(\alpha) = 0$. This contradicts to the assumption that f_1, f_2 have no common zeros. Therefore $f_1^n + (af_1^m(\omega_k(f_1,f_2))^l - bf_2^{m+l(k+1)})f_2^{n-m-l(k+1)}$ and $af_1^m(\omega_k(f_1,f_2))^l - bf_2^{m+l(k+1)}$ have no common zeros. Combining this with the previous fact that $f_1^n+(af_1^m(\omega_k[f_1,f_2])^l-bf_2^{m+l(k+1)})f_2^{n-m-l(k+1)}$ and f_2 have no common zeros, we have $f_1^n+(af_1^m(\omega_k(f_1,f_2))^l-bf_2^{m+l(k+1)})f_2^{n-m-l(k+1)}$ and $(af_1^m(\omega_k(f_1,f_2))^l - bf_2^{m+l(k+1)})f_2^{n-m-l(k+1)}$ have no common zeros. So we have

$$E_F(1) = E_{f_1^n+(af_1^m(\omega_k[f_1,f_2])^l-bf_2^{m+l(k+1)})f_2^{n-m-l(k+1)}}(0). \tag{3.5}$$

Similarly, we get

$$E_G(1) = E_{g_1^n+(ag_1^m(\omega_k[g_1,g_2])^l-bg_2^{m+l(k+1)})g_2^{n-m-l(k+1)}}(0). \tag{3.6}$$

From (3.2) to (3.6) and the assumption that $E_{\phi_f}(b) = E_{\phi_g}(b)$, we have

$$E_F(1) = E_G(1).$$

Applying Lemma 2.9 to F and G with the following cases:

Case 1.

$$T(r,F) + T(r,G) \leq 2\{N_2(r,F) + N_2(r,\frac{1}{F}) + N_2(r,G) + N_2(r,\frac{1}{G})\} + \\ + S(r,F) + S(r,G). \tag{3.7}$$

By (3.4), we obtain

$$N_2(r,\frac{1}{F}) \leq 2\overline{N}(r,\frac{1}{f}).$$

Hence

$$N_2(r,\frac{1}{F}) \leq 2T(r,f). \tag{3.8}$$

Similarly, we get

$$N_2(r,\frac{1}{G}) \leq 2T(r,g). \tag{3.9}$$

On the other hand, we have

$$\begin{aligned} N_2(r,F) &= N_2(r,\frac{-f^n}{af^m(f^{(k)})^l-b}) \leq \\ &\leq N_2(r,\frac{1}{af^m(f^{(k)})^l-b}) + N_2(r,f^n) \leq \\ &\leq N_2(r,\frac{1}{af^m(f^{(k)})^l-b}) + 2\overline{N}(r,f) \leq \\ &\leq T(r,f^m(f^{(k)})^l) + 2T(r,f) \leq \\ &\leq mT(r,f) + lT(r,f^{(k)}) + 2T(r,f) + O(1) \leq \\ &\leq mT(r,f) + l(k+1)T(r,f) + 2T(r,f) + S(r,f), \end{aligned}$$

which implies

$$N_2(r,F) \leq (m+2+l(k+1))T(r,f) + S(r,f). \tag{3.10}$$

Similarly, we get

$$N_2(r,G) \leq (m+2+l(k+1))T(r,g) + S(r,g). \tag{3.11}$$

From (3.7) to (3.11), we have

$$\begin{aligned} T(r,F)+T(r,G) \leq 2\{(l(k+1)+m+4)T(r,f) + (l(k+1)+m+4)T(r,g)\}+ \\ + S(r,f) + S(r,g). \end{aligned}$$

Therefore

$$\begin{aligned} nT(r,f)+nT(r,g) &= T(r,f^n)+T(r,g^n)+O(1) \leq \\ &\leq T(r,F)+T(r,\frac{1}{af^m(f^{(k)})^l-b}) + O(1) + \\ &\quad +T(r,G)+T(r,\frac{1}{ag^m(g^{(k)})^l-b}) + O(1) = \\ &= T(r,F)+T(r,f^m(f^{(k)})^l)+O(1) + \\ &\quad +T(r,G)+T(r,g^m(g^{(k)})^l)+O(1) \leq \\ &\leq T(r,F)+T(r,f^m)+T(r,(f^{(k)})^l)+O(1) + \\ &\quad +T(r,G)+T(r,g^m)+T(r,(g^{(k)})^l)+O(1) \leq \\ &\leq 2\{(l(k+1)+m+4)T(r,f) + \\ &\quad +(l(k+1)+4+m)T(r,g)\} + \\ &\quad +(m+l(k+1))T(r,f)+(m+l(k+1))T(r,g) + \\ &\quad +S(r,f)+S(r,g), \end{aligned}$$

which implies

$$n(T(r,f)+T(r,g)) \leq (3l(k+1)+3m+8)(T(r,f)+T(r,g))+S(r,f)+S(r,g).$$

Thus

$$(n-3l(k+1)-3m-8)(T(r,f)+T(r,g)) \leq S(r,f)+S(r,g),$$

which contradicts to the assumption that $n \geq 3l(k+1)+3m+9$.

Case 2. $F=G$. Then

$$\frac{-f^n}{af^m(f^{(k)})^l-b} = \frac{-g^n}{ag^m(g^{(k)})^l-b}.$$

Therefore

$$\frac{\phi_f-b}{\phi_g-b} = \frac{f^n}{g^n} = \frac{af^m(f^{(k)})^l-b}{ag^m(g^{(k)})^l-b}.$$

Case 3. $FG=1$. Thus

$$\frac{\phi_f-b}{\phi_g-b} = \frac{f^n}{ag^m(g^{(k)})^l-b} = \frac{af^m(f^{(k)})^l-b}{g^n}.$$

This completes the proof of Theorem 1.1. ■

Proof. [Proof of Theorem 1.2]

We claim that $af^m(f^{(k)})^l-b \not\equiv 0$. If $af^m(f^{(k)})^l-b \equiv 0$, we have $f^{(k)} \not\equiv 0$ and

$$\begin{aligned} mT(r,f) &= T(r,f^m)+O(1) = \\ &= T(r,(f^{(k)})^l)+O(1) = \\ &= lT(r,f^{(k)})+O(1) \leq \\ &\leq lT(r,f)+S(r,f), \end{aligned}$$

which implies

$$(m-l)T(r,f) \leq S(r,f),$$

which contradicts to the assumption that $m \geq l+3$. Similarly, we have $ag^m(g^{(k)})^l-$ $-b \not\equiv 0$.

We define the functions F and G as in the proof of Theorem 1.1. Proceeding as in the proof of Theorem 1.1, we can obtain that F and G share 1 CM. Applying Lemma 2.9 to F and G, we have the following three cases:

Case 1.

$$T(r,F)+T(r,G) \leq 2\{N_2(r,F)+N_2(r,\frac{1}{F})+N_2(r,G)+N_2(r,\frac{1}{G})\}+ \\ +S(r,F)+S(r,G).$$

We have

$$N_2(r, \frac{1}{F}) \le 2T(r, f),$$
$$N_2(r, \frac{1}{G}) \le 2T(r, g).$$

By Lemma 2.3, we get

$$\begin{aligned} N_2(r, F) &= N_2(r, \frac{-f^n}{af^m(f^{(k)})^l - b}) \le \\ &\le N_2(r, \frac{1}{af^m(f^{(k)})^l - b}) \le \\ &\le T(r, f^m(f^{(k)})^l) \le \\ &\le mT(r, f) + lT(r, f^{(k)}) + O(1), \end{aligned}$$

which implies

$$N_2(r, \frac{1}{F}) \le (m + l)T(r, f) + S(r, f).$$

Similarly, we have

$$N_2(r, \frac{1}{G}) \le (m + l)T(r, g) + S(r, g).$$

Hence

$$T(r, F) + T(r, G) \le 2\{(m+l+2)T(r, f) + (m+l+2)T(r, g)\} + S(r, f) + S(r, g).$$

Therefore

$$\begin{aligned} nT(r, f) + nT(r, f) &= T(r, f^n) + T(r, f^n) + O(1) \le \\ &\le T(r, F) + T(r, \frac{1}{af^m(f^{(k)})^l - b}) + T(r, G) + \\ &\quad +T(r, \frac{1}{af^m(g^{(k)})^l - b}) + O(1) = \\ &= T(r, F) + T(r, \frac{1}{f^m(f^{(k)})^l}) + T(r, G) + \\ &\quad +T(r, \frac{1}{f^m(g^{(k)})^l}) + O(1) = \\ &= T(r, F) + T(r, f^m(f^{(k)})^l) + T(r, G) + \\ &\quad +T(r, f^m(g^{(k)})^l) + O(1) \le \\ &\le T(r, F) + T(r, f^m) + T(r, (f^{(k)})^l) + T(r, G) + \\ &\quad +T(r, g^m) + T(r, (g^{(k)})^l) + O(1) \le \\ &\le 2\{(m + l + 2)T(r, f) + (m + l + 2)T(r, g)\} + \\ &\quad +(m + l)T(r, f) + (m + l)T(r, g) + S(r, f) + S(r, g), \end{aligned}$$

which implies

$$n(T(r,f)+T(r,g)) \le (3l+3m+4)(T(r,f)+T(r,g))+S(r,f)+S(r,g).$$

Thus

$$(n-3l-3m-4)(T(r,f)+T(r,g)) \le S(r,f)+S(r,g),$$

which contradicts to the assumption that $n \ge 3l+3m+5$.

Case 2. $F=G$. Then

$$\frac{-f^n}{af^m(f^{(k)})^l-b}=\frac{-g^n}{af^m(g^{(k)})^l-b}.$$

Therefore (1.5) holds.

Case 3. $FG=1$. Then

$$\frac{\phi_f-b}{\phi_g-b}=\frac{f^n}{ag^m(g^{(k)})^l-b}=\frac{af^m(f^{(k)})^l-b}{g^n}. \tag{3.12}$$

We will prove that (3.12) cannot occur. Since ϕ_f, ϕ_g share the value b CM and f, g are entire functions, $\frac{\phi_f-b}{\phi_g-b}$ has no zero or pole at all. From this and (3.12), we have

$$N(r,\frac{1}{f})=\frac{1}{n}N(r,\frac{1}{g^m(g^{(k)})^l-\frac{b}{a}}), N(r,\frac{1}{g})=\frac{1}{n}N(r,\frac{1}{f^m(f^{(k)})^l-\frac{b}{a}}), \tag{3.13}$$

$$\overline{N}(r,\frac{1}{f})=\overline{N}(r,\frac{1}{g^m(g^{(k)})^l-\frac{b}{a}}), \overline{N}(r,\frac{1}{g})=\overline{N}(r,\frac{1}{f^m(f^{(k)})^l-\frac{b}{a}}). \tag{3.14}$$

We will show that $f^{(k)} \not\equiv 0$. Suppose for contradiction that $f^{(k)} \equiv 0$. Then f is a non-constant polynomial. Combining this and (3.13), we have g has no zero at all. This implies fg is a non-constant function and $g^{(k)} \not\equiv 0$. From (3.12), we have

$$(fg)^n=-b(ag^m(g^{(k)})^l-b). \tag{3.15}$$

From this and Lemma 2.6, we have

$$\begin{aligned} nT(r,fg) &\le \overline{N}(r,\frac{1}{(fg)^n})+\overline{N}(r,\frac{1}{g^m(g^{(k)})^l})+S(r,fg) \le \\ &\le N(r,\frac{1}{fg})+\overline{N}(r,\frac{1}{g^{(k)}})+S(r,fg) \le \\ &\le T(r,fg)+T(r,g^{(k)})+S(r,fg), \end{aligned}$$

which implies

$$(n-1)T(r,fg) \leq T(r,g) + S(r,fg) + S(r,g). \tag{3.16}$$

On the other hand, by Lemma 2.6 and (3.15), we have

$$\begin{aligned}
mT(r,g) &= T(r,g^m) + O(1) \leq \\
&\leq T(r,g^m(g^{(k)})^l) + T(r,\frac{1}{(g^{(k)})^l}) + O(1) \leq \\
&\leq \overline{N}(r,\frac{1}{(fg)^n}) + \overline{N}(r,\frac{1}{g^m(g^{(k)})^l}) + lT(r,g) + S(r,g) \leq \\
&\leq T(r,fg) + T(r,g) + lT(r,g) + S(r,g),
\end{aligned}$$

which yields

$$(m-l-1)T(r,g) \leq T(r,fg) + S(r,g). \tag{3.17}$$

From (3.16) and (3.17), we have

$$(n-1)T(r,fg) + (m-l-2)T(r,g) \leq S(r,fg) + S(r,g),$$

which contradicts to the assumptions that $n \geq 3l + 3m + 5$ and $m \geq l + 3$. Hence $f^{(k)} \not\equiv 0$ and similarly, we have $g^{(k)} \not\equiv 0$.

By Lemma 2.8, we have

$$\begin{aligned}
mT(r,f) &= T(r,f^m) + O(1) \leq \\
&\leq T(r,f^m(f^{(k)})^l) + T(r,\frac{1}{(f^{(k)})^l}) + O(1) \leq \\
&\leq N(r,\frac{1}{f}) + \overline{N}(r,\frac{1}{af^m(f^{(k)})^l - b}) + \\
&\quad + T(r,f^{(k)}) + T(r,(f^{(k)})^l) + S(r,f) \leq \\
&\leq N(r,\frac{1}{f}) + \overline{N}(r,\frac{1}{af^m(f^{(k)})^l - b}) + \\
&\quad + T(r,f^{(k)}) + lT(r,f^{(k)}) + S(r,f) \leq \\
&\leq N(r,\frac{1}{f}) + \overline{N}(r,\frac{1}{af^m(f^{(k)})^l - b}) + (l+1)T(r,f) + S(r,f),
\end{aligned}$$

which implies

$$(m-l-1)T(r,f) \leq N(r,\frac{1}{f}) + \overline{N}(r,\frac{1}{af^m(f^{(k)})^l - b}) + S(r,f).$$

From this and (3.13), (3.14), we have

$$\begin{aligned}
(m-l-1)T(r,f) &\leq \frac{1}{n}N(r,\frac{1}{ag^m(g^{(k)})^l-b})+\overline{N}(r,\frac{1}{g})+S(r,f)\leq \\
&\leq \frac{1}{n}N(r,\frac{1}{ag^m(g^{(k)})^l-b})+N(r,\frac{1}{g})+S(r,f)= \\
&= \frac{1}{n}N(r,\frac{1}{ag^m(g^{(k)})^l-b})+\frac{1}{n}N(r,\frac{1}{af^m(f^{(k)})^l-b})+ \\
&\quad +S(r,f)\leq \\
&\leq \frac{1}{n}T(r,ag^m(g^{(k)})^l-b)+\frac{1}{n}T(r,af^m(f^{(k)})^l-b)+ \\
&\quad +S(r,f)+S(r,g)= \\
&= \frac{1}{n}T(r,g^m(g^{(k)})^l)+\frac{1}{n}T(r,f^m(f^{(k)})^l)+ \\
&\quad +S(r,f)+S(r,g)\leq \\
&\leq \frac{1}{n}\left(mT(r,g)+lT(r,g^{(k)})\right)+ \\
&\quad +\frac{1}{n}\left(mT(r,f)+lT(r,f^{(k)})\right)+S(r,g)\leq \\
&\leq \frac{1}{n}\left(mT(r,g)+lT(r,g)\right)+\frac{1}{n}\left(mT(r,f)+lT(r,f)\right)+ \\
&\quad +S(r,f)+S(r,g),
\end{aligned}$$

which implies

$$(m-l-1)T(r,f)\leq \frac{(m+l)}{n}T(r,g)+\frac{m+l}{n}T(r,f)+S(r,f)+S(r,g).$$

Similarly, we get

$$(m-l-1)T(r,g)\leq \frac{(m+l)}{n}T(r,f)+\frac{m+l}{n}T(r,g)+S(r,g).$$

Hence

$$(m-l-1)(T(r,f)+T(r,g))\leq \frac{2(m+l)}{n}(T(r,f)+T(r,g))+S(r,f)+S(r,g).$$

Therefore

$$(n(m-l-1)-2(m+l))(T(r,f)+T(r,g))\leq S(r,f)+S(r,g),$$

which contradicts to the assumptions that $m\geq l+3$ and $n\geq 3l+3m+5$. This proves Theorem 1.2. ■

Proof. [Proof of Theorem 1.3]

We define the functions F and G as in the proof of Theorem 1.1. Proceeding as in the proof of Theorem 1.1, we obtain that F and G share 1 IM. Applying Lemma 2.10 to F and G, we have the following three cases:

Case 1.

$$\begin{aligned} T(r,F)+T(r,G) &\leq 2N_2(r,F)+3\overline{N}(r,F)+2N_2(r,G)+3\overline{N}(r,G)+\\ &+2N_2(r,\frac{1}{F})+3\overline{N}(r,\frac{1}{F})+2N_2(r,\frac{1}{G})+3\overline{N}(r,\frac{1}{G})+\\ &+S(r,F)+S(r,G). \end{aligned}$$

Proceeding as in the proof of Theorem 1.1, we have

$$N_2(r,\frac{1}{F}) \leq 2T(r,f),$$

$$N_2(r,\frac{1}{G}) \leq 2T(r,g),$$

$$\overline{N}(r,\frac{1}{F}) \leq T(r,f),$$

$$\overline{N}(r,\frac{1}{G}) \leq T(r,g),$$

$$N_2(r,F) \leq (m+2+l(k+1))T(r,f)+S(r,f),$$

$$N_2(r,G) \leq (m+2+l(k+1))T(r,g)+S(r,g).$$

By Lemma 2.3, we have

$$\begin{aligned} \overline{N}(r,F) &= \overline{N}(r,\frac{-f^n}{af^m(f^{(k)})^l-b}) \leq \\ &\leq \overline{N}(r,\frac{1}{af^m(f^{(k)})^l-b})+\overline{N}(r,f) \leq \\ &\leq T(r,af^m(f^{(k)})^l-b)+\overline{N}(r,f) \leq \\ &\leq T(r,f^m(f^{(k)})^l)+\overline{N}(r,f) \leq \\ &\leq mT(r,f)+lT(r,f^{(k)})+T(r,f) \leq \\ &\leq mT(r,f)+l(k+1)T(r,f)+T(r,f)+S(r,f). \end{aligned}$$

which implies

$$\overline{N}(r,\frac{1}{F}) \leq (l(k+1)+m+1)T(r,f)+S(r,f).$$

Similarly, we have

$$\overline{N}(r,\frac{1}{G}) = (l(k+1)+m+1)T(r,g)+S(r,g).$$

Hence

$$\begin{aligned} T(r,F)+T(r,G) \quad &\leq \quad (5l(k+1)+5m+14)T(r,f)+ \\ &+(5l(k+1)+5m+14)T(r,g)+S(r,f)+S(r,g). \end{aligned}$$

Therefore

$$\begin{aligned} nT(r,f)+nT(r,g) &= T(r,f^n)+T(r,g^n)+O(1) \leq \\ &\leq T(r,F)+T(r,\frac{1}{af^m(f^{(k)})^l-b})+ \\ &\quad +T(r,G)+T(r,\frac{1}{af^m(g^{(k)})^l-b})+O(1) \leq \\ &\leq (5l(k+1)+5m+14)T(r,f)+ \\ &\quad +(5l(k+1)+5m+14)T(r,g)+ \\ &\quad +(l(k+1)+m)T(r,f)+(l(k+1)+m)T(r,g)+ \\ &\quad +S(r,f)+S(r,g), \end{aligned}$$

which implies

$$n(T(r,f)+T(r,g)) \leq (6l(k+1)+6m+14)(T(r,f)+T(r,g))+S(r,f)+S(r,g).$$

Thus

$$(n-6l(k+1)-6m-14)(T(r,f)+T(r,g)) \leq S(r,f)+S(r,g),$$

which contradicts to the assumption that $n \geq 6l(k+1)+6m+15$.

Case 2. $F=G$. Then

$$\frac{\phi_f-b}{\phi_g-b} = \frac{f^n}{g^n} = \frac{af^m(f^{(k)})^l-b}{ag^m(g^{(k)})^l-b}.$$

Case 3. $FG=1$. Then

$$\frac{\phi_f-b}{\phi_g-b} = \frac{f^n}{ag^m(g^{(k)})^l-b} = \frac{af^m(f^{(k)})^l-b}{g^n}.$$

This completes the proof of Theorem 1.3. ■

Proof. [Proof of Theorem 1.4]

Proceeding as the proof of Theorem 1.2, we have $af^m(f^{(k)})^l - b \not\equiv 0$ and $ag^m(g^{(k)})^l - b \not\equiv 0$. We define the functions F and G as in the proof of Theorem 1.1. Proceeding as in the proof of Theorem 1.1, we have F and G share 1 IM. Applying Lemma 2.10 to F and G, we have the following three cases:

Case 1.

$$\begin{aligned} T(r,F)+T(r,G) \quad \leq \quad & 2N_2(r,F)+3\overline{N}(r,F)+2N_2(r,G)+3\overline{N}(r,G)+ \\ & +2N_2(r,\frac{1}{F})+3\overline{N}(r,\frac{1}{F})+2N_2(r,\frac{1}{G})+3\overline{N}(r,\frac{1}{G})+ \\ & +S(r,F)+S(r,G). \end{aligned}$$

Proceeding as the proof of Theorem 1.2, we have

$$N_2(r,\frac{1}{F}) \leq 2T(r,f),$$

$$N_2(r,\frac{1}{G}) \leq 2T(r,g),$$

$$\overline{N}(r,\frac{1}{F}) \leq T(r,f),$$

$$\overline{N}(r,\frac{1}{G}) \leq T(r,g),$$

$$N_2(r,F) \leq (m+l)T(r,f)+S(r,f),$$

$$N_2(r,G) \leq (m+l)T(r,g)+S(r,g),$$

$$\overline{N}(r,F) \leq (m+l)T(r,f)+S(r,f),$$

$$\overline{N}(r,G) \leq (m+l)T(r,g)+S(r,g).$$

Hence

$$T(r,F)+T(r,G) \leq (5l+5m+7)T(r,f)+(5l+5m+7)T(r,g)+S(r,f)+S(r,g).$$

Therefore

$$\begin{aligned} nT(r,f)+nT(r,g) &= T(r,f^n)+T(r,g^n)+O(1)\le \\ &\le T(r,F)+T(r,\frac{1}{af^m(f^{(k)})^l-b})+T(r,G)+ \\ &\quad +T(r,\frac{1}{ag^m(g^{(k)})^l-b})+O(1)= \\ &= T(r,F)+T(r,f^m(f^{(k)})^l)+T(r,G)+ \\ &\quad +T(r,g^m(g^{(k)})^l)+O(1)\le \\ &\le T(r,F)+mT(r,f)+lT(r,(f^{(k)}))+T(r,G)+ \\ &\quad +mT(r,g)+lT(r,(g^{(k)}))+O(1)\le \\ &\le (5l+5m+7)T(r,f)+(5l+5m+7)T(r,g)+ \\ &\quad +(m+l)T(r,f)+(m+l)T(r,g)+S(r,f)+S(r,g). \end{aligned}$$

Thus

$$(n-6l-6m-7)(T(r,f)+T(r,g))\le S(r,f)+S(r,g),$$

which contradicts to the assumption that $n\ge 6l+6m+8$.

Case 2. $F=G$. Then

$$\frac{-f^n}{af^m(f^{(k)})^l-b}=\frac{-g^n}{af^m(g^{(k)})^l-b}.$$

Therefore (1.5) holds.

Case 3. $FG=1$. Then

$$\frac{\phi_f-b}{\phi_g-b}=\frac{f^n}{a(g^{(k)})^l-b}=\frac{a(f^{(k)})^l-b}{g^n}. \tag{3.18}$$

We will prove that (3.18) cannot occur. Since ϕ_f,ϕ_g share the value b IM and f,g are entire functions, we have

$$\overline{N}(r,\frac{1}{f})=\overline{N}(r,\frac{1}{g^m(g^{(k)})^l-\frac{b}{a}}),\overline{N}(r,\frac{1}{g})=\overline{N}(r,\frac{1}{f^m(f^{(k)})^l-\frac{b}{a}}). \tag{3.19}$$

We will show that $f^{(k)}\not\equiv 0$. Suppose for contradiction that $f^{(k)}\equiv 0$. This implies f is a non-constant polynomial. Combining this and (3.19), we have fg is non-constant and $g^{(k)}\not\equiv 0$. From (3.18), we have

$$(fg)^n=-b(ag^m(g^{(k)})^l-b). \tag{3.20}$$

From this and Lemma 2.6, we have

$$\begin{aligned} nT(r,fg) &\le \overline{N}(r,\frac{1}{(fg)^n})+\overline{N}(r,\frac{1}{g^m(g^{(k)})^l})+S(r,fg)\le \\ &\le N(r,\frac{1}{fg})+\overline{N}(r,\frac{1}{g^m})+\overline{N}(r,\frac{1}{g^{(k)}})+S(r,fg)\le \\ &\le T(r,fg)+T(r,g)+T(r,g^{(k)})+S(r,fg), \end{aligned}$$

which implies

$$(n-1)T(r,fg)\le 2T(r,g)+S(r,fg)+S(r,g). \tag{3.21}$$

On the other hand, by (3.20) we have

$$\begin{aligned} mT(r,g) &= T(r,g^m)+O(1)\le \\ &\le T(r,g^m(g^{(k)})^l)+T(r,\frac{1}{(g^{(k)})^l})+O(1)\le \\ &\le \overline{N}(r,\frac{1}{(fg)^n})+\overline{N}(r,\frac{1}{g^m(g^{(k)})^l})+lT(r,g)+S(r,g)\le \\ &\le T(r,fg)+2T(r,g)+lT(r,g)+S(r,g), \end{aligned}$$

which yields

$$(m-l-2)T(r,g)\le T(r,fg)+S(r,g). \tag{3.22}$$

From (3.21) and (3.22), we have

$$(n-1)T(r,fg)+(m-l-3)T(r,g)\le S(r,fg)+S(r,g),$$

which contradicts to the assumptions that $n\ge 6l+6m+5$ and $m\ge l+4$. Hence $f^{(k)}\not\equiv 0$ and similarly we have $g^{(k)}\not\equiv 0$.

By Lemma 2.8, we have

$$\begin{aligned} T(r,f^m(f^{(k)})^l) &\le N(r,\frac{1}{f})+\overline{N}(r,\frac{1}{f^m(f^{(k)})^l-\frac{b}{a}})+T(r,f^{(k)})+S(r,f)= \\ &= N(r,\frac{1}{f})+\overline{N}(r,\frac{1}{g})+T(r,f^{(k)})+S(r,f)\le \\ &\le 2T(r,f)+T(r,g)+S(r,f). \end{aligned}$$

Hence

$$\begin{aligned} mT(r,f) &= T(r,f^m)+O(1)\le \\ &\le T(r,f^m(f^{(k)})^l)+T(r,\frac{1}{(f^{(k)})^l})+O(1)\le \\ &\le 2T(r,f)+T(r,g)+T(r,(f^{(k)})^l)+S(r,f)\le \\ &\le 2T(r,f)+T(r,g)+lT(r,f)+S(r,f)+S(r,g), \end{aligned}$$

which implies

$$mT(r,f) \leq (l+2)T(r,f) + T(r,g) + S(r,f) + S(r,g).$$

Similarly, we have

$$mT(r,g) \leq (l+2)T(r,g) + T(r,f) + S(r,f) + S(r,g).$$

Hence

$$(m-l-3)(T(r,f)+T(r,g)) \leq S(r,f) + S(r,g),$$

which contradicts to the assumption that $m \geq l+4$.

This proves Theorem 1.4. ■

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