ON THE EQUATION $f(n^2 + Dm^2) = f(n)^2 + Df(m)^2$

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Abstract. Let $D = 2$ or 3, $E := \{n^2 + Dm^2 | n, m \in \mathbb{N}\}$, $\epsilon(n) = 1$ if $n \in E$ and $\epsilon(n) \in \{-1, 1\}$ if $n \in \mathbb{N} \setminus E$. Let $f : \mathbb{N} \to \mathbb{C}$ be such a function for which

$$f(n^2 + Dm^2) = f(n)^2 + Df(m)^2 \quad \text{for every} \quad n, m \in \mathbb{N}.$$

Then either $f(n) = 0$, or $f(n) = \dfrac{\epsilon(n)}{D+1}$, or $f(n) = \epsilon(n)n$ for every $n \in \mathbb{N}$.

1. Introduction

Let, as usual, $\mathcal{P}$, $\mathbb{N}$, $\mathbb{C}$ be the set of primes, positive integers and complex numbers, respectively.

Let us consider an arithmetical function $f : \mathbb{N} \to \mathbb{C}$ satisfying the Cauchy's functional equation

$$f(n + m) = f(n) + f(m) \quad \text{for every} \quad n, m \in \mathbb{N}.$$

It is obvious that $f(n) = nf(1)$ holds for all $n \in \mathbb{N}$.

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In 1992, C. Spiro [9] proved that if a multiplicative function $f:\mathbb{N}\to\mathbb{C}$ satisfies the relations

$$f(p_0)\neq 0 \quad \text{for some} \quad p_0\in\mathcal{P}$$

and

$$f(p+q)=f(p)+f(q) \quad \text{for every} \quad p,q\in\mathcal{P},$$

then $f(n)=n$ for all $n\in\mathbb{N}$.

In 1997 J.-M. De Koninck, I. Kátai and B. M. Phong [4] proved that if a multiplicative function $f:\mathbb{N}\to\mathbb{C}$ satisfies the relation

$$f(p+n^2)=f(p)+f(n^2) \quad \text{for every} \quad p\in\mathcal{P},\ n\in\mathbb{N},$$

then f is the identity function. K.-H. Indlekofer and B. M. Phong [5] proved that if $k\in\mathbb{N}$, $f\in\mathcal{M}$ satisfy $f(2)f(5)\neq 0$ and $f\left(n^2+m^2+k+1\right)=f(n^2+ +1)+f(m^2+k)$ for all $n,m\in\mathbb{N}$, then $f(n)=n$ for all $n\in\mathbb{N}$, $(n,2)=1$.

For some generalizations of the above results, we refer the other works of P. V. Chung [2], B. M. Phong [6], [7], [8].

Let $D\in\mathbb{N}$. We are interested in all solutions of those $f:\mathbb{N}\to\mathbb{C}$ for which

$$f(n^2+Dm^2)=f(n)^2+Df(m)^2 \quad \text{for every} \quad n,m\in\mathbb{N}. \tag{1.1}$$

In the case $D=1$, the solutions of (1.1) were given in [1]. I. Kátai and B. M. Phong posed the following conjecture:

Conjecture. (I. Kátai and B. M. Phong [3]) *Assume that the arithmetical function $f:\mathbb{N}\to\mathbb{C}$ satisfies (1.1). Then one of the following assertions holds:*

$$\begin{aligned} &a) \quad f(n)=0 \quad \textit{for every} \quad n\in\mathbb{N},\\ &b) \quad f(n)=\frac{\epsilon(n)}{D+1} \quad \textit{for every} \quad n\in\mathbb{N},\\ &c) \quad f(n)=\epsilon(n)n \quad \textit{for every} \quad n\in\mathbb{N}, \end{aligned}$$

where $E:=\{n^2+Dm^2|n,m\in\mathbb{N}\}$, $\epsilon(n)=1$ if $n\in E$ and $\epsilon(n)\in\{-1,1\}$ if $n\in\mathbb{N}\setminus E$.

Our purpose in this note is to prove this conjecture for $D=2$ and $D=3$.

Theorem 1. *The conjecture is true for $D=2$.*

Theorem 2. *The conjecture is true for $D=3$.*

2. Proof of Theorem 1.

In this section, we assume that $D=2$ and $f:\mathbb{N}\to\mathbb{C}$ satisfies

$$f(n^2+2m^2)=f(n)^2+2f(m)^2 \quad \text{for every} \quad n,m\in\mathbb{N}. \tag{2.1}$$

First we prove the following

Lemma 1. *Let*

$$S_n := f(n)^2 \quad \textit{for every} \quad n\in\mathbb{N}.$$

Then

$$S_k = Ak^2+Bk+C(k) \quad \textit{for every} \quad k\in\mathbb{N}, \tag{2.2}$$

where

$$A := \frac{1}{4}(S_4-S_3-S_2+S_1), B := \frac{1}{2}(-2S_4+3S_3+2S_2-3S_1)$$

and

$$C(k) := \frac{1}{8}[(7S_4-13S_3-3S_2+17S_1)+(S_4-3S_3+3S_2-S_1)(-1)^k].$$

Proof. Since

$$(n+4)^2+2(n+1)^2=n^2+2(n+3)^2 \quad \text{for every} \quad n\in\mathbb{N},$$

we infer from (2.1) that

$$S_{n+4}=2S_{n+3}-2S_{n+1}+S_n \quad \text{for every} \quad n\in\mathbb{N}. \tag{2.3}$$

Assume that A, B and $C(k)$ are defined in Lemma 1. Then we have

$$C(k)=\begin{cases} \frac{1}{4}(3S_4-5S_3-3S_2+9S_1) & \text{if} \quad 2\nmid k \\ S_4-2S_3+2S_1 & \text{if} \quad 2\mid k \end{cases} \tag{2.4}$$

and it is easy to check that

$$A+B+C(1)=S_1, 4A+2B+C(2)=S_2,$$

and

$$9A+3B+C(3)=S_3, 16A+4B+C(4)=S_4.$$

These prove that (2.2) holds for $k\le 4$.

Assume that (2.2) holds for $k = n, n+1, n+2, n+3$, where $n \geq 1$. Then we get from (2.3) and our assumptions that

$$\begin{aligned}
&S_{n+4} = 2S_{n+3} - 2S_{n+1} + S_n = 2[A \cdot (n+3)^2 + B \cdot (n+3) + C(n+3)]- \\
&- 2[A \cdot (n+1)^2 + B \cdot (n+1) + C(n+1)] + [An^2 + Bn + C(n)] = \\
&= A[2(n+3)^2 - 2(n+1)^2 + n^2] + B[2(n+3) - 2(n+1) + n] + C(n+4) = \\
&= A \cdot (n+4)^2 + B \cdot (n+4) + C(n+4).
\end{aligned}$$

Here, we have used (2.4) to get $C(n+3) = C(n+1)$ and $C(n) = C(n+4)$. Thus, we proved that (2.2) is true for $k = n+4$ and the proof of Lemma 1 is complete. ■

Lemma 2. *One of the following holds:*

(2.5) $$f(n) = 0 \quad \text{for every} \quad n \in \mathbb{N},$$

(2.6) $$S_n = f(n)^2 = \frac{1}{9} \quad \text{for every} \quad n \in \mathbb{N},$$

(2.7) $$S_n = f(n)^2 = n^2 \quad \text{for every} \quad n \in \mathbb{N}.$$

Proof. Let $S_1 = f(1)^2 := a$ and $S_2 = f(2)^2 := b$.

It follows from (2.1) that if $n = u^2 + 2v^2$, then

$$f(n) = f(u^2 + 2v^2) = f(u)^2 + 2f(v)^2 = S_u + 2S_v,$$

consequently

(2.8) $$S_n = (S_u + 2S_v)^2 \quad \text{if} \quad n = u^2 + 2v^2.$$

Since $(n, u, v) \in \{3,1,1), (6,2,1), (9,1,2), (11,3,1)\}$ satisfies the equation $n = u^2 + 2v^2$, we get from (2.8) that

(2.9) $$\begin{cases} S_3 = (S_1 + 2S_1)^2 = 9a^2, \\ S_6 = (S_2 + 2S_1)^2 = (b + 2a)^2, \\ S_9 = (S_1 + 2S_2)^2 = (a + 2b)^2, \\ S_{11} = (S_3 + 2S_1)^2 = (9a^2 + 2a)^2. \end{cases}$$

It is obvious from (2.1) that if $x^2 + 2y^2 = z^2 + 2t^2$ then

$$S_x + 2S_y = S_z + 2S_t.$$

Consequently, the relations $5^2+2\cdot 1^2=3^2+2\cdot 3^2$, $5^2+2\cdot 2^2=1^2+2\cdot 4^2$ and $7^2+2\cdot 1^2=1^1+2\cdot 5^2$ imply

$$(2.10)\qquad \begin{cases} S_5=S_3+2S_3-2S_1=27a^2-2a,\\ S_4=\frac{S_5+2S_2-S_1}{2}=\frac{27}{2}a^2-\frac{3}{2}a+b,\\ S_7=-S_1+2S_5=54a^2-5a. \end{cases}$$

Thus, by using (2.9)-(2.10), we infer from the relations $6^2+2\cdot 3^2=2^2+2\cdot 5^2$, $9^2+2\cdot 3^2=1^1+2\cdot 7^2$ and $1^2+2\cdot 11^2=9^2+2\cdot 9^2$ that

$$(2.11)\qquad (8a+b-1)(4a-b)=S_6+2S_3-(S_2+2S_5)=0,$$

$$(2.12)\qquad -89a^2+9a+4ba+4b^2=S_9+2S_3-(S_7+2S_5)=0$$

and

$$(2.13)\qquad a(a-1)(9a-1)(9a+14)=S_{11}+2S_1-(S_5+2S_7)=0.$$

From (2.11) we have

$$b\in\{1-8a,\ 4a\}.$$

Case I: $b=1-8a$

First we prove that $a=b=\frac{1}{9}$. From (2.12), we have

$$-89a^2+9a+4ba+4b^2=-89a^2+9a+4(1-8a)a+4(1-8a)^2=(9a-1)(15a-4)=0.$$

This relation with (2.13) proves that $a=\frac{1}{9}$ and $b=1-8a=1-\frac{8}{9}=\frac{1}{9}$.

Finally, the above relations imply

$$S_1=a=\frac{1}{9}, S_2=b=\frac{1}{9}, S_3=9a^2=\frac{1}{9}, S_4=\frac{1}{2}(7a^2-3a+2b)=\frac{1}{9}.$$

It is easy to check that in this case we have

$$A=B=0$$

and

$$\begin{aligned} C(k)=&\frac{1}{8}[(7S_4-13S_3-3S_2+17S_1)+(S_4-3S_3+3S_2-S_1)(-1)^k]=\\ &=\frac{1}{72}[(7-13-3+17)+(1-3+3-1)(-1)^k]=\frac{1}{9}. \end{aligned}$$

The proof of (2.6) follows from (2.2) of Lemma 1.

Case II: $b = 4a$

We obtain from (2.12) that

$$-89a^2 + 9a + 4ba + 4b^2 = -89a^2 + 9a + 4(4a)a + 4(4a)^2 = -9a(a-1) = 0.$$

This implies

$$a \in \Big\{0, 1\Big\}.$$

If $a = 0$, then $b = 4a = 0$. Then $S_1 = a = 0$, $S_2 = b = 0$, $S_3 = 9a^2 = 0$. By (2.10) we have $S_4 = \frac{1}{2}(27a^2 - 3a + 2b) = 0$, and so

$$A = B = C(k) = 0 \quad \text{for every} \quad k \in \mathbb{N}.$$

It follows from (2.2) that $S_n = f(n)^2 = 0$ for all $n \in \mathbb{N}$, which proves (2.5).

Finally we consider the case $a = 1$. Then we have

$$S_1 = a = 1^2, S_2 = b = 4a = 2^2, S_3 = 9a^2 = 3^2, S_4 = \frac{1}{2}(27 - 3 + 8) = 4^2$$

and

$$A = \frac{1}{4}(S_4 - S_3 - S_2 + S_1) = \frac{1}{4}(4^2 - 3^2 - 2^2 + 1) = 1,$$

$$B := \frac{1}{2}(-2S_4 + 3S_3 + 2S_2 - 3S_1) = \frac{1}{2}(-2 \cdot 4^2 + 3 \cdot 3^2 + 2 \cdot 2^2 - 3) = 0$$

and

$$C(k) := \frac{1}{8}[(7S_4 - 13S_3 - 3S_2 + 17S_1) + (S_4 - 3S_3 + 3S_2 - S_1)(-1)^k] =$$
$$= \frac{1}{8}[(7 \cdot 4^2 - 13 \cdot 3^2 - 3 \cdot 2^2 + 17) + (4^2 - 3 \cdot 3^2 + 3 \cdot 2^2 - 1)(-1)^k] = 0$$

for all $k \in \mathbb{N}$.

Thus we get from (2.2) that

$$S_k = Ak^2 + Bk + C(k) = k^2 \quad \text{for every} \quad k \in \mathbb{N}.$$

The proof of (2.7) and of Lemma 2 is complete. ■

Theorem 1 follows from Lemma 2, because from (2.6)–(2.7) it follows that

$$f(n^2 + 2m^2) = f(n)^2 + 2f(m)^2 = \begin{cases} \frac{1}{3} & \text{if} \quad f(k)^2 = \frac{1}{9} \ (\forall k \in \mathbb{N}) \\ n^2 + 2m^2 & \text{if} \quad f(k)^2 = k^2 \ (\forall k \in \mathbb{N}). \end{cases}$$

3. Proof of Theorem 2.

In this section, we assume that $D=3$ and $f:\mathbb{N}\to\mathbb{C}$ satisfies

$$f(n^2+3m^2)=f(n)^2+3f(m)^2 \quad \text{for every} \quad n,m\in\mathbb{N}. \tag{3.1}$$

First we prove the following

Lemma 3. *Let*

$$S_n := f(n)^2 \quad \textit{for every} \quad n\in\mathbb{N}.$$

Then

$$\begin{cases} S_2 &= \frac{4}{5}S_1+\frac{16}{5}S_1^2,\\ S_3 &= \frac{7}{15}S_1+\frac{128}{15}S_1^2,\\ S_4 &= 16S_1^2,\\ S_5 &= -\frac{3}{5}S_1+\frac{128}{5}S_1^2,\\ S_6 &= -\frac{4}{3}S_1+\frac{112}{3}S_1^2 \end{cases} \tag{3.2}$$

and

$$S_1\in\{0,\frac{1}{16},1\}. \tag{3.3}$$

Proof. It is obvious from (3.1) that if $x^2+3y^2=z^2+3t^2$ then

$$S_x+3S_y=S_z+3S_t.$$

Consequently, the relations $4^2+3\cdot 2^2=1^2+3\cdot 3^2$, $5^2+3\cdot 1^2=1^2+3\cdot 3^2$ and $6^2+3\cdot 4^2=3^2+3\cdot 5^2$ imply

$$\begin{cases} S_4 &= S_1-3S_2+3S_3\\ S_5 &= -2S_1+3S_3\\ S_6 &= -3S_4+S_3+3S_5=-9S_1+9S_2+S_3 \end{cases} \tag{3.4}$$

From (3.4) and from $5^2+3\cdot 3^2=2^2+3\cdot 4^2$, we get

$$0=S_5+3S_3-S_2-3S_4=(-2S_1+3S_3)+3S_3-S_2-3(S_1-3S_2+3S_3)=$$
$$=-5S_1+8S_2-3S_3,$$

which gives

$$S_3=\frac{-5S_1+8S_2}{3} \quad \text{and} \quad S_4=S_1-3S_2+3S_3=-4S_1+5S_2. \tag{3.5}$$

Finally, we infer from (3.5) and from the fact $f(4) = f(1^2 + 3 \cdot 1^2) = 4S_1$ that

$$S_4 = (f(4))^2 = 16S_1^2 \quad \text{and} \quad 16S_1^2 + 4S_1 - 5S_2 = 0.$$

This implies

$$S_2 = \frac{4S_1 + 16S_1^2}{5}. \tag{3.6}$$

Therefore the proof of (3.2) follows from (3.4)-(3.6).

Now we prove (3.3). It follows from the relations $7^2 + 3 \cdot 3^2 = 1^2 + 3 \cdot 5^2$, $12^2 + 3 \cdot 2^2 = 3^2 + 3 \cdot 7^2$ that

$$S_7 = S_1 - 3S_3 + 3S_5 = -\frac{11}{5}S_1 + \frac{256}{5}S_1^2$$

and

$$S_{12} = -3S_2 + S_3 + 3S_7 = -\frac{128}{15}S_1 + \frac{2288}{15}S_1^2.$$

But

$$f(7) = f(2^2 + 3 \cdot 1^2) = S_2 + 3S_1 = \frac{19}{5}S_1 + \frac{16}{5}S_1^2$$

and

$$f(12) = f(3^2 + 3 \cdot 1^2) = S_3 + 3S_1 = \frac{52}{15}S_1 + \frac{128}{15}S_1^2.$$

We get the following two equations

$$-\frac{1}{25}S_1(S_1 - 1)(16S_1 - 1)(16S_1 + 55) = S_7 - \Big(\frac{19}{5}S_1 + \frac{16}{5}S_1^2\Big)^2 = 0$$

and

$$-\frac{128}{225}S_1(S_1 - 1)(16S_1 - 1)(8S_1 + 15) = S_{12} - \Big(\frac{52}{15}S_1 + \frac{128}{15}S_1^2\Big)^2 = 0.$$

These show that (3.3) is true. Lemma 3 is proved. ∎

Proof of Theorem 2.

Since

$$(n + 6)^2 + 3(n + 2)^2 = n^2 + 3(n + 4)^2 \quad \text{for every} \quad n \in \mathbb{N},$$

we infer from (3.1) that

$$S_{n+6} = 3S_{n+4} - 3S_{n+2} + S_n \quad \text{for every} \quad n \in \mathbb{N}. \tag{3.7}$$

We shall prove that

$$S_n = \frac{1}{15} S_1(16S_1 - 1)n^2 - \frac{16}{15} S_1(S_1 - 1) \quad \text{for every} \quad n \in \mathbb{N}. \tag{3.8}$$

By using (3.2), one can check that (3.8) holds for $n \in \{1, 2, \cdots, 6\}$. Let

$$A := \frac{1}{15} S_1(16S_1 - 1) \quad \text{and} \quad B := -\frac{16}{15} S_1(S_1 - 1). \tag{3.9}$$

Assume that $S_n = An^2 + B$ holds for $k = n, n+1, n+2, n+3, n+4, n+5$, where $n \geq 1$. Then we get from (3.7) and our assumptions that

$$\begin{aligned} S_{n+6} &= 3S_{n+4} - 3S_{n+2} + S_n = \\ &= 3[A \cdot (n+4)^2 + B] - 3[A \cdot (n+2)^2 + B] + [A \cdot n^2 + B] = \\ &= A \cdot (n+6)^2 + B. \end{aligned}$$

Thus, (3.8) is proved.

From (3.3), we have $S_1 \in \{0, \frac{1}{16}, 1\}$.

If $S_1 = 0$, then from (3.8)-(3.9) we have $A = B = 0$ and $S_n = 0$ for all $n \in \mathbb{N}$. Consequently $f(n) = 0$ for all $n \in \mathbb{N}$.

If $S_1 = \frac{1}{16}$, then from (3.8)-(3.9) we have $A = 0, B = \frac{1}{16}$ and $S_n = \frac{1}{16}$ for all $n \in \mathbb{N}$. Therefore $f(n)^2 = \frac{1}{16}$ and

$$f(n^2 + 3m^2) = f(n)^2 + 3f(m)^2 = \frac{1}{4}$$

for all $n, m \in \mathbb{N}$, which proves Theorem 2.

If $S_1 = 1$, then from (3.8)-(3.9) we have $A = 1, B = 0$ and $S_n = n^2$ for all $n \in \mathbb{N}$. In this case we also have $f(n)^2 = n^2$ and

$$f(n^2 + 3m^2) = f(n)^2 + 3f(m)^2 = n^2 + 3m^2$$

for all $n, m \in \mathbb{N}$, from which the proof of Theorem 2 is completed. ■

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