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ON THE UNIFORM DISTRIBUTION OF CERTAIN SEQUENCES INVOLVING THE EULER TOTIENT FUNCTION AND THE SUM OF DIVISORS FUNCTION

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Abstract. We examine the uniform distribution of certain sequences involving the Euler totient function and the sum of divisors function.

1. Introduction and notation

Let us denote by $\phi(n)$ the well known Euler totient function and by $\sigma(n)$ the sum of the positive divisors of n.

Let also $\mathcal{M}$ (resp. $\mathcal{A}$) be the set of multiplicative (resp. additive) functions and $\mathcal{M}_1$ the set of those $f \in \mathcal{M}$ such that $|f(n)| = 1$ for all positive integers n. For each $y \in \mathbb{R}$, we set $e(y) := e^{2\pi i y}$.

A famous result of H. Daboussi (see Daboussi and Delange [2], [3]) asserts that

$$\sup_{f\in\mathcal{M}_1} \frac{1}{x}\left|\sum_{n\le x} f(n)e(n\alpha)\right| \to 0 \qquad \text{as } x\to\infty \tag{1.1}$$

for every $\alpha \in \mathbb{R}\setminus\mathbb{Q}$.

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The proof of (1.1) is based on the large sieve inequality. Another proof follows from a general form of the Turán-Kubilius inequality.

Here, we examine the uniform distribution of certain sequences involving the Euler totient function and the sum of divisors function.

From here on, we let $\wp$ stand for the set of all primes and we let $\{y\}$ be the fractional part of y. We also let $P(n)$ stand for the largest prime factor of n.

2. Background results

The following result was obtained by the second author [7].

Theorem A. *Let $t:\mathbb{N}\to\mathbb{R}$. Assume that for every real number $K>0$, there exists a finite set $\wp_K$ of primes $p_1<p_2<\cdots<p_k$ such that*

$$A_K := \sum_{i=1}^{k} \frac{1}{p_i} > K \tag{2.1}$$

and that, given any pair $i\neq j$, $i,j\in\{1,2,\dots,k\}$, the corresponding sequence

$$\eta_{i,j}(m) = t(p_i m) - t(p_j m) \qquad (m\in\mathbb{N})$$

satisfies the relation

$$\frac{1}{x}\sum_{m\le x} e(\eta_{i,j}(m)) \to 0 \qquad \text{as } x\to\infty.$$

Then there exists a function ρ_x for which $\rho_x\to 0$ as $x\to\infty$ and such that

$$\sup_{f\in\mathcal{M}_1} \frac{1}{x}\left|\sum_{n\le x} f(n)e(t(n))\right| \le \rho_x.$$

Observe that Theorem A holds in particular if one chooses $t(n):=\alpha_r n^r + {}+\cdots+\alpha_1 n$, a polynomial with real coefficients where at least one the α_i's is irrational.

Recall that the *discrepancy* of a set of N real numbers $x_1,\dots,x_N$ is the quantity

$$D(x_1,\dots,x_N) := \sup_{[a,b)\subseteq[0,1)}\left|\frac{1}{N}\sum_{\{x_\nu\}\in[a,b)} 1-(b-a)\right|.$$

We now consider the set $\mathcal{T}$ of all those real valued arithmetic functions t for which the sequence

$$\eta_n(F) := F(n) + t(n) \qquad (n \in \mathbb{N})$$

satisfies

$$D((\eta_1(F), \eta_2(F), \ldots, \eta_N(F)) \to 0 \qquad \text{as } N \to \infty$$

for every arithmetic function F.

The following result is then a consequence of Theorem A.

Corollary 1. *Assume that for every real number $K > 0$, one can choose a set of primes $\wp_K = \{p_1, p_2, \ldots, p_k\}$ for which (2.1) holds, and let $t : \mathbb{N} \to \mathbb{R}$ be a function such that the sequence $(t(p_i m) - t(p_j m))_{m \geq 1}$ is uniformly distributed modulo 1 for every pair of integers $i \neq j$, $i, j \in \{1, 2, \ldots, k\}$. Then $t \in \mathcal{T}$.*

Remark 1. Observe that it is clear that if $t \in \mathcal{T}$, then the sequence $(t(n))_{n \geq 1}$ is uniformly distributed modulo 1.

Note also that, letting $\|x\|$ stand for the distance between x and the nearest integer, we proved in [4] the following.

Theorem B. *If α is a positive irrational number such that for each real number $\kappa > 1$ there exists a positive constant $c = c(\kappa, \alpha)$ for which the inequality*

$$\|\alpha q\| > \frac{c}{q^\kappa} \qquad \text{holds for every positive integer } q,$$

and let $Q(x) = a_r x^r + \cdots + a_0 \in \mathbb{R}[x]$, where $a_r > 0$. Assume that h is an integer valued function belonging to $\mathcal{M}_1$ such that $h(p) = Q(p)$ for every $p \in \wp$ and that for some fixed $d > 0$ we have $h(p^a) = O(p^{da})$ for every prime power p^a. Then the function $t(n) = \alpha h(n)$ belongs to $\mathcal{T}$.

It follows from Theorem B and Remark 1 that the sequence $(\{\alpha\sigma(n)\})_{n \geq 1}$ is uniformly distributed modulo 1.

Remark 2. Observe that one can construct an irrational number α for which the corresponding sequence $(\{\alpha\sigma(n)\})_{n \geq 1}$ is not uniformly distributed modulo 1. Indeed, consider the sequence of integers $(\ell_k)_{k \geq 1}$ defined by $\ell_1 = 1$ and $\ell_{k+1} = 2^{2^{2^{\ell_k}}}$ for each integer $k \geq 1$. Then consider the number

$$\alpha := \sum_{i=1}^{\infty} \frac{1}{2^{\ell_i}}.$$

It is clear that, letting $A_k := \sum_{i=1}^{k} 1/2^{\ell_i}$ for each integer $k \geq 1$, we have

$$\left| \alpha - \frac{A_k}{2^{\ell_k}} \right| < \frac{2}{2^{\ell_{k+1}}} \qquad (k \geq 1).$$

For each integer $k \geq 1$, define $Y_k := 2^{\frac{1}{2}\cdot \ell_{k+1}}$. With a technique used by Wijsmuller [11], one can prove that, for any fixed $\varepsilon > 0$, setting $T_x := \lfloor (2 - -\varepsilon) \log\log x \rfloor$, then

$$\frac{1}{x}\#\{n \leq x : \sigma(n) \equiv 0 \pmod{2^{T_x}}\} \to 1 \qquad \text{as } x \to \infty. \tag{2.2}$$

It follows from (2.2) that, for every fixed $\delta > 0$,

$$\frac{1}{Y_k}\#\{n \leq Y_k : \|\alpha\sigma(n)\| < \delta\} \to 1 \qquad \text{as } k \to \infty.$$

Indeed, if for some integer $n \leq Y_k$, we have $\sigma(n) \equiv 0 \pmod{2^{T_x}}$, then $T_{Y_k} > \ell_k$, in which case we have

$$\|\alpha\sigma(n)\| < \frac{2\sigma(n)}{2^{\ell_{k+1}}} \leq \frac{2Y_k \log Y_k}{2^{\ell_{k+1}}},$$

which tends to 0 as $k \to \infty$. Hence, for every $\delta > 0$, we have

$$\frac{1}{x}\#\{n \leq x : \|\alpha\sigma(n)\| < \delta\} \to 1 \qquad \text{as } x \to \infty,$$

thus proving our claim.

Further such constructions are given in Kátai [8]. Finally, observe that the same is also true for the sequence $(\{\alpha\phi(n)\})_{n\geq 1}$.

Now, let $\phi_k(n)$ (resp. $\sigma_k(n)$) stand for the k-th iterate of the ϕ (resp. σ) function. We first state two conjectures regarding these functions.

Conjecture 1. *Let $k \in \mathbb{N}$ be fixed. Then, for almost all real numbers $\alpha \in [0,1)$,*

$$\sup_{f\in\mathcal{M}_1} \frac{1}{x}\left|\sum_{n\leq x} f(n)e(\alpha\phi_k(n))\right| \to 0 \qquad \textit{as } x \to \infty, \tag{2.3}$$

$$\sup_{f\in\mathcal{M}_1} \frac{1}{x}\left|\sum_{n\leq x} f(n)e(\alpha\sigma_k(n))\right| \to 0 \qquad \textit{as } x \to \infty, \tag{2.4}$$

and in particular, for almost all $\alpha \in [0,1)$, both sequences $(\alpha\phi_k(n))_{n\geq 1}$ and $(\alpha\sigma_k(n))_{n\geq 1}$ are in $\mathcal{T}$.

Unfortunately this conjecture is still out of reach when $k \geq 2$. The main difficulty is that we cannot obtain a good upper bound for the quantities

$$\begin{aligned} A_k(n) &:= \#\{m \in \mathbb{N} : \phi_k(m) = n\}, \\ B_k(n) &:= \#\{m \in \mathbb{N} : \sigma_k(m) = n\}, \end{aligned}$$

when $k \geq 2$. Observe that, in the case $k = 1$, it is known (see Pomerance [10]) that

$$A_1(n) \leq n \exp\{-(1 + o(1))L(n)\} \qquad (n \to \infty), \tag{2.5}$$

where

$$L(n) = \frac{(\log n)(\log \log \log n)}{\log \log n}.$$

Conjecture 2. *Let $k \geq 2$ be a fixed integer. There exists a positive constant c_k such that, for all integers $n \geq 2$,*

$$A_k(n) \leq c_k \frac{n}{\log^9 n}, \tag{2.6}$$

$$B_k(n) \leq c_k \frac{n}{\log^9 n}. \tag{2.7}$$

Remark 3. Observe that (2.6) holds in the case $k = 1$, since it is a consequence of (2.5). On the other hand, (2.7) is also true in the case $k = 1$, as it can be proved using the same technique developed by Pomerance [10].

3. Main results

Theorem 1. *Conjecture 2 implies Conjecture 1.*

Theorem 2. *Given a real number α and a prime p, let $\xi_p := \{\alpha\phi(p + a)\}$. Then, for almost all real numbers α, the corresponding sequence $(\xi_p)_{p \in \wp}$ is uniformly distributed modulo 1.*

Theorem 3. *Let α be a positive irrational number such that for each real number $\kappa > 1$ there exists a positive constant $c = c(\kappa, \alpha)$ for which the inequality*

$$\|\alpha q\| > \frac{c}{q^\kappa} \qquad \text{holds for every positive integer } q.$$

Then, the sequence $(\{\alpha\phi(n)\}, \{\alpha\sigma(n)\})_{n \geq 1}$ is uniformly distributed modulo $[0, 1)^2$.

4. Proof of Theorems 1 and 2

We begin with Theorem 1. We shall consider only the case of ϕ_k since the case of σ_k can be handled in a similar way.

Let $N \geq 1$ be a fixed integer. Set

$$u_N = e^N, \qquad y_{h,N} = y_h = e^N + \frac{he^N}{N} \qquad (h = 1, 2, \ldots, \lfloor eN \rfloor)$$

and, for $\alpha \in \mathbb{R}$,

$$K_{N,h}(\alpha) = \sum_{u_N \leq n \leq y_h} e(\alpha \phi_k(n)).$$

Let $S = S(N,h) = \{\phi_k(n) : n \in (u_N, y_h)\}$. Given $s \in S$, let

$$U(s) = \#\{n \in (u_N, y_h) : \phi_k(n) = s\}.$$

It is clear that $U(s) \leq A_k(s)$ for $s \leq y_h$. Hence, using (2.6), we have

$$\begin{aligned} \int_0^1 |K_{N,h}(\alpha)|^2 \, d\alpha &= \sum_{s \in S} U^2(s) \leq \max_{s \in S} A_k(s) \sum_{s \in S} U(s) \leq \\ &\leq \max_{s \in S} A_k(s) \sum_{n \in [u_N, y_h]} 1 \leq \\ &\leq c_k \frac{e^N}{N^9}(y_h - u_N) \leq 3c_k \frac{e^{2N}}{N^9}. \end{aligned} \tag{4.1}$$

Let

$$A_{N,h} := \left\{ \alpha \in [0,1) : \left| \frac{K_{N,h}(\alpha)}{y_h - u_N} \right| > \frac{1}{N^3} \right\}.$$

It follows from (4.1) that, letting $\lambda(S)$ stand for the Lebesgue measure of a real set S,

$$\lambda(A_{N,h}) \leq \frac{3c_k}{N^3},$$

so that

$$\lambda\left(\bigcup_{h=1}^{\lfloor eN \rfloor} A_{N,h} \right) \leq \frac{5c_k}{N^2}. \tag{4.2}$$

Therefore, since $\displaystyle\sum_{N \geq 1} \frac{5c_k}{N^2} < \infty$, it follows from (4.2) that

$$\sum_{N=1}^{\infty} \lambda\left(\bigcup_{h=1}^{\lfloor eN \rfloor} A_{N,h} \right) < \infty.$$

Hence, using the well known Borel-Cantelli lemma, we have that if E is the set of all those real α which belong to $\bigcup_{h=1}^{\lfloor eN \rfloor} A_{N,h}$ for infinitely many N, then $\lambda(E) = 0$.

Now, let $\alpha \notin E$. Then, for every $N > N_0(\alpha)$, we have

$$|K_{N,h}(\alpha)| \leq \frac{1}{N^3(y_h - u_N)}.$$

We shall use this to prove that

$$\frac{1}{x} \sum_{n \leq x} e(\alpha\phi_k(n)) \to 0 \qquad \text{as } x \to \infty. \tag{4.3}$$

For $x \in [y_{h,N}, y_{h+1,N})$, letting T_N be a function tending to infinity arbitrarily slowly with N, we have

$$\begin{aligned}
\sum_{n \leq x} e(\alpha\phi_k(n)) &= \sum_{n \leq e^{N-T_N}} e(\alpha\phi_k(n)) + \sum_{e^{N-T_N} < n \leq e^N} e(\alpha\phi_k(n)) + \\
&\quad + \sum_{e^N < n \leq y_{h,N}} e(\alpha\phi_k(n)) + \sum_{y_{h,N} < n \leq x} e(\alpha\phi_k(n)) = \\
&= S_1 + S_2 + S_3 + S_4,
\end{aligned}$$

say. Trivially we have

$$|S_1| \leq \frac{x}{e^{T_N}}. \tag{4.4}$$

From (4.2), we have

$$|S_2| \leq \sum_{N-T_N \leq M \leq N} \frac{5c_k e^M}{M^2} \leq \frac{d_k x}{N - T_N} \tag{4.5}$$

for some constants d_k. Finally,

$$|S_3| \leq \frac{5c_k x}{N}, \tag{4.6}$$

and

$$|S_4| \leq y_{h+1,N} - y_{h,N} \leq \frac{e^N}{N} \leq \frac{x}{N}. \tag{4.7}$$

Gathering (4.4), (4.5), (4.6) and (4.7), estimate (4.3) follows.

On the other hand, letting E_ℓ be the set of those α for which $\{\alpha\ell\} \in E$, then $\lambda(E_\ell) = 0$, while if $\alpha \notin E_\ell$, then

$$\frac{1}{x}\sum_{n\le x} e(\alpha\ell\phi_k(n)) \to 0 \qquad \text{as } x\to\infty.$$

Let $q(n)$ be the smallest prime Q such that $Q \nmid n$. In order to complete the proof of the Theorem 1, we need the following result.

Lemma 1. *Let $k \in \mathbb{N}$. There exists a function y_x which tends to infinity with x such that*

$$\frac{1}{x}\#\{n \le x : q(\phi_k(n)) \le y_x\} \to 0 \qquad \textit{as } x\to\infty. \tag{4.8}$$

Proof. By choosing $y_x = (\log\log x)^{k(1-\varepsilon)}$ for a fixed small $\varepsilon > 0$, and by using the same techniques as in Erdős, Granville, Pomerance and Spiro [5] or as in Bassily and Kátai [1], one can easily obtain (4.8). ∎

We may now complete the proof of Theorem 1. Let $\wp_K = \{p_1, p_2, \dots, p_k\}$ be a set of primes satisfying (2.1) and let $t(m) = \alpha\phi_k(m)$. Observe that in general we have that if $u \mid \phi(v)$, then $\phi(u\phi(v)) = u\phi(\phi(v))$. Using this observation and Lemma 1, we have that $t(p_jm) = \alpha p_j\phi_k(m)$, so that

$$\eta_{i,j}(m) = t(p_im) - t(p_jm) = \alpha(p_i - p_j)\phi_k(m).$$

Hence, the sequence $(\eta_{i,j}(m))_{m\ge1}$ is uniformly distributed modulo 1 if $\alpha(p_i - -p_j) \notin E$. We can drop those α which belong to the set

$$F = \bigcup_{K=1}^{\infty} \bigcup_{\substack{i,j=1,\dots,R_K\\ i\ne j}} E_{K(p_i-p_j)},$$

where $R_K = \#\wp_k$, since $\lambda(F) = 0$. On the other hand, if $\alpha \notin F$, then the statement of Theorem 1 certainly holds. Thus, the proof of Theorem 1 is complete.

We will omit the proof of Theorem 2 since it can be obtained by repeating the arguments used in the proof of Theorem 1 and the techniques used in the proof of (2.5).

5. Proof of Theorem 3

In order to prove that a given sequence $((u_n, v_n))_{n\ge1}$ is uniformly distributed mod $[0,1)^2$, it is clear that we only need to prove that the sequence

$(ku_n + \ell v_n)_{n\geq 1}$ is uniformly distributed modulo 1 for all $(k,\ell) \in \mathbb{Z}\times\mathbb{Z}$ with $(k,\ell) \neq (0,0)$.

Given a fixed $(k,\ell) \in \mathbb{Z}\times\mathbb{Z}$ with $(k,\ell) \neq (0,0)$, consider the functions

$$A(n) = \alpha(k\sigma(n) + \ell\phi(n)), \qquad B(n) = \alpha(k\sigma(n) - \ell\phi(n)).$$

To prove the theorem, it is sufficient to establish that

$$\frac{1}{x}\sum_{n\leq x} e(A(n)) \to 0 \qquad \text{as } x\to\infty. \tag{5.1}$$

One can easily establish that, for each $\varepsilon > 0$, there exists $c = c(\varepsilon)$ such that $\lim_{\varepsilon\to 0} c(\varepsilon) = 0$ and such that

$$\frac{1}{x}\#\{n\leq x : P(n)\leq x^{\varepsilon}\} + \frac{1}{x}\#\{n\leq x : P(n)\geq x^{1-\varepsilon}\} \leq c(\varepsilon).$$

Therefore, in order to prove (5.1), it is sufficient to prove that

$$\frac{1}{x}\sum_{\substack{n\leq x\\ x^{\varepsilon}<P(n)<x^{1-\varepsilon}}} e(A(n)) \to 0 \qquad \text{as } x\to\infty. \tag{5.2}$$

Now, given an integer $n\leq x$, we write $n = mp$, where $p = P(n)$. Since

$$\#\{n\leq x : P(n) > x^{\varepsilon} \text{ and } p \mid m\} \leq x\sum_{p>x^{\varepsilon}}\frac{1}{p^2} = o(x),$$

in order to prove (5.2), we only need to prove that

$$\frac{1}{x}\sum_{\substack{n\leq x\\ P(n)<x^{1-\varepsilon}}} e(A(n)) \to 0 \qquad \text{as } x\to\infty. \tag{5.3}$$

Now, observe that if $(p,m) = 1$, then clearly,

$$A(pm) = pA(m) + B(m),$$

so that

$$\begin{aligned}\sum_{\substack{n\leq x\\ P(n)<x^{1-\varepsilon}}} e(A(n)) &= \sum_{m\leq x^{1-\varepsilon}} e(B(m))\left\{\sum_{p<x/m} e(pA(m)) - \sum_{p\leq P(m)} e(pA(m))\right\}\\ &= S_A(m) + S_B(m), \qquad (5.4)\end{aligned}$$

say.

We consider the two cases:

(a) $A(m) = 0$;

(b) $A(m) \neq 0$.

In case (a), we have that $k\sigma(m) + \ell\phi(m) = 0$, so that $\dfrac{\sigma(m)}{\phi(m)} = -\dfrac{\ell}{k}$.

We will prove that

$$\frac{1}{y}\#\left\{m \in [y, 2y],\ \frac{\sigma(m)}{\phi(m)} = -\frac{\ell}{k}\right\} \to 0 \quad \text{as } y \to \infty. \tag{5.5}$$

Now, according to a result of Lévy [9], if g is an additive function for which the three series

$$\sum_{|g(p)|<1} \frac{g(p)}{p}, \qquad \sum_{|g(p)|<1} \frac{g^2(p)}{p}, \qquad \sum_{|g(p)|\geq 1} \frac{1}{p}$$

are convergent, then if $(\xi_p)_{p\in\wp}$ is a sequence of independent random variables such that

$$P(\xi_p = g(p^a)) = \left(1 - \frac{1}{p}\right)\frac{1}{p^a} \qquad (a = 1, 2, \ldots). \tag{5.6}$$

then, the distribution F_η of $\eta = \sum \xi_p$ is everywhere continuous if and only if

$$\sum_{p\in\wp} P(\xi_p \neq 0) = \infty \tag{5.7}$$

Choosing $g(n) := \log \dfrac{\sigma(m)}{\phi(m)}$, we then have

$$g(p) = \log \frac{p+1}{p-1} \quad \text{and} \quad g(p^a) = \log \frac{1 + p + \cdots + p^a}{p^{a-1}(p-1)}.$$

For this function g and ξ_p as in (5.6), one can see that condition (5.7) is satisfied. Hence, using Lévy's result, we may conclude that (5.5) is satisfied.

Let D be the set of those positive integers m for which $\dfrac{\sigma(m)}{\phi(m)} = -\dfrac{\ell}{k}$ and let us estimate the right hand side of (5.4) as m running over D. We have that the right hand side of (5.4) is

$$
\begin{aligned}
&\ll \sum_{\substack{m\le x^{1-\varepsilon}\\ m\in D}} \pi(x/m) \le \\
&\le \sum_{2^\nu \le x^{1-\varepsilon}/\log x} \sum_{\substack{\frac{x^{1-\varepsilon}}{2^{\nu+1}}\le m<\frac{x^{1-\varepsilon}}{2^\nu}\\ m\in D}} \pi(x/m) \le \\
&\le \frac{c_\varepsilon x}{\log x} \sum_{2^\nu \le x^{1-\varepsilon}/\log x} \sum_{\substack{\frac{x^{1-\varepsilon}}{2^{\nu+1}}\le m<\frac{x^{1-\varepsilon}}{2^\nu}\\ m\in D}} \frac{1}{m} \le \\
&\le o(1)\frac{c_\varepsilon x}{\log x}\log x = o(1),
\end{aligned}
$$

where we use (5.5) with $y = \dfrac{x^{1-\varepsilon}}{2^{\nu+1}}$. Hence, the contribution of those $n = pm \le x$ for which $m \in D$ to the sum in (5.3) is $o(x)$ as $x \to \infty$.

It remains to consider case (b), that is when $A(m) \ne 0$. First, we set $\tau = x/(\log x)^{30}$. Then, there exists a sequence of rational numbers $(a_m/q_m)_{m\ge1}$ such that

$$
\left|A(m) - \frac{a_m}{q_m}\right| \le \frac{1}{q_m\tau} \qquad (m = 1, 2, \ldots), \tag{5.8}
$$

where $1 \le q_m \le \tau$ for each integer $m \ge 1$.

If $q_m > \log^{40} x$, arguing as in [1], we obtain that

$$
S_A(m) \ll \frac{x/m}{\log^2(x/m)},
$$

so that

$$
\sum_{\substack{m\le x^{1-\varepsilon}\\ m\notin D}} e(B(m))S_A(m) = o(x). \tag{5.9}
$$

On the other hand,

$$
\sum_{\substack{m\le x^{1-\varepsilon}\\ m\notin D}} e(B(m))S_B(m) \ll \sum_{\substack{mP(m)\le x\\ m\le x^{1-\varepsilon}}} \frac{P(m)}{\log P(m)} = o(x), \tag{5.10}
$$

where the fact that this last sum is $o(x)$ was proved in our 2005 paper [4]). Thus, combining (5.9) and (5.10) shows that the contribution of those $n = pm \le x$ for which $m \notin D$ to the sum in (5.3) is $o(x)$ as $x \to \infty$.

On the other hand, if $q_m \le \log^{40} x$, then it follows from (5.8) that

$$\left|\alpha - \frac{a_m}{q_m(k\sigma(n)+\ell\phi(n))}\right| < \frac{1}{q_m(k\sigma(n)+\ell\phi(n))\tau}.$$

Setting

$$\frac{a_m}{q_m(k\sigma(n)+\ell\phi(n))} := \frac{A}{Q}, \quad (A,Q)=1,$$

it is clear that

$$Q < (\log x)^{40}\,(|k|\log x + |\ell|)\,x^{1-\varepsilon} < x^{1-\varepsilon/2},$$

provided x is large enough. Using this and (5.8), we may conclude that, for some function $\delta_x \to 0$ as $x \to \infty$, we have

$$\|Q\alpha\| Q^{1+\varepsilon/4} \le \delta(x),$$

thus contradicting our assumption (2.3). This fully establishes (5.3) and thereby completes the proof of Theorem 3.

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