DIFFERENTIAL POLYNOMIALS AND VALUE-SHARING

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Dedicated to Professor Ha Huy Khoai
on the occasion of his 70-th birthday

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Abstract. In this paper, we give some theorems on uniqueness problem of differential polynomials of meromorphic functions. Let $a, b$ be non-zero constants and let $n, m, l, k$ be positive integers satisfying $n \geq 3l(k + 1) + 3m + 9$ and $m \geq l(k + 1) + 1$. If $f^n + af'(f^k)^l$ and $g^n + ag'(g^k)^l$ share the value $b$ CM, then $f$ and $g$ are closely related. We also consider the case sharing the value IM.

1. Introduction and main results

Let $\mathbb{C}$ denote the complex plane and $f(z)$ be a non-constant meromorphic function in $\mathbb{C}$. It is assumed that the reader is familiar with the standard notion used in Nevanlinna value distribution theory such as $T(r, f), m(r, f), N(r, f), \ldots$ (see [9, 24]), and $S(r, f)$ denotes any quantity that satisfies the condition $S(r, f) = o(T(r, f))$ as $r \rightarrow \infty$, outside of a possible exceptional set of finite linear measure.

In 1959, Hayman considered the problem which was motivated by Picard exceptional values and proved the following result in [10].

Theorem A (Hayman’s Theorem). For all $z \in \mathbb{C}$, each complex meromorphic function $f$ satisfying

$$f^n(z) + af'(z) \neq b$$

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is constant if \( n \geq 5 \) and \( a, b \in \mathbb{C}, a \neq 0 \). However, if \( f \) is entire, this holds also for \( n \geq 3 \) and for \( n = 2, b = 0 \).

As a consequence, if \( n \geq 3 \) then \( f^n(z)f'(z) \) assumes all finite values except possibly zero and infinitely often unless \( f \) is a rational function. When \( f \) is an entire function, the remain case is only \( n = 1 \), which was proved later by Cluine in [4]. In 1982, Döringer has shown, Hayman’s theorem remains valid for \( f^n + af^m(f^{(k)}) \) instead of \( f^n(z) + af'(z) \) provided that \( n \geq 3 + (k+1)l + m \) in [5]. These results are related to the value sharing problem of meromorphic functions and their derivatives. Let us first recall some basic definitions.

For \( f \) be a non-constant meromorphic function and \( S \subset \mathbb{C} \cup \{\infty\} \), we define
\[
E_f(S) = \bigcup_{a \in S} \{(z, m) \mid f(z) = a \text{ with multiplicity } m\},
\]
\[
E_f^{-1}(S) = f^{-1}(S) = \bigcup_{a \in S} \{z \mid f(z) = a\}.
\]

Let \( F \) be a non-empty set of meromorphic functions. Two functions \( f \) and \( g \) of \( F \) are said to share \( S \), counting multiplicity (share \( S \) CM), if \( E_f(S) = E_g(S) \). Similarly, two functions \( f \) and \( g \) are said to share \( S \), ignoring multiplicity (share \( S \) IM), if \( E(S) = E_g(S) \).

In 1997, Yang-Hua studied the unicity problem for meromorphic functions and the differential monomials of the form \( u^n u' \), when they share only one value, and obtained the following result in [22].

**Theorem B.** Let \( f \) and \( g \) be two non-constant meromorphic functions, \( n \geq 11 \) be an integer and \( a \in \mathbb{C} \setminus \{0\} \). If \( f^n f' \) and \( g^n g' \) share the value \( a \) CM, then either \( f = dg \) for some \((n+1)\)-th root of unity \( d \) or \( g(z) = c_1 e^{cz} \) and \( f(z) = c_2 e^{-cz} \) where \( c, c_1 \) and \( c_2 \) are constants and satisfy \((c_1 c_2)^{n+1} c^2 = -a^2\).

Since then, several authors study the uniqueness of meromorphic functions by considering differential polynomials like \((u^n)^{(k)}\), \(u^n (u-1) u'\), \(u^n (u-1)^2 u'\), ... (see [6, 7, 15, 16, 17, 18]).

In 2011, Grahl-Nevo studied the unicity problem for meromorphic functions and the differential polynomial of the form \( u^n + au^{(k)} \) and obtained the following theorems in [8].

**Theorem C.** Let \( f \) and \( g \) be non-constant meromorphic functions on \( \mathbb{C}, a, b \in \mathbb{C} \setminus \{0\} \) and let \( n \) and \( k \) be positive integers satisfying \( n \geq 5k + 17 \). Assume that the functions
\[
\psi_f := f^n + af^{(k)} \text{ and } \psi_g := g^n + ag^{(k)}
\]
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share the value b CM. Then

\[(1.2) \quad \frac{\psi_f - b}{\psi_g - b} = \frac{f^n}{g^n} = \frac{af^{(k)} - b}{ag^{(k)} - b}\]

or

\[(1.3) \quad \frac{\psi_f - b}{\psi_g - b} = \frac{f^n}{ag^{(k)} - b} = \frac{af^{(k)} - b}{g^n}\]

or

\[(1.4) \quad f = g, f^{(k)} = g^{(k)} \equiv b/a.\]

Theorem D. Let f and g be two non-constant entire functions on \( \mathbb{C} \), \( a, b \in \mathbb{C} \setminus \{0\} \) and let \( n, k \) be positive integers satisfying \( n \geq 11 \) and \( n \geq k + 2 \). Assume that the functions \( \psi_f \) and \( \psi_g \) defined as in (1.1) share the value b CM. Then (1.2) or (1.4) holds.

In 2014, Zhang-Yang added an assumption that "the b-point of \( \psi_f \) are not the zeros of \( f \) and \( g \)” and proved the following theorems in [27].

Theorem E. Let f and g be two non-constant meromorphic functions on \( \mathbb{C} \), \( a, b \in \mathbb{C} \setminus \{0\} \) and let \( n, k \) be positive integers satisfying \( n \geq 3k + 12 \). Assume that \( \psi_f \) and \( \psi_g \) defined as in (1.1) share the value b CM and the b-point of \( \psi_f \) are not the zeros of \( f \) and \( g \). Then (1.2) or (1.3) holds.

Theorem F. Let f and g be two non-constant entire functions on \( \mathbb{C} \), \( a, b \in \mathbb{C} \setminus \{0\} \) and let \( n, k \) be positive integers satisfying \( n \geq 8 \). Assume that \( \psi_f \) and \( \psi_g \) defined as in (1.1) share the value b CM and the b-point of \( \psi_f \) are not the zeros of \( f \) and \( g \). Then (1.2) holds.

In this paper, we study the unicity problem for \( f^n + af^m(f^{(k)})^l \), where \( n, m, k, l \geq 1 \), which is related to this kind of differential polynomial. Namely, we prove the following theorems.

Theorem 1.1. Let f and g be non-constant meromorphic functions on \( \mathbb{C} \), \( a, b \in \mathbb{C} \setminus \{0\} \) and let \( n, m, k, l \) be positive integers satisfying \( n \geq 3(l(k+1)+3m+9) \) and \( m \geq l(k+1)+1 \). Assume that the functions \( \phi_f := f^n + af^m(f^{(k)})^l \) and \( \phi_g := g^n + ag^m(g^{(k)})^l \) share the value b CM. Then

\[(1.5) \quad \frac{\phi_f - b}{\phi_g - b} = \frac{f^n}{g^n} = \frac{af^m(f^{(k)})^l - b}{ag^m(g^{(k)})^l - b}\]

or

\[(1.6) \quad \frac{\phi_f - b}{\phi_g - b} = \frac{f^n}{ag^m(g^{(k)})^l - b} = \frac{af^m(f^{(k)})^l - b}{g^n}.\]
**Theorem 1.2.** Let \( \phi_f \) and \( \phi_g \) be given as in Theorem 1.1, where \( f \) and \( g \) be non-constant entire functions. Assume that \( \phi_f \) and \( \phi_g \) share the value \( b \) CM. If \( n \geq 3l + 3m + 5 \) and \( m \geq l + 3 \), then (1.5) holds.

**Theorem 1.3.** Let \( \phi_f \) and \( \phi_g \) be given as in Theorem 1.1. Assume that \( \phi_f \) and \( \phi_g \) share the value \( b \) IM. If \( n \geq 6l(k+1) + 6m + 15 \) and \( m \geq l(k+1) + 1 \), then (1.5) or (1.6) holds.

**Theorem 1.4.** Let \( \phi_f \) and \( \phi_g \) be given as in Theorem 1.1, where \( f \) and \( g \) be non-constant entire functions. Assume that \( \phi_f \) and \( \phi_g \) share the value \( b \) IM. If \( n \geq 6l + 6m + 8 \) and \( m \geq l + 4 \), then (1.5) holds.

2. Some basic lemmas

Let us recall a few classical lemmas.

**Lemma 2.1.** [9] Let \( f, g \) be non-constant meromorphic functions on \( \mathbb{C} \), \( a \in \mathbb{C} \). Then

\[
T(r, f + g) \leq T(r, f) + T(r, g) + O(1),
\]

\[
T(r, fg) \leq T(r, f) + T(r, g) + O(1),
\]

\[
T(r, f - a) = T(r, f) + O(1),
\]

\[
T(r, \frac{1}{f}) = T(r, f) + O(1).
\]

**Lemma 2.2.** [9] Let \( f \) be a non-constant meromorphic function on \( \mathbb{C} \) and let \( P(z) \in \mathbb{C}[x] \) be a polynomial of degree \( q \). Then

\[
T(r, P(z)) = qT(r, f) + O(1).
\]

**Lemma 2.3.** [9] Let \( f \) be a non-constant meromorphic function on \( \mathbb{C} \). Then for any positive integer \( k \), we have

\[
T(r, f^{(k)}) \leq T(r, f) + kN(r, f) + S(r, f) \leq (k + 1)T(r, f) + S(r, f).
\]

Moreover, if \( f \) be a non-constant entire function, then

\[
T(r, f^{(k)}) \leq T(r, f) + S(r, f).
\]

**Lemma 2.4** (Lemma of Logarithmic Derivative). [9] Let \( f \) be a non-constant meromorphic function on \( \mathbb{C} \). Then for any positive integer \( k \), we have

\[
m(r, \frac{f^{(k)}}{f}) = S(r, f).
\]
Lemma 2.5 (First Main Theorem). [9] Let \( f \) be a non-constant meromorphic function on \( \mathbb{C} \). Then for \( a \in \mathbb{C} \), we have

\[
T(r, \frac{1}{f - a}) = T(r, f) + O(1).
\]

Lemma 2.6 (Second Main Theorem). [9] Let \( a_1, \ldots, a_n \in \mathbb{C} \) with \( n \geq 2, n \in \mathbb{N} \), and let \( f \) be a non-constant meromorphic function on \( \mathbb{C} \). Then for \( r > 0 \), we have

\[
(n - 1)T(r, f) \leq N(r, f) + \sum_{i=1}^{n} N(r, \frac{1}{f - a_i}) + S(r, f).
\]

Suppose that \( f_1, \ldots, f_l \) be meromorphic functions on \( \mathbb{C} \). Let \( n_{ij} (0 \leq i \leq l, 1 \leq j \leq k_i) \) be non-negative integers. We denote by

\[
M[f_1, \ldots, f_l] = f_1^{a_{11}}(f'_1)^{n_{11}} \cdots f_l^{a_{l1}}(f'_l)^{n_{l1}}
\]

the differential monomial in \( f_1, \ldots, f_l \).

Let \( f_1, \ldots, f_l \) be meromorphic functions on \( \mathbb{C} \), \( M_1[f_1, \ldots, f_l], \ldots, M_k[f_1, \ldots, f_l] \) be differential monomials in \( f_1, \ldots, f_l \) and \( a_1, \ldots, a_k \in \mathbb{C} \setminus \{0\} \). The summation

\[
P[f_1, \ldots, f_l] = a_1M_1[f_1, \ldots, f_l] + \cdots + a_kM_k[f_1, \ldots, f_l]
\]

is said to be a differential polynomial in \( f_1, \ldots, f_l \).

Lemma 2.7. Let \( f \) be a meromorphic function on \( \mathbb{C} \). Suppose that \( f = \frac{f_1}{f_2} \), where \( f_1 \) and \( f_2 \) be entire functions that have no common zeros and let \( k \) be a positive integer number. Then there exists a differential polynomial \( \omega_k[f_1, f_2] \) in \( f_1, f_2 \) such that

\[
f^{(k)} = \frac{\omega_k[f_1, f_2]}{f_2^{k+1}}.
\]

Proof. We prove by induction. With \( k = 1 \), we have

\[
f' = f'_1f_2 - f'_2f_1 = \frac{\omega_1[f_1, f_2]}{f_2^2}.
\]

Assume

\[
f^{(k)} = \frac{\omega_k[f_1, f_2]}{f_2^{k+1}}.
\]
We have
\[
J_{k+1} = \frac{f_{k+1}}{f_{k}} = \frac{\alpha_k[f_1, f_2] - (k+1)\beta f_{k+1} \alpha_k[f_1, f_2]}{\beta f_{k+2}} = \frac{\alpha_{k+1}[f_1, f_2]}{f_{k+2}}.
\]
This completes the proof of Lemma 2.7.

Lemma 2.8. Let \( f \) be an entire function on \( \mathbb{C} \), \( a, b \in \mathbb{C} \setminus \{0\} \) and \( m, l, k \) be positive integers. Suppose that \( f^m(f^{(k)})^l \) is a non-constant function. Then we have
\[
T\left(r, f^m(f^{(k)})^l\right) \leq N\left(r, \frac{1}{f} \right) + N\left(r, \frac{1}{af^m(f^{(k)})^l - b} \right) + T\left(r, f^{(k)}\right) + S(r, f).
\]

Proof. By Lemma 2.6 and the assumption that \( f \) is a non-constant entire function, we have
\[
T(r, f^m(f^{(k)})^l) \leq N(r, f^m(f^{(k)})^l) + N(r, f^m(f^{(k)})^l - \frac{b}{a}) + S(r, f) \leq N(r, f^m(f^{(k)})^l) + S(r, f) \leq N\left(r, f^{(k)}\right) + S(r, f),
\]
which implies,
\[
T(r, f^m(f^{(k)})^l) \leq N(r, f^{(k)}) + S(r, f) + T(r, f^{(k)}) + S(r, f).
\]
Lemma 2.8 is proved.

Lemma 2.9 ([22], Lemma 3). Let \( f \) and \( g \) be non-constant meromorphic functions on \( \mathbb{C} \). If \( f \) and \( g \) share \( 1 \) CM, then one of the following cases holds:
1) \( T(r, f) + T(r, g) \leq 2\{N_2(r, f) + N_2(r, g) + N_2(r, f^{1/2}) + N_2(r, g^{1/2})\} + S(r, f) + S(r, g); \)
2) \( f \equiv g; \)
3) \( fg \equiv 1. \)
**Lemma 2.10** ([24], Theorem 1). Let \( f \) and \( g \) be non-constant meromorphic functions on \( \mathbb{C} \). If \( f \) and \( g \) share 1 IM, then one of the following three cases holds:

1) \( T(r,f) + T(r,g) \leq 2N_2(r,f) + 3N(r,f) + 2N_2(r,g) + 3N(r,g) + 2N_2(r,\frac{1}{f}) + 3N(r,\frac{1}{f}) + 2N_2(r,\frac{1}{g}) + 3N(r,\frac{1}{g}) + S(r,f) + S(r,g); \)
2) \( f \equiv g; \)
3) \( fg \equiv 1. \)

3. **Proof of the Theorems**

**Proof.** [Proof of Theorem 1.1]

We claim that \( af^m(f^{(k)})^l - b \not\equiv 0 \). Suppose that \( af^m(f^{(k)})^l - b \equiv 0 \), we have \( f^{(k)} \not\equiv 0 \) and

\[
mT(r,f) = T(r,f^m) + O(1) = lT(r,f^{(k)}) + O(1) \leq (k+1)T(r,f) + S(r,f).
\]

Hence

\[
(m -(k+1))T(r,f) \leq S(r,f),
\]

which contradicts the assumption that \( m \geq l(k+1) + 1 \). Similarly, we have \( ag^m(g^{(k)})^l - b \not\equiv 0 \).

Setting

\[
F = \frac{-f^n}{af^m(f^{(k)})^l - b}, \quad G = \frac{-g^n}{ag^m(g^{(k)})^l - b}.
\]

By Lemma 2.1 and Lemma 2.3, we have

\[
nT(r,f) = T(r,-f^n) + O(1) \leq T(r,\frac{-f^n}{af^m(f^{(k)})^l - b}) + T(r,af^m(f^{(k)})^l - b) + O(1) \leq T(r,F) + mT(r,f) + lT(r,f^{(k)}) + O(1) \leq T(r,F) + (m + l(k+1))T(r,f) + S(r,f) \leq T(r,F) + (m + l(k+1))T(r,f) + S(r,f).
\]

Hence

\[
(n - m - l(k+1))T(r,f) \leq T(r,F) + S(r,f).
\]
From this and the assumption that \( n \geq 3l(k+1) + 3m + 9 \), we have \( F \) is non-constant. Similarly, we have \( G \) is non-constant.

Suppose that \( f = \frac{f_1}{f_2} \), where \( f_1, f_2 \) are entire functions which have no common zeros and \( g = \frac{g_1}{g_2} \), where \( g_1, g_2 \) are entire functions which have no common zeros. By Lemma 2.7, there exists a differential polynomial \( \omega [f_1, f_2] \) such that \( f^{(k)} = \frac{\omega [f_1, f_2]}{f_2^{l+1}} \) and \( g^{(k)} = \frac{\omega [g_1, g_2]}{g_2^{m+1}} \). So

\[
\phi_f - b = \frac{f_1^n + (af_1^n(\omega_k[f_1, f_2])^l - bf_2^{m+l(k+1)})f_2^{n-m-l(k+1)}}{f_2^n}
\]

and

\[
\phi_g - b = \frac{g_1^n + (ag_1^n(\omega_k[g_1, g_2])^l - bg_2^{m+l(k+1)})g_2^{n-m-l(k+1)}}{g_2^n}.
\]

In the following we prove that the functions

\[
f_1^n + (af_1^n(\omega_k[f_1, f_2])^l - bf_2^{m+l(k+1)})f_2^{n-m-l(k+1)}
\]

and \( f_2 \) have no common zeros. Suppose that there exists a constant \( \gamma \) such that

\[
(f_1(\gamma))^n + \left(a(f_1(\gamma))^m(\omega_k[f_1, f_2](\gamma))^l - b(f_2(\gamma))^{m+l(k+1)}\right)(f_2(\gamma))^{n-m-l(k+1)} = 0
\]

and

\[
f_2(\gamma) = 0.
\]

This implies \( f_1(\gamma) = 0 \) and \( f_2(\gamma) = 0 \), which contradicts to the assumption that \( f_1 \) and \( f_2 \) have no common zeros. Hence \( f_1^n + (af_1^n(\omega_k[f_1, f_2])^l - bf_2^{m+l(k+1)})f_2^{n-m-l(k+1)} \) and \( f_2 \) have no common zeros. Therefore

\[
E_{\phi_f}(b) = E_{f_1^n + (af_1^n(\omega_k[f_1, f_2])^l - bf_2^{m+l(k+1)})f_2^{n-m-l(k+1)}(0)}.
\]

Similarly, we have

\[
E_{\phi_g}(b) = E_{g_1^n + (ag_1^n(\omega_k[g_1, g_2])^l - bg_2^{m+l(k+1)})g_2^{n-m-l(k+1)}(0)}.
\]

On the other hand, we have

\[
F = \frac{f_1^n}{-(af_1^n(\omega_k(f_1, f_2))^l - bf_2^{m+l(k+1)})f_2^{n-m-l(k+1)}}.
\]

Hence

\[
F - 1 = \frac{f_1^n + (af_1^n(\omega_k(f_1, f_2))^l - bf_2^{m+l(k+1)})f_2^{n-m-l(k+1)}}{-(af_1^n(\omega_k(f_1, f_2))^l - bf_2^{m+l(k+1)})f_2^{n-m-l(k+1)}}.
\]
We will show that the functions $f_1^n + (af_1^n(\omega_k(f_1, f_2))^l - bf_2'^{m+l(k+1)})f_2'^{n-m-l(k+1)}$ and $bf_2'^{n-m-l(k+1)}$ have no common zeros. Suppose that there exists a constant $\alpha \in \mathbb{C}$ such that

$$
\begin{align*}
(f_1(\alpha))^n + (af_1(\alpha))^m(\omega_k(f_1, f_2)(\alpha))^l - b(f_2(\alpha))^{m+l(k+1)}), \\
(f_1(\alpha))^n - b(f_2(\alpha))^{m+l(k+1)} = 0.
\end{align*}
$$

From this and the assumption that $m \geq 1$, we have $f_1(\alpha) = f_2(\alpha) = 0$. This contradicts to the assumption that $f_1, f_2$ have no common zeros. Therefore $f_1^n + (af_1^n(\omega_k(f_1, f_2))^l - bf_2'^{m+l(k+1)})f_2'^{n-m-l(k+1)}$ and $bf_2'^{n-m-l(k+1)}$ have no common zeros. Combining this with the previous fact that $f_1^n + (af_1^n(\omega_k(f_1, f_2))^l - bf_2'^{m+l(k+1)})f_2'^{n-m-l(k+1)}$ and $f_2'^{n-m-l(k+1)}$ have no common zeros. So we have

$$
(3.5) \quad EF(1) = E_{f_1^n + (af_1^n(\omega_k(f_1, f_2))^l - bf_2'^{m+l(k+1)})f_2'^{n-m-l(k+1)}}(0).
$$

Similarly, we get

$$
(3.6) \quad EG(1) = E_{g_1^n + (af_1^n(\omega_k(g_1, g_2))^l - bf_2'^{m+l(k+1)})f_2'^{n-m-l(k+1)}}(0).
$$

From (3.2) to (3.6) and the assumption that $E_{\phi_f}(b) = E_{\phi_g}(b)$, we have

$$
EF(1) = EG(1).
$$

Applying Lemma 2.9 to $F$ and $G$ with the following cases:

**Case 1.**

$$
(3.7) \quad T(r, F) + T(r, G) \leq 2\{N_2(r, F) + N_2(r, \frac{1}{F}) + N_2(r, G) + N_2(r, \frac{1}{G})\} + S(r, F) + S(r, G).
$$

By (3.4), we obtain

$$
N_2(r, \frac{1}{F}) \leq 2N(r, \frac{1}{F}).
$$

Hence

$$
(3.8) \quad N_2(r, \frac{1}{F}) \leq 2T(r, f).
$$

Similarly, we get

$$
(3.9) \quad N_2(r, \frac{1}{G}) \leq 2T(r, g).
$$
On the other hand, we have

\[
N_2(r, F) = N_2(r, \frac{-f^n}{a f^m(f^{(k)}l - b)}) \leq \\
\leq N_2(r, \frac{1}{a f^m(f^{(k)}l - b)}) + N_2(r, f^n) \leq \\
\leq N_2(r, \frac{1}{a f^m(f^{(k)}l - b)}) + 2N(r, f) \leq \\
\leq T(r, f^m(f^{(k)}l)^l + 2T(r, f) \leq \\
\leq mT(r, f) + lT(r, f^{(k)}) + 2T(r, f) + O(1) \leq \\
\leq mT(r, f) + l(k + 1)T(r, f) + 2T(r, f) + S(r, f),
\]

which implies

\[
N_2(r, F) \leq (m + 2 + l(k + 1))T(r, f) + S(r, f).
\]

Similarly, we get

\[
N_2(r, G) \leq (m + 2 + l(k + 1))T(r, g) + S(r, g).
\]

From (3.7) to (3.11), we have

\[
T(r, F) + T(r, G) \leq 2\{(l(k + 1) + m + 4)T(r, f) + (l(k + 1) + m + 4)T(r, g)\} + \\
+ S(r, f) + S(r, g).
\]

Therefore

\[
nT(r, f) + nT(r, g) = T(r, f^n) + T(r, g^n) + O(1) \leq \\
\leq T(r, F) + T(r, \frac{1}{a f^m(f^{(k)}l - b)}) + O(1) + \\
+ T(r, G) + T(r, \frac{1}{a g^m(g^{(k)}l - b)}) + O(1) = \\
= T(r, F) + T(r, f^m(f^{(k)}l)^l) + O(1) + \\
+ T(r, G) + T(r, g^m(g^{(k)}l)^l) + O(1) \leq \\
\leq T(r, F) + T(r, f^m) + T(r, f^{(k)}l) + O(1) + \\
+ T(r, G) + T(r, g^m) + T(r, g^{(k)}l) + O(1) \leq \\
\leq m(k + 1) + l + 4 + m)T(r, f) + \\
+ (m + l(k + 1) + 4 + m)T(r, g) + \\
S(r, f) + S(r, g).
\]
which implies
\[ n(T(r, f) + T(r, g)) \leq (3l(k + 1) + 3m + 8)(T(r, f) + T(r, g)) + S(r, f) + S(r, g). \]

Thus
\[ (n - 3l(k + 1) - 3m - 8)(T(r, f) + T(r, g)) \leq S(r, f) + S(r, g), \]

which contradicts to the assumption that \( n \geq 3l(k + 1) + 3m + 9. \)

**Case 2.** \( F = G. \) Then
\[
-\frac{fn}{a f^m(f(k)^l - b} = -\frac{gn}{a g^m(g(k)^l - b}.
\]

Therefore
\[
\frac{\phi_f - b}{\phi_g - b} = \frac{fn}{g^n} = \frac{a f^m(f(k)^l - b}{a g^m(g(k)^l - b}.
\]

**Case 3.** \( FG = 1. \) Thus
\[
\frac{\phi_f - b}{\phi_g - b} = \frac{fn}{g^n} = \frac{a f^m(f(k)^l - b}{a g^m(g(k)^l - b}.
\]

This completes the proof of Theorem 1.1.

**Proof.** [Proof of Theorem 1.2]

We claim that \( a f^m(f(k)^l - b \not\equiv 0. \) If \( a f^m(f(k)^l - b \equiv 0, \) we have \( f(k) \equiv 0 \) and
\[
mT(r, f) = T(r, f^m) + O(1) = T(r, (f(k)^l) + O(1) = lT(r, f(k) + O(1) \leq lT(r, f) + S(r, f),
\]

which implies
\[
(m - l)T(r, f) \leq S(r, f),
\]

which contradicts to the assumption that \( m \geq l + 3. \) Similarly, we have \( a g^m(g(k)^l - b \neq 0. \)

We define the functions \( F \) and \( G \) as in the proof of Theorem 1.1. Proceeding as in the proof of Theorem 1.1, we can obtain that \( F \) and \( G \) share 1 CM. Applying Lemma 2.9 to \( F \) and \( G, \) we have the following three cases:

**Case 1.**
\[
T(r, F) + T(r, G) \leq 2\{N_2(r, F) + N_2(r, \frac{1}{F}) + N_2(r, G) + N_2(r, \frac{1}{G})\} + S(r, F) + S(r, G).
\]
We have

\[ N_2(r, \frac{1}{F}) \leq 2T(r, f), \]
\[ N_2(r, \frac{1}{G}) \leq 2T(r, g). \]

By Lemma 2.3, we get

\[ N_2(r, F) = N_2(r, \frac{-f^m}{a f^m(f^{(k)})^l - b}) \leq \]
\[ \leq N_2(r, \frac{1}{a f^m(f^{(k)})^l - b}) \leq \]
\[ \leq T(r, \frac{f^m(f^{(k)})^l}{1}) \leq \]
\[ \leq mT(r, f) + lT(r, f^{(k)}) + O(1), \]

which implies

\[ N_2(r, \frac{1}{F}) \leq (m + l)T(r, f) + S(r, f). \]

Similarly, we have

\[ N_2(r, \frac{1}{G}) \leq (m + l)T(r, g) + S(r, g). \]

Hence

\[ T(r, F) + T(r, G) \leq 2\{(m + l + 2)T(r, f) + (m + l + 2)T(r, g)\} + S(r, f) + S(r, g). \]

Therefore

\[ nT(r, f) + nT(r, f) = T(r, f^n) + T(r, f^n) + O(1) \leq \]
\[ \leq T(r, F) + T(r, \frac{1}{a f^m(f^{(k)})^l - b}) + T(r, G) + \]
\[ + T(r, \frac{1}{a f^m(g^{(k)})^l - b}) + O(1) = \]
\[ = T(r, F) + T(r, \frac{f^m(f^{(k)})^l}{1}) + T(r, G) + \]
\[ + T(r, \frac{1}{f^m(g^{(k)})^l}) + O(1) = \]
\[ = T(r, F) + T(r, f^m(f^{(k)})^l) + T(r, G) + \]
\[ + T(r, f^m(g^{(k)})^l) + O(1) \leq \]
\[ \leq T(r, F) + T(r, f) + T(r, f^{(k)}) + T(r, G) + \]
\[ + T(r, g^m) + T(r, g^{(k)}) + O(1) \leq \]
\[ \leq 2\{(m + l + 2)T(r, f) + (m + l + 2)T(r, g)\} + \]
\[ + (m + l)T(r, f) + (m + l)T(r, g) + S(r, f) + S(r, g), \]
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which implies

\[ n(T(r,f) + T(r,g)) \leq (3l + 3m + 4)(T(r,f) + T(r,g)) + S(r,f) + S(r,g). \]

Thus

\[ (n - 3l - 3m - 4)(T(r,f) + T(r,g)) \leq S(r,f) + S(r,g), \]

which contradicts to the assumption that \( n \geq 3l + 3m + 5 \).

**Case 2.** \( F = G \). Then

\[
\frac{-f^n}{af^m(f(k))^{l} - b} = \frac{-g^n}{ag^m(g(k))^{l} - b}.
\]

Therefore (1.5) holds.

**Case 3.** \( FG = 1 \). Then

\[
(3.12) \quad \frac{\phi_f - b}{\phi_g - b} = \frac{f^n}{ag^m(g(k))^{l} - b} = \frac{a f^m(f(k))^{l} - b}{g^n}.
\]

We will prove that (3.12) cannot occur. Since \( \phi_f, \phi_g \) share the value \( b \) CM and \( f, g \) are entire functions, \( \frac{\phi_f - b}{\phi_g - b} \) has no zero or pole at all. From this and (3.12), we have

\[
(3.13) \quad N(r, \frac{1}{f}) = \frac{1}{n} N(r, \frac{1}{g^m(g(k))^{l} - \frac{b}{a}}), \quad N(r, \frac{1}{g}) = \frac{1}{n} N(r, \frac{1}{f^m(f(k))^{l} - \frac{b}{a}}),
\]

\[
(3.14) \quad N(r, \frac{1}{f^m(f(k))^{l} - \frac{b}{a}}), \quad N(r, \frac{1}{g}) = N(r, \frac{1}{g^m(g(k))^{l} - \frac{b}{a}}).
\]

We will show that \( f^{(k)} \not\equiv 0 \). Suppose for contradiction that \( f^{(k)} \equiv 0 \). Then \( f \) is a non-constant polynomial. Combining this and (3.13), we have \( g \) has no zero at all. This implies \( fg \) is a non-constant function and \( g^{(k)} \not\equiv 0 \). From (3.12), we have

\[
(3.15) \quad (fg)^n = -b(ag^m(g^{(k)})^{l} - b).
\]

From this and Lemma 2.6, we have

\[
nT(r, fg) \leq N(r, \frac{1}{(fg)^{n}}) + N(r, \frac{1}{g^m(g^{(k)})^{l}}) + S(r, fg) \leq N(r, \frac{1}{fg}) + N(r, \frac{1}{g^{(k)}}) + S(r, fg) \leq T(r, fg) + T(r, g^{(k)}) + S(r, fg),
\]
which implies

\[(3.16) \quad (n - 1)T(r, fg) \leq T(r, g) + S(r, f) + S(r, g).\]

On the other hand, by Lemma 2.6 and (3.15), we have

\[
mT(r, g) = T(r, g^n) + O(1) \leq T(r, g^n(g^{(k)})^l) + O(1) \leq N(r, \frac{1}{g^n}) + IT(r, g) + S(r, g) \leq T(r, f) + T(r, g) + IT(r, g) + S(r, g),
\]

which yields

\[(3.17) \quad (m - l - 1)T(r, g) \leq T(r, f) + S(r, g).\]

From (3.16) and (3.17), we have

\[(n - 1)T(r, f) + (m - l - 2)T(r, g) \leq S(r, f) + S(r, g),\]

which contradicts to the assumptions that \(n \geq 3l + 3m + 5\) and \(m \geq l + 3\).

Hence \(f^{(k)} \not\equiv 0\) and similarly, we have \(g^{(k)} \not\equiv 0\).

By Lemma 2.8, we have

\[
mT(r, f) = T(r, f^m) + O(1) \leq T(r, f^m(f^{(k)})^l) + O(1) \leq N(r, \frac{1}{f}) + IT(r, f^{(k)}) + (l + 1)T(r, f) + S(r, f),
\]

which implies

\[(m - l - 1)T(r, f) \leq N(r, \frac{1}{f}) + IT(r, f^{(k)}) + (l + 1)T(r, f) + S(r, f).\]
From this and (3.13), (3.14), we have

\[(m - l - 1)T(r, f) \leq \frac{1}{n} N(r, \frac{1}{ag^m(g^{(k)})^{l} - b}) + \frac{1}{n} N(r, \frac{1}{a} - \frac{1}{g}) + S(r, f) \leq \frac{1}{n} N(r, g) + S(r, f) = \]

\[= \frac{1}{n} N(r, \frac{1}{ag^m(g^{(k)})^{l} - b}) + \frac{1}{n} N(r, \frac{1}{af^m(f^{(k)})^{l} - b}) + S(r, f) \leq \frac{1}{n} T(r, ag^m(g^{(k)})^{l} - b) + \frac{1}{n} T(r, af^m(f^{(k)})^{l} - b) + S(r, f) + S(r, g) = \]

\[= \frac{1}{n} T(r, g^m(g^{(k)})^{l}) + \frac{1}{n} T(r, f^m(f^{(k)})^{l}) + S(r, f) + S(r, g) \leq \]

\[\leq \frac{1}{n} \left( mT(r, g) + lT(r, g^{(k)}) \right) + \frac{1}{n} \left( mT(r, f) + lT(r, f^{(k)}) \right) + S(r, g) \leq \]

\[\leq \frac{1}{n} \left( mT(r, g) + lT(r, g) \right) + \frac{1}{n} \left( mT(r, f) + lT(r, f) \right) + S(r, f) + S(r, g), \]

which implies

\[(m - l - 1)T(r, f) \leq \frac{m + l}{n} T(r, g) + \frac{m + l}{n} T(r, f) + S(r, f) + S(r, g). \]

Similarly, we get

\[(m - l - 1)T(r, g) \leq \frac{m + l}{n} T(r, f) + \frac{m + l}{n} T(r, g) + S(r, g). \]

Hence

\[(m - l - 1)(T(r, f) + T(r, g)) \leq \frac{2(m + l)}{n} (T(r, f) + T(r, g)) + S(r, f) + S(r, g). \]

Therefore

\[(n(m - l - 1) - 2(m + l))(T(r, f) + T(r, g)) \leq S(r, f) + S(r, g), \]

which contradicts to the assumptions that \( m \geq l + 3 \) and \( n \geq 3l + 3m + 5 \). This proves Theorem 1.2. ■
**Proof.** [Proof of Theorem 1.3]

We define the functions $F$ and $G$ as in the proof of Theorem 1.1. Proceeding as in the proof of Theorem 1.1, we obtain that $F$ and $G$ share 1 IM. Applying Lemma 2.10 to $F$ and $G$, we have the following three cases:

**Case 1.**

\[
T(r, F) + T(r, G) \leq 2N_2(r, F) + 3N(r, F) + 2N_2(r, G) + 3N(r, G) + 2N_2(r, \frac{1}{F}) + 3N(r, \frac{1}{F}) + 2N_2(r, \frac{1}{G}) + 3N(r, \frac{1}{G}) + S(r, F) + S(r, G).
\]

Proceeding as in the proof of Theorem 1.1, we have

\[
N_2(r, \frac{1}{F}) \leq 2T(r, f),
\]

\[
N_2(r, \frac{1}{G}) \leq 2T(r, g),
\]

\[
\overline{N}(r, \frac{1}{F}) \leq T(r, f),
\]

\[
\overline{N}(r, \frac{1}{G}) \leq T(r, g),
\]

\[
N_2(r, F) \leq (m + 2 + l(k + 1))T(r, f) + S(r, f),
\]

\[
N_2(r, G) \leq (m + 2 + l(k + 1))T(r, g) + S(r, g).
\]

By Lemma 2.3, we have

\[
\overline{N}(r, F) = \overline{N}(r, \frac{-f^n}{af^m(f(k))^l - b}) \leq \overline{N}(r, \frac{1}{af^m(f(k))^l - b}) + \overline{N}(r, f) \leq T(r, af^m(f(k))^l - b) + \overline{N}(r, f) \leq T(r, f^m(f(k))^l) + \overline{N}(r, f) \leq mT(r, f) + lT(r, f^k) + T(r, f) \leq mT(r, f) + l(k + 1)T(r, f) + T(r, f) + S(r, f).
\]

which implies

\[
\overline{N}(r, \frac{1}{F}) \leq (l(k + 1) + m + 1)T(r, f) + S(r, f).
\]
Similarly, we have
\[ N(r, \frac{1}{G}) = (l(k + 1) + m + 1)T(r, g) + S(r, g). \]

Hence
\[ T(r, F) + T(r, G) \leq (5l(k + 1) + 5m + 14)T(r, f) + (5l(k + 1) + 5m + 14)T(r, g) + S(r, f) + S(r, g). \]

Therefore
\[ nT(r, f) + nT(r, g) = T(r, f^n) + T(r, g^n) + O(1) \leq T(r, F) + T(r, G) + O(1) \leq (5l(k + 1) + 5m + 14)T(r, f) + (5l(k + 1) + 5m + 14)T(r, g) + S(r, f) + S(r, g), \]

which implies
\[ n(T(r, f) + T(r, g)) \leq (6l(k + 1) + 6m + 14)(T(r, f) + T(r, g)) + S(r, f) + S(r, g). \]

Thus
\[ (n - 6l(k + 1) - 6m - 14)(T(r, f) + T(r, g)) \leq S(r, f) + S(r, g), \]

which contradicts the assumption that \( n \geq 6l(k + 1) + 6m + 15. \)

**Case 2.** \( F = G. \) Then
\[ \frac{\phi_f - b}{\phi_g - b} = \frac{f^n}{g^n} = \frac{af^m(f^{(k)}y^l - b)}{ag^m(g^{(k)}y^l - b)}. \]

**Case 3.** \( FG = 1. \) Then
\[ \frac{\phi_f - b}{\phi_g - b} = \frac{f^n}{ag^m(g^{(k)}y^l - b)} = \frac{af^m(f^{(k)}y^l - b)}{g^n}. \]

This completes the proof of Theorem 1.3. \( \square \)
Proof. [Proof of Theorem 1.4]

Proceeding as the proof of Theorem 1.2, we have $a f^m(f^{(k)})^l - b \neq 0$ and $a g^m(g^{(k)})^l - b \neq 0$. We define the functions $F$ and $G$ as in the proof of Theorem 1.1. Proceeding as in the proof of Theorem 1.1, we have $F$ and $G$ share 1 IM.

Applying Lemma 2.10 to $F$ and $G$, we have the following three cases:

Case 1.

$$T(r, F) + T(r, G) \leq 2N_2(r, F) + 3\overline{N}(r, F) + 2N_2(r, G) + 3\overline{N}(r, G) +$$
$$+ 2N_2(r, \frac{1}{F}) + 3\overline{N}(r, \frac{1}{F}) + 2N_2(r, \frac{1}{G}) + 3\overline{N}(r, \frac{1}{G}) +$$
$$S(r, F) + S(r, G).$$

Proceeding as the proof of Theorem 1.2, we have

$$N_2(r, \frac{1}{F}) \leq 2T(r, f),$$

$$N_2(r, \frac{1}{G}) \leq 2T(r, g),$$

$$\overline{N}(r, \frac{1}{F}) \leq T(r, f),$$

$$\overline{N}(r, \frac{1}{G}) \leq T(r, g),$$

$$N_2(r, F) \leq (m + l)T(r, f) + S(r, f),$$

$$N_2(r, G) \leq (m + l)T(r, g) + S(r, g),$$

$$\overline{N}(r, F) \leq (m + l)T(r, f) + S(r, f),$$

$$\overline{N}(r, G) \leq (m + l)T(r, g) + S(r, g).$$

Hence

$$T(r, F) + T(r, G) \leq (5l + 5m + 7)T(r, f) + (5l + 5m + 7)T(r, g) + S(r, f) + S(r, g).$$
Therefore

\[ nT(r, f) + nT(r, g) = T(r, f^n) + T(r, g^n) + O(1) \leq \]
\[ \leq T(r, F) + T(r, \frac{1}{af^m(f^{(k)})^l - b}) + T(r, G) + \]
\[ + T(r, \frac{1}{ag^m(g^{(k)})^l - b}) + O(1) = \]
\[ = T(r, F) + T(r, f^m(f^{(k)})^l) + T(r, G) + \]
\[ + T(r, g^m(g^{(k)})^l) + O(1) \leq \]
\[ \leq T(r, F) + mT(r, f) + \ell T(r, (f^{(k)})) + T(r, G) + \]
\[ + mT(r, g) + \ell T(r, (g^{(k)})) + O(1) \leq \]
\[ \leq (5l + 5m + 7)T(r, f) + (5l + 5m + 7)T(r, g) + \]
\[ + (m + l)T(r, f) + (m + l)T(r, g) + S(r, f) + S(r, g). \]

Thus

\[ (n - 6l - 6m - 7)(T(r, f) + T(r, g)) \leq S(r, f) + S(r, g), \]

which contradicts to the assumption that \( n \geq 6l + 6m + 8 \).

**Case 2.** \( F = G \). Then

\[ \frac{-f^n}{af^m(f^{(k)})^l - b} = \frac{-g^n}{ag^m(g^{(k)})^l - b}. \]

Therefore (1.5) holds.

**Case 3.** \( FG = 1 \). Then

(3.18) \[ \frac{\phi_f - b}{\phi_g - b} = \frac{f^n}{a(g^{(k)})^l - b} = \frac{a(f^{(k)})^l - b}{g^n}. \]

We will prove that (3.18) cannot occur. Since \( \phi_f, \phi_g \) share the value \( b \) IM and \( f, g \) are entire functions, we have

(3.19) \[ N(r, \frac{1}{f}) = N(r, \frac{1}{g^m(g^{(k)})^l - b}), N(r, \frac{1}{g}) = N(r, \frac{1}{f^m(f^{(k)})^l - b}). \]

We will show that \( f^{(k)} \not\equiv 0 \). Suppose for contradiction that \( f^{(k)} \equiv 0 \). This implies \( f \) is a non-constant polynomial. Combining this and (3.19), we have \( fg \) is non-constant and \( g^{(k)} \not\equiv 0 \). From (3.18), we have

(3.20) \[ (fg)^n = -b(af^m(g^{(k)})^l - b). \]
From this and Lemma 2.6, we have
\[
\begin{align*}
T(r, fg) &\leq N(r, \frac{1}{fg^n}) + N(r, \frac{1}{g^{m(g(k)^l)}}) + S(r, fg) \\
&\leq N(r, \frac{1}{fg^n}) + N(r, \frac{1}{g^m}) + S(r, fg) \\
&\leq T(r, fg) + T(r, g) + T(r, g^{(k)}) + S(r, fg),
\end{align*}
\]
which implies
\[
(n - 1)T(r, fg) \leq 2T(r, g) + S(r, fg) + S(r, g).
\]

On the other hand, by (3.20) we have
\[
mT(r, g) = T(r, g^m) + O(1) \leq T(r, g^{m(g^{(k)})^l}) + T(r, \frac{1}{g^{(g^{(k)})^l}}) + O(1) \leq N(r, \frac{1}{fg^n}) + N(r, \frac{1}{g^{m(g^{(k)})^l}}) + IT(r, g) + S(r, g) \leq T(r, fg) + 2T(r, g) + IT(r, g) + S(r, g),
\]
which yields
\[
(m - l - 2)T(r, g) \leq T(r, fg) + S(r, g).
\]

From (3.21) and (3.22), we have
\[
(n - 1)T(r, fg) + (m - l - 3)T(r, g) \leq S(r, fg) + S(r, g),
\]
which contradicts to the assumptions that \( n \geq 6l + 6m + 5 \) and \( m \geq l + 4 \).

Hence \( f^{(k)} \neq 0 \) and similarly we have \( g^{(k)} \neq 0 \).

By Lemma 2.8, we have
\[
T(r, f^m(f^{(k)})^l) \leq N(r, \frac{1}{f}) + N(r, \frac{1}{f^m(f^{(k)})^l} - \frac{1}{2}) + T(r, f^{(k)}) + S(r, f) = N(r, \frac{1}{f}) + N(r, \frac{1}{g}) + T(r, f^{(k)}) + S(r, f) \leq 2T(r, f) + T(r, g) + S(r, f).
\]
Hence
\[
mT(r, f) = T(r, f^m) + O(1) \leq T(r, f^m(f^{(k)})^l) + T(r, \frac{1}{(f^{(k)})^l}) + O(1) \leq 2T(r, f) + T(r, g) + T(r, f^{(k)})^l + S(r, f) \leq 2T(r, f) + T(r, g) + IT(r, f) + S(r, f) + S(r, g),
\]
which implies
\[ mT(r, f) \leq (l + 2)T(r, f) + T(r, g) + S(r, f) + S(r, g). \]

Similarly, we have
\[ mT(r, g) \leq (l + 2)T(r, g) + T(r, f) + S(r, f) + S(r, g). \]

Hence
\[ (m - l - 3)(T(r, f) + T(r, g)) \leq S(r, f) + S(r, g), \]

which contradicts to the assumption that \( m \geq l + 4 \).

This proves Theorem 1.4. ■

References


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