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A CONVERGENCE ANALYSIS FOR A CERTAIN FAMILY OF EXTENDED ITERATIVE METHODS: PART II. APPLICATIONS TO FRACTIONAL CALCULUS

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Abstract. We present some choices of the operators involved in fractional calculus where the operators satisfy the convergence conditions of some iterative methods given in Part I [4]. Moreover, we provide a corrected version of the generalized fractional Taylor's formula given in [14].

1. Introduction

In this study we are concerned with the problem of approximating a locally unique solution x^* of the nonlinear equation

$$F(x) = 0, \tag{1.1}$$

where F is a continuous operator defined on a subset D of a Banach space X with values in a Banach space Y.

A lot of problems in Computational Sciences and other disciplines can be brought in a form like (1.1) using Mathematical Modelling [7], [11], [15]. The

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solutions of such equations can be found in closed form only in special cases. That is why most solution methods for these equations are iterative. Iterative algorithms are usually studied based on semilocal and local convergence. The semilocal convergence matter is, based on the information around the initial point to give hypotheses ensuring the convergence of the method; while the local one is, based on the information around a solution, to find estimates of the radii of convergence balls as well as error bounds on the distances involved.

We introduce the method defined for each $n = 0, 1, 2, ...$ by

$$x_{n+1} = x_n - A(x_n)^{-1} F(x_n), \tag{1.2}$$

where $x_0 \in D$ is an initial point and $A(x) \in L(X, Y)$ the space of bounded linear operators from X into Y. There is a plethora on local as well as semilocal convergence theorems for method (1.2) provided that the operator A is an approximation to the Fréchet-derivative F' [1], [2], [5]–[15]. In the present study we do not assume that operator A is not necessarily related to F'. This way we expand the applicability of method (1.2). Notice that many well known methods are special case of method (1.2).

Newton's method: Choose $A(x) = F'(x)$ for each $x \in D$.

Steffensen's method: Choose $A(x) = [x, G(x); F]$, where $G : X \to X$ is a known operator and $[x, y; F]$ denotes a divided difference of order one [7], [11], [15].

The so called Newton-like methods and many other methods are special cases of method (1.2).

The semilocal as well as the local convergence analysis of method (1.2) was given in Part I [4]. Some applications from fractional calculus are given in Part II. In particular, we first correct the generalized fractional Taylor's formula, the integral version extracted from [14]. Then, we use the corrected formula in our applications.

2. Applications to fractional calculus

Remark 2.1. We present some choices and properties of operator $A(y, x)$ from fractional calculus satisfying the crucial estimate (2.15) (see Part I) [4], in the special case when,

$$g_1(t) = c\psi \text{ for some } c > 0 \text{ and each } t \geq 0.$$

(see the end of Section 2 for a possible definition of the constant c).

Hence, Theorem 2.6 in [4] can apply to solve equation $F(x) = 0$. Other choices for operator $A(x)$ or operator $A(y,x)$ can be found in [5]–[7], [9]–[15].

Let $f : [a,b] \to \mathbb{R}$ such that $f^{(m)} \in L_\infty([a,b])$, the left Caputo fractional derivative of order $\alpha \notin \mathbb{N}$, $\alpha > 0$, $m = \lceil \alpha \rceil$ ($\lceil \cdot \rceil$ ceiling) is defined as follows:

$$(D_a^\alpha f)(x) = \frac{1}{\Gamma(m-\alpha)} \int_a^x (x-t)^{m-\alpha-1} f^{(m)}(t)\, dt, \tag{2.1}$$

where Γ is the gamma function, $\forall\, x \in [a,b]$.

We observe that

$$|(D_a^\alpha f)(x)| \leq \frac{1}{\Gamma(m-\alpha)} \int_a^x (x-t)^{m-\alpha-1} \left| f^{(m)}(t) \right| dt \leq$$

$$\leq \frac{\left\| f^{(m)} \right\|_\infty}{\Gamma(m-\alpha)} \left(\int_a^x (x-t)^{m-\alpha-1}\, dt \right) = \frac{\left\| f^{(m)} \right\|_\infty}{\Gamma(m-\alpha)} \frac{(x-a)^{m-\alpha}}{(m-\alpha)} =$$

$$= \frac{\left\| f^{(m)} \right\|_\infty}{\Gamma(m-\alpha+1)} (x-a)^{m-\alpha}. \tag{2.2}$$

We have proved that

$$|(D_a^\alpha f)(x)| \leq \frac{\left\| f^{(m)} \right\|_\infty}{\Gamma(m-\alpha+1)} (x-a)^{m-\alpha} \leq \frac{\left\| f^{(m)} \right\|_\infty}{\Gamma(m-\alpha+1)} (b-a)^{m-\alpha}. \tag{2.3}$$

Clearly then $(D_a^\alpha f)(a) = 0$.

Let $n \in \mathbb{N}$ we denote $D_a^{n\alpha} = D_a^\alpha D_a^\alpha \cdots D_a^\alpha$ (n-times).

Let us assume now that

$$D_a^{k\alpha} f \in C([a,b]), \quad k = 0, 1, \ldots, n+1; \quad n \in \mathbb{N}, \quad 0 < \alpha \leq 1. \tag{2.4}$$

By [14], we are able to extract the following interesting generalized fractional Caputo type Taylor's formula: (there it is assumed that $D_a^{k\alpha} f(x) \in C((a,b])$, $k = 0, 1, \ldots, n+1$; $0 < \alpha \leq 1$)

$$f(x) = \sum_{i=0}^{n} \frac{(x-a)^{i\alpha}}{\Gamma(i\alpha+1)} \left(D_a^{i\alpha} f\right)(a) + \tag{2.5}$$

$$+\frac{1}{\Gamma((n+1)\alpha)} \int_a^x (x-t)^{(n+1)\alpha-1} \left(D_a^{(n+1)\alpha} f\right)(t)\, dt, \quad \forall\, x \in (a,b].$$

Notice that [14] has lots of typos or minor errors, which we fixed.

Under our assumption and conclusion, see (2.4), Taylor's formula (2.5) becomes

$$f(x) - f(a) = \sum_{i=2}^{n} \frac{(x-a)^{i\alpha}}{\Gamma(i\alpha+1)} \left(D_a^{i\alpha} f\right)(a) +$$

$$\frac{1}{\Gamma((n+1)\alpha)} \int_a^x (x-t)^{(n+1)\alpha-1} \left(D_a^{(n+1)\alpha} f\right)(t)\, dt, \tag{2.6}$$

$$\forall\, x \in (a,b],\ 0 < \alpha < 1.$$

Here we are going to operate more generally. Again we assume $0 < \alpha \leq 1$, and $f : [a,b] \to \mathbb{R}$, such that $f' \in C([a,b])$. We define the following left Caputo fractional derivatives:

$$\left(D_y^{\alpha} f\right)(x) = \frac{1}{\Gamma(1-\alpha)} \int_y^x (x-t)^{-\alpha} f'(t)\, dt, \tag{2.7}$$

for any $x \geq y$; $x, y \in [a,b]$, and

$$\left(D_x^{\alpha} f\right)(y) = \frac{1}{\Gamma(1-\alpha)} \int_x^y (y-t)^{-\alpha} f'(t)\, dt, \tag{2.8}$$

for any $y \geq x$; $x, y \in [a,b]$.

Notice $D_y^1 f = f'$, $D_x^1 f = f'$ by convention.

Clearly here $\left(D_y^{\alpha} f\right)$, $\left(D_x^{\alpha} f\right)$ are continuous functions over $[a,b]$, see [2], p. 388. We also make the convention that $\left(D_y^{\alpha} f\right)(x) = 0$, for $x < y$, and $\left(D_x^{\alpha} f\right)(y) = 0$, for $y < x$.

Here we assume that $D_y^{k\alpha} f$, $D_x^{k\alpha} f \in C([a,b])$, $k = 0, 1, \ldots, n+1$, $n \in \mathbb{N}$; $\forall\, x, y \in [a,b]$.

Then by (2.6) we obtain

$$f(x)-f(y)=\sum_{i=2}^{n}\frac{(x-y)^{i\alpha}}{\Gamma(i\alpha+1)}\left(D_y^{i\alpha}f\right)(y)+$$

$$+\frac{1}{\Gamma((n+1)\alpha)}\int_y^x (x-t)^{(n+1)\alpha-1}\left(D_y^{(n+1)\alpha}f\right)(t)\,dt, \tag{2.9}$$

$\forall\ x>y;\ x,y\in[a,b],\ 0<\alpha<1,$
and also it holds

$$f(y)-f(x)=\sum_{i=2}^{n}\frac{(y-x)^{i\alpha}}{\Gamma(i\alpha+1)}\left(D_x^{i\alpha}f\right)(x)+$$

$$+\frac{1}{\Gamma((n+1)\alpha)}\int_x^y (y-t)^{(n+1)\alpha-1}\left(D_x^{(n+1)\alpha}f\right)(t)\,dt, \tag{2.10}$$

$\forall\ y>x;\ x,y\in[a,b],\ 0<\alpha<1.$

We define the following linear operator

$$(A(f))(x,y)=$$

(2.11)
$$\begin{cases} \sum_{i=2}^{n}\frac{(x-y)^{i\alpha-1}}{\Gamma(i\alpha+1)}\left(D_y^{i\alpha}f\right)(y)+\left(D_y^{(n+1)\alpha}f(x)\right)\frac{(x-y)^{(n+1)\alpha-1}}{\Gamma((n+1)\alpha+1)}, & x>y, \\ \sum_{i=2}^{n}\frac{(y-x)^{i\alpha-1}}{\Gamma(i\alpha+1)}\left(D_x^{i\alpha}f\right)(x)+\left(D_x^{(n+1)\alpha}f(y)\right)\frac{(y-x)^{(n+1)\alpha-1}}{\Gamma((n+1)\alpha+1)}, & y>x, \\ f'(x), \text{ when } x=y, \end{cases}$$

$\forall\ x,y\in[a,b],\ 0<\alpha<1.$

We may assume that

$$|(A(f))(x,x)-(A(f))(y,y)|=|f'(x)-f'(y)|\leq \tag{2.12}$$

$$\leq\Phi|x-y|,\ \forall\ x,y\in[a,b],\ \text{with }\Phi>0.$$

We estimate and have:

i) case of $x > y$:

$$|f(x) - f(y) - (A(f))(x,y)(x-y)| =$$

$$= \left| \frac{1}{\Gamma((n+1)\alpha)} \int_y^x (x-t)^{(n+1)\alpha-1} \left(D_y^{(n+1)\alpha} f\right)(t)\,dt - \right. \tag{2.13}$$

$$\left. - \left(D_y^{(n+1)\alpha} f(x)\right) \frac{(x-y)^{(n+1)\alpha}}{\Gamma((n+1)\alpha+1)} \right| =$$

$$= \frac{1}{\Gamma((n+1)\alpha)} \left| \int_y^x (x-t)^{(n+1)\alpha-1} \left(\left(D_y^{(n+1)\alpha} f\right)(t) - \left(D_y^{(n+1)\alpha} f\right)(x)\right) dt \right| \leq$$

$$\leq \frac{1}{\Gamma((n+1)\alpha)} \int_y^x (x-t)^{(n+1)\alpha-1} \left| D_y^{(n+1)\alpha} f(t) - \left(D_y^{(n+1)\alpha} f\right)(x) \right| dt \leq$$

(we assume here that

$$\left| D_y^{(n+1)\alpha} f(t) - D_y^{(n+1)\alpha} f(x) \right| \leq \lambda_1 |t-x|, \tag{2.14}$$

$\forall\ t,x,y \in [a,b] : x \geq t \geq y$, where $\lambda_1 > 0$)

$$\leq \frac{\lambda_1}{\Gamma((n+1)\alpha)} \int_y^x (x-t)^{(n+1)\alpha-1} (x-t)\,dt =$$

$$= \frac{\lambda_1}{\Gamma((n+1)\alpha)} \int_y^x (x-t)^{(n+1)\alpha}\,dt = \frac{\lambda_1}{\Gamma((n+1)\alpha)} \frac{(x-y)^{(n+1)\alpha+1}}{((n+1)\alpha+1)}. \tag{2.15}$$

We have proved that
(2.16)

$$|f(x) - f(y) - (A(f))(x,y)(x-y)| \leq \frac{\lambda_1}{\Gamma((n+1)\alpha)} \frac{(x-y)^{(n+1)\alpha+1}}{((n+1)\alpha+1)},$$

for any $x, y \in [a,b] : x > y,\ 0 < \alpha < 1$.

ii) case of $x < y$:

$$|f(x) - f(y) - (A(f))(x,y)(x-y)| =$$

$$= |f(y) - f(x) - (A(f))(x,y)(y-x)| =$$

$$= \left| \frac{1}{\Gamma((n+1)\alpha)} \int\limits_x^y (y-t)^{(n+1)\alpha-1} \left(D_x^{(n+1)\alpha} f \right)(t)\, dt - \right. \tag{2.17}$$

$$\left. - \left(D_x^{(n+1)\alpha} f(y) \right) \frac{(y-x)^{(n+1)\alpha}}{\Gamma((n+1)\alpha+1)} \right| =$$

$$= \frac{1}{\Gamma((n+1)\alpha)} \left| \int_x^y (y-t)^{(n+1)\alpha-1} \left(\left(D_x^{(n+1)\alpha} f \right)(t) - \left(D_x^{(n+1)\alpha} f \right)(y) \right) dt \right| \leq$$

$$\leq \frac{1}{\Gamma((n+1)\alpha)} \int_x^y (y-t)^{(n+1)\alpha-1} \left| \left(D_x^{(n+1)\alpha} f \right)(t) - \left(D_x^{(n+1)\alpha} f \right)(y) \right| dt \leq$$

(we assume that

$$\left| \left(D_x^{(n+1)\alpha} f \right)(t) - \left(D_x^{(n+1)\alpha} f \right)(y) \right| \leq \lambda_2 |t-y|, \tag{2.18}$$

$\forall\ t, y, x \in [a,b] : y \geq t \geq x$, where $\lambda_2 > 0$)

$$\leq \frac{\lambda_2}{\Gamma((n+1)\alpha)} \int\limits_x^y (y-t)^{(n+1)\alpha-1} (y-t)\, dt =$$

$$= \frac{\lambda_2}{\Gamma((n+1)\alpha)} \int\limits_x^y (y-t)^{(n+1)\alpha}\, dt = \frac{\lambda_2}{\Gamma((n+1)\alpha)} \frac{(y-x)^{(n+1)\alpha+1}}{((n+1)\alpha+1)}. \tag{2.19}$$

We have proved that

$$|f(x) - f(y) - A(f)(x,y)(x-y)| \leq \frac{\lambda_2}{\Gamma((n+1)\alpha)} \frac{(y-x)^{(n+1)\alpha+1}}{((n+1)\alpha+1)}, \tag{2.20}$$

$\forall\ x, y \in [a,b] : y > x,\ 0 < \alpha < 1.$

Conclusion. Let $\lambda := \max(\lambda_1, \lambda_2)$. It holds

$$|f(x) - f(y) - (A(f))(x,y)(x-y)| \leq \frac{\lambda}{\Gamma((n+1)\alpha)} \frac{|x-y|^{(n+1)\alpha+1}}{((n+1)\alpha+1)}, \tag{2.21}$$

$\forall\ x, y \in [a,b]$, where $0 < \alpha < 1$, $n \in \mathbb{N}$.

One may assume that $\frac{\lambda}{\Gamma((n+1)\alpha)} < 1$.

(Above notice that (2.21) is trivial when $x = y$.)

Now based on (2.12) and (2.21), we can apply our numerical methods presented in this article, to solve $f(x) = 0$.

To have $(n+1)\alpha + 1 \geq 2$, we need to take $1 > \alpha \geq \frac{1}{n+1}$, where $n \in \mathbb{N}$.

Then, returning back to Remark 2.1, we see that the constant c can be defined by

$$c = \frac{\lambda}{\Gamma((n+1)\alpha)[(n+1)\alpha+1]}$$

provided that $n = p$, $(p+1)\alpha \leq p$ and

$$|y - x| \leq 1 \quad \text{for each} \quad x, y \in [a,b]. \tag{2.22}$$

Notice that condition (2.22) can always be satisfied by choosing x, y (i.e. a, b) sufficiently close to each other.

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