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SOME RELATIONS AMONG ARITHMETICAL FUNCTIONS

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Abstract. We consider some possible relations among q-additive and completely multiplicative functions. We proved that if f is completely multiplicative and q-additive function, then either $f(n) = n$ for every $n \in \mathbb{N}$ or $f(n)$ is the Dirichlet character $\pmod{q_0}$, where $q_0 | q$.

1. Notation and some notions

Let, as usual, $\mathcal{P}$, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{C}$ be the set of primes, positive integers, integers, rational numbers and complex numbers, respectively. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ be the set of non-negative integers.

Let $\mathcal{A}$ (resp. $\mathcal{A}^*$) be the class of additive (completely additive) functions, $\mathcal{M}$ (resp. $\mathcal{M}^*$) be the class of multiplicative (completely multiplicative) functions.

For some integer $q \geq 2$ let $\mathcal{A}_q$ be the set of q-additive function. Every $n \in \mathbb{N}_0$ can be uniquely represented in the form

$$n = \sum_{r=0}^{\infty} a_r(n) q^r \quad \text{with} \quad a_r(n) \in \{0, 1, ..., q-1\} (= A_q)$$

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and $a_r(n) = 0$ if $q^r > n$. We say that $f \in \mathcal{A}_q$, if $f : \mathbb{N}_0 \to \mathbb{C}$

$$f(0) = 0 \quad \text{and} \quad f(n) = \sum_{r=0}^{\infty} f\Big(\epsilon_r(n)q^r\Big) \quad \text{for every} \quad n \in \mathbb{N}.$$

These functions were first introduced and studied by A. O. Gelfond [6].

Definition 1. (Set of uniqueness for completely additive functions.) We say that $A \subseteq \mathbb{N}$ is a *set of uniqueness for completely additive functions* if $f \in \mathcal{A}^*$, $f(a) = 0$ for every $a \in A$ implies that $f(n) = 0$ for all $n \in \mathbb{N}$.

Definition 2. (Set of uniqueness for completely additive functions (mod 1).) We say that $B \subseteq \mathbb{N}$ is a *set of uniqueness for completely additive functions* (mod 1) if $f \in \mathcal{A}^*$, $f(b) \equiv 0 \pmod 1$ for all $b \in B$ implies that $f(n) \equiv 0 \pmod 1$ for every $n \in \mathbb{N}$.

D. Wolke [16] proved that A is a set of uniqueness if and only if every $n \in \mathbb{N}$ can be written as $n = \prod_{i=1}^{k} a_i^{r_i}$, where $a_i \in A$ and $r_i \in \mathbb{Q}$.

K.-H. Indlekofer [8], [9], P. Hoffman [7], F. Dress and B. Volkmann [2] proved independently that B is a set of uniqueness for completely additive functions (mod 1) if and only if every $n \in \mathbb{N}$ can be written as

$$n = \prod_{j=1}^{k} b_j^{\ell_j}, \quad \text{where} \quad \ell_j \in \mathbb{Z},\ b_j \in B.$$

I. Kátai [11], [12] formulated the conjecture in 1969 that

$$\mathcal{P}_{+1} = \{p + 1 \mid p \in \mathcal{P}\}$$

is a set of uniqueness for additive functions, and proved that there exists such a finite set Q of primes for which $\mathcal{P}_{+1} \cup Q$ is a set of uniqueness (mod 1). P. D. T. A. Elliott [3] proved that $Q = \{p \mid p \le 10^{387}, p \in \mathcal{P}\}$ is an appropriate choice, that is every $n \in \mathbb{N}$ can be written as

$$n = t \cdot \prod_{i=1}^{k} (p_i + 1)^{\epsilon_i}, \quad \epsilon_i \in \{-1, 1\},$$

and t is such a rational number the largest prime factor of which does not exceed 10^{387}.

Furthermore, in [4] he proved that for every rational number r, we can find such primes $p_1, \cdots, p_k$ and $\epsilon_j \in \{-1, 1\}$ $(j = 1, 2, \cdots, k)$, for which

$$r^g = \prod_{i=1}^{k} (p_i + 1)^{\epsilon_i}. \tag{1.1}$$

Here g is a constant, $g \in \{1, 2, 3\}$.

A direct consequence of this assertion is

Theorem 1. *Let* $f \in \mathcal{M}^*$, $f(p+1) = p+1 \quad (\forall p \in \mathcal{P})$. *Then*

$$f(n) = nH(n), \quad H \in \mathcal{M}^*, \quad H(n)^g = 1 \quad \textit{for every} \quad n \in \mathbb{N}.$$

Especially, if $f(n)$ *is a positive real number for every* $n \in \mathbb{N}$, *then*

$$f(n) = n \quad \textit{for every} \quad n \in \mathbb{N}.$$

Proof. Since, for every $q \in \mathcal{P}$ there is at least one $p \in \mathcal{P}$ such that $q | p+1$, therefore $f(q) \neq 0$, and so $f(n) \neq 0 \quad (n \in \mathbb{N})$. From (1.1) we obtain that

$$f(r)^g = \prod_{i=1}^{k} f(p_i+1)^{\epsilon_i} = \prod_{i=1}^{k} (p_i+1)^{\epsilon_i} = r^g.$$

Thus

$$1 = \left(\frac{f(r)}{r}\right)^g = H(r)^g \quad \text{for every positive rational number.}$$

The proof of Theorem 1 is complete. ■

The conjecture that $\mathcal{P}_{+1}$ is a set of uniqueness (mod 1) is formulated by several mathematicians. It would follow from the conjecture of A. Schinzel and W. Sierpinnski [14], namely that every positive integer has infinitely many representations of the form $\frac{p+1}{q+1} \quad (p, q \in \mathcal{P})$.

T. Csajbók, A. Járai and J. Kasza [1] proved that every prime $Q \in [2, 10^{14}]$ can be written as $\frac{p+1}{q+1} \quad (p, q \in \mathcal{P})$, and every $n \in [2, 10^{11}]$ can be written also as $\frac{p+1}{q+1} \quad (p, q \in \mathcal{P})$.

2. On q-additive functions

Definition 3. (Set of uniqueness for q-additive functions.) We say that $D \subseteq \mathbb{N}_0$ is a *set of uniqueness for the set of q-additive functions*, if $f \in \mathcal{A}_q$, $f(d) = 0$ for all $d \in D$ implies that $f(n) = 0$ for all $n \in \mathbb{N}_0$.

The functions $f(n) = cn$ belong to $\mathcal{A}_q$ for every $q \geq 2$. J. C. Puchta and J. Spilker [13] gave all the functions belonging to $\mathcal{A}_{q_1} \cap \mathcal{A}_{q_2}$.

The following question seems to be interesting. Let $(q_1, q_2) = 1$, $q_1, \ q_2 \geq 2$. Let $\mathcal{K}$ be such a subset of $\mathbb{N}_0$ for which the following assertion holds:

If

$$f_1 \in \mathcal{A}_{q_1}, f_2 \in \mathcal{A}_{q_2} \quad \textit{and} \quad f_1(k) = f_2(k) \quad \textit{for every} \quad k \in \mathcal{K},$$

then

$$f_1(n) = f_2(n) = cn \quad \textit{for every} \quad n \in \mathbb{N}_0,$$

where $c \in \mathbb{C}$ is a suitable number.

Assume that $q_2 > q_1 \geq 2$,

$$E_1 = \{aq_1^n | a = 1, \cdots, q_1 - 1, n = 0, 1, \cdots\}$$

and

$$E_2 = \{bq_2^m | b = 1, \cdots, q_2 - 1, m = 0, 1, \cdots\}.$$

Let $bq_2^m \in E_2$ with $m \geq 1$. Let $L(bq_2^m) = aq_1^n$ be the largest element of E_1, for which

$$aq_1^n < bq_2^m.$$

Observe that $aq_1^n = bq_2^m$ or $(a+1)q_1^n = bq_2^m$ cannot occur. Let

$$J_{bq_2^m} = \left(bq_2^m, (a+1)q_1^n\right), \quad \text{where} \quad L(bq_2^m) = aq_1^n.$$

It is obvious that $a + 1 = q_1$ can be occur.

We know that the interval $J_{bq_2^m}$ is quite a large interval. This follows from an important theorem of R. Tijdeman [15], which is stated now as

Theorem A. *Let p be a prime, $p \geq 3$ and let $1 = n_1 < n_2 < \cdots$ be the sequence of all positive integers composed of primes $\leq p$. Then there exists an effectively computable constant $C = C(p)$ such that*

$$n_{i+1} - n_i > \frac{n_i}{(\log n_i)^C} \quad \textit{for every} \quad n_i \geq 3.$$

Corollary. *Let p be the largest prime factor of* $q_2!$, $C = C(p)$ *(defined in Theorem A). Let $\mathcal{K}$ be such a set for which*

$$\mathcal{K} \cap J_{bq_2^m} \neq \emptyset \quad \textit{if} \quad bq_2^m > K.$$

Assume that $g_1 \in \mathcal{A}_{q_1}$, $g_2 \in \mathcal{A}_{q_2}$,

$$g_1(n) = g_2(n) \quad \textit{if} \quad n \in \mathcal{K}$$

and

$$g_1(n) = g_2(n) = cn \quad \textit{for every} \quad n \leq K \quad (K > q_2).$$

Then

$$g_1(n) = g_2(n) = cn \quad \textit{for every} \quad n \in \mathbb{N}_0.$$

Proof. We can see it by using induction. ■

Assume that $g_1(k) = g_2(k) = ck$ holds for every $k < bq_2^m$. Let $k < (a+1)q_1^n$. Since

$$g_1(uq_1^\nu) = \begin{cases} cuq_1^\nu & \text{for} \quad \nu < n \quad \text{and} \quad u \in \{1, \cdots, q_1 - 1\} \\ cuq_1^n & \text{for every} \quad u \leq a, \end{cases}$$

therefore

$$g_1(k) = ck \quad \text{if} \quad k < (a+1)q_1^n.$$

Let $\kappa \in \mathcal{K} \cap J_{bq_2^m}$. Thus

$$g_1(\kappa) = g_2(\kappa) = c\kappa, \kappa = bq_2^m + h, h < q_2^m,$$

and so

$$g_2(\kappa) = g_2(bq_2^m) + g_2(h) = g_2(bq_2^m) + ch.$$

Consequently

$$g_2(bq_2^m) = cbq_2^m.$$

Remark. Let $\mathcal{E} = \{a_1 < a_2 < \cdots\}$ be such a sequence of integers for which $a_{n+1} - a_n < a_n^{1-\epsilon}$ for some $\epsilon > 0$. Then $\mathcal{E}$ is a $\mathcal{K}$-sequence, that is, if K is a suitable constant, $f_1 \in \mathcal{A}_{q_1}$, $f_2 \in \mathcal{A}_{q_2}$, and

$$f_1(n) = f_2(n) \quad (\forall n < K) \quad \text{and} \quad f_1(a_j) = f_2(a_j) \quad (j = 1, 2, \cdots),$$

then

$$f_1(n) = f_2(n) = c \cdot n \quad \text{for every} \quad n \in \mathbb{N}_0.$$

Specially, if $Q(x) \in \mathbb{Z}[x]$,

$$\mathcal{E}_1 = \{Q(n) | n = 1, 2, \cdots\} \quad \text{and} \quad \mathcal{E}_2 = \{Q(p) | p \in \mathcal{P}\},$$

then both of $\mathcal{E}_1$ and $\mathcal{E}_2$ are $\mathcal{K}$-sequences.

3. What are the q-additive multiplicative functions?

In this section let $I(n)$ be the identity function and $\chi_q(n)$ be a Dirichlet character function. It is clear that

$$\{I, \chi_q\} \subseteq \mathcal{M}^* \cap \mathcal{A}_q.$$

We shall prove the following

Theorem 2. *Let $\mathcal{M}^*$ the set of all completely multiplicative functions f with $f(1) = 1$. Then there exists a $q_0|q$ such that*

$$\mathcal{M}^* \cap \mathcal{A}_q \subseteq \{I, \chi_{q_0}\}.$$

Proof. Let $f \in \mathcal{M}^* \cap \mathcal{A}_q$. First we consider the case $f(q) = 0$. Let $q = q_0 q_1$, where $f(q_1) \neq 0$ and $f(p) = 0$ for all primes $p, p|q_0$. It is clear that

$$f(q_1)f(q_0 m + 1) = f(qm + q_1) = f(q_1) + f(qm) = f(q_1) + f(q)f(m) = f(q_1)$$

for every $m \in \mathbb{N}$. This implies that $f(q_0 m + 1) = 1$, which with Lemma 19.3 of [5] proves that $f = \chi_{q_0}$.

Now we assume that $\xi := f(q) \neq 0$. For each $a \in \{0, 1, \cdots q - 2\}$, we have

$$f(q + 1) = f(1) + f(q) = 1 + \xi, \quad f(q + a) = f(a) + f(q) = f(a) + \xi$$

and

$$(q + 1)(q + a) = q^2 + (a + 1)q + a.$$

Thus, we infer that

$$f\Big((q + 1)(q + a)\Big) = f\Big(q^2 + (a + 1)q + a\Big) =$$

$$= f(q^2) + f\Big((a + 1)q\Big) + f(a) = \xi^2 + f(a + 1)\xi + f(a),$$

consequently

$$(1 + \xi)(f(a) + \xi) = \xi^2 + f(a + 1)\xi + f(a).$$

This shows that

$$f(a + 1) = f(a) + 1 \quad \text{for} \quad a \in \{0, 1, \cdots q - 2\},$$

and so

$$f(m) = m \quad \text{for} \quad m \in \{0, 1, \cdots q - 1\}.$$

Since $f \in \mathcal{A}_{q^2}$, therefore the above result shows that

$$f(M) = M \quad \text{for every} \quad M < q^2,$$

and so $\xi = f(q) = q$, because $q < q^2$.

Theorem 2 is proved. ■

The following problem seems to be interesting. Characterize those subsets $\mathcal{D}$ of $\mathbb{N}_0$ for which if $f \in \mathcal{M}^*$, $g \in \mathcal{A}_q$, and

$$f(d) = g(d) \quad (\forall d \in \mathcal{D}),$$

then

$$f(n) = g(n) \quad \text{for every} \quad n \in \mathbb{N}.$$

Theorem 3. *Let c, N_0 be given numbers, $J_N = (2^N, 2^{N+1})$. Let $\mathcal{D} \subseteq \mathbb{N}_0$ be such a set of integers for which if $N > N_0$, then there exist $m_1, m_2 \in J_N \cap \mathcal{D}$, $m_1 \neq m_2$ such that*

$$\frac{m_1}{m_2} = \frac{A}{B}, \quad A, B < c, A, B \in \mathbb{N}.$$

Assume that

$$f \in \mathcal{M}^*, \quad g \in \mathcal{A}_2, \quad f(n) = g(n) = n \quad \textit{if} \quad n \leq \max(c, 2^{N_0}).$$

Then

$$f(d) = g(d) \quad \textit{for every} \quad d \in \mathcal{D},$$

implies that

$$f(n) = n \quad \textit{for every} \quad n \in \mathcal{D}.$$

Proof. We shall use induction. Assume that

$$g(2^n) = 2^n \quad \text{for every} \quad n = 0, 1, \cdots, N-1.$$

Then clearly

$$f(u) = g(u) = u \quad \text{if} \quad u < 2^N.$$

Let $m_1 = 2^N + l, m_2 = 2^N + r, 0 < l, r < 2^N$, $l \neq r$. We have

$$g(m_1) = g(2^N) + g(l) = g(2^N) + l,$$

similarly

$$g(m_2) = g(2^N) + g(r) = g(2^N) + r.$$

We can choose m_1, m_2 such that $\frac{m_1}{m_2} = \frac{A}{B}$, $A, B < c$. Thus $m_1 B = m_2 A$, $Bf(m_1) = Af(m_2)$, $f(m_j) = g(m_j)$, and so

$$B\Big(g(2^N) + l\Big) = A\Big(g(2^N) + r\Big).$$

Since $B(2^N + l) = A(2^N + r)$, the above relation implies that $g(2^N) = 2^N$. The proof of Theorem 3 is complete. ■

We guess that the following conjecture is true.

Conjecture 1. *There exist such constants $c > 0$ and N_0 such that if $N \geq N_0$, $N \in \mathbb{N}$, then in the interval $[2^N, 2^{N+1})$ there exist $p-1, q-1$ $(p, q \in \mathcal{P})$ for which*

$$\frac{p-1}{q-1} = \frac{A}{B}, \quad A, B \in \mathbb{N}, A, B < c.$$

Corollary. *Assume that Conjecture 1 is true. Let*

$$f \in \mathcal{M}^*, \quad g \in \mathcal{A}_2, \quad f(n) = g(n) = n \quad \text{if} \quad n \leq \max(c, 2^{N_0}).$$

If

$$f(p-1) = g(p-1) \quad \text{for every} \quad p \in \mathcal{P},$$

then

$$f(p-1) = p - 1 \quad \text{for every} \quad p \in \mathcal{P}.$$

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