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UNSOLVED PROBLEMS SECTION

SOME UNSOLVED PROBLEMS ON ARITHMETICAL FUNCTIONS

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Let, as usual, $\mathcal{P}$, $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$ be the set of primes, positive integers, rationals and real numbers, respectively. Let $\mathbb{Q}_\times$ = multiplicative group of positive rationals.

1. Let α, β distinct positive numbers, at least one of which is irrational. Let $\gamma_n = \dfrac{[\alpha n]}{[\beta n]}$, $\mathcal{B}$ be the multiplicative group generated by $\{\gamma_n \mid n \in \mathbb{N}\}$.

Conjecture 1. $\mathcal{B} = \mathbb{Q}_\times$.

Conjecture 2. *Let f be a completely additive function for which*

$$f([\alpha n]) - f([\beta n]) \to C \quad (n \to \infty).$$

Then $f(n) = A\log(n),\ A = \dfrac{C}{\log \frac{\alpha}{\beta}}$.

Note. We proved these conjectures in the case $\alpha = \sqrt{2}$, $\beta = 1$ (see [2], [3], [4], [5]).

2. Let $\mathcal{A}, \mathcal{B}$ be the subsets of $\mathbb{N}$, $\mathcal{C} = \mathcal{A} \oplus \mathcal{B} = \{a + b \mid a \in \mathcal{A},\ b \in \mathcal{B}\}$. Let $\mathcal{F}$ be the set of those collection of functions $f : \mathcal{C} \to \mathbb{R}$, $g : \mathcal{A} \to \mathbb{R}$, $h : \mathcal{C} \to \mathbb{R}$, for which

$$(*) \qquad f(a + b) = g(a) + h(b) \quad \text{for all} \quad a \in \mathcal{A},\ b \in \mathcal{B}$$

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is satisfied.

It is clear that for each choice of $A, B, C \in \mathbb{R}$, the functions

$$\begin{cases} g(n) = An + B \quad (\forall n \in \mathcal{A}), \\ h(m) = Am + C \quad (\forall m \in \mathcal{B}), \\ f(k) = Ak + (B + C) \quad (\forall k \in \mathcal{C}) \end{cases}$$

give a solution of $(*)$.

Problem. *Under what condition is true that $\mathcal{F}$ does not contain more elements?*

3.

Conjecture 3. *If $a, b \in \mathbb{N}, a \neq b$, f_1, f_2 are real-valued completely additive functions and $f_1(p+a) \equiv f_2(p+b) \pmod 1$ for every $p \in \mathcal{P}$, then $f_1(n) \equiv \equiv f_2(n) \equiv 0 \pmod 1$ holds for every $n \in \mathbb{N}$.*

Conjecture 4. *If $a, b \in \mathbb{N}, a \neq b$, f_1, f_2 are real-valued completely additive functions and $f_1(p+a) = f_2(p+b)$ for every $p \in \mathcal{P}$, then $f_1(n) = f_2(n) = 0$ holds for every $n \in \mathbb{N}$.*

Conjecture 5. *Let D be a positive integer. Assume that the arithmetical function $f : \mathbb{N} \to \mathbb{C}$ satisfy*

$$f(n^2 + Dm^2) = f(n)^2 + Df(m)^2 \quad \textit{for all} \quad n, m \in \mathbb{N}.$$

Then one of the following assertions holds:

$$\begin{aligned} &a) \quad f(n) = 0 \quad \textit{for all} \quad n \in \mathbb{N}, \\ &b) \quad f(n) = \frac{\epsilon(n)}{D+1} \quad \textit{for all} \quad n \in \mathbb{N}, \\ &c) \quad f(n) = \epsilon(n)n \quad \textit{for all} \quad n \in \mathbb{N}, \end{aligned}$$

where $\epsilon(n) = 1$ if $n \in E$ and $\epsilon(n) = \pm 1$ if $n \notin E$, $E := \{n^2 + Dm^2 \mid n, m \in \mathbb{N}\}$.

This is proved in [1] for $D = 1$ and in [6] for $D = 2, 3$.

4. Let f, g be arithmetical functions and $a, b \in \mathbb{N}$. Assume that

$$f(p + q + a + b) = g(p + a) + g(q + b) \quad \text{for all} \quad p, q \in \mathcal{P}.$$

Let

$$S_p := g(p + a) \quad \text{for all} \quad p \in \mathcal{P}.$$

Conjecture 6. *We have*

$$S_p = \frac{p-3}{2}S_5 - \frac{p-5}{2}S_3 \quad \textit{for all} \quad p \in \mathcal{P} \setminus \{2\},$$

that is

$$g(p+a) = \frac{p-3}{2}g(5+a) - \frac{p-5}{2}g(3+a) \quad \textit{for all} \quad p \in \mathcal{P} \setminus \{2\}.$$

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