Annales Univ. Sci. Budapest., Sect. Comp. 44 (2015) 29–36

SPECTRAL SYNTHESIS ON SPECIAL VARIETIES

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Communicated by Zoltán Daróczy

(Received October 1, 2014; accepted April 16, 2015)

Abstract. Spectral synthesis on varieties deals with the description of translation invariant function spaces on groups. In this paper we show that spectral synthesis holds for a variety, if the factor ring with respect to its annihilator in the group algebra is a Noetherian semi-local ring with exponential maximal ideals.

1. Introduction and preliminaries

In this paper $\mathbb{C}$ denotes the set of complex numbers. If G is an Abelian group, then $\mathcal{C}(G)$ denotes the locally convex topological vector space of all complex valued functions defined on G, equipped with the pointwise operations and the topology of pointwise convergence.

The dual of $\mathcal{C}(G)$ can be identified with $\mathcal{M}_c(G)$, the space of all finitely supported complex measures on G. This space is also identified with the set of all finitely supported complex valued functions on G in the obvious way that the pairing between $\mathcal{C}(G)$ and $\mathcal{M}_c(G)$ is given by the formula

$$\langle \mu, f\rangle = \int f\, d\mu = \sum_{x\in G} f(x)\mu(x)\,.$$

Key words and phrases: Spectral synthesis, Noetherian ring, semi-local ring.
2010 Mathematics Subject Classification: 43A45, 16P40, 39A70.
The Project is supported by the Hungarian National Foundation for Scientific Research (OTKA), Grant No. K111651 and by the University of Botswana Grant "CDU".

Convolution on $\mathcal{M}_c(G)$ is defined by

$$\mu * \nu(x) = \int \mu(x-y)\, d\nu(y)$$

for any μ, ν in $\mathcal{M}_c(G)$ and x in G. Convolution converts the linear space $\mathcal{M}_c(G)$ into a commutative algebra with unit δ_0, 0 being the zero element in G. In general, δ_x is the characteristic function of the singleton $\{x\}$. We realize that $\mathcal{M}_c(G)$ is the so-called *group algebra* of G, hence we shall use the notation $\mathbb{C}G$ for it.

We also define convolution of measures in $\mathbb{C}G$ with arbitrary functions in $\mathcal{C}(G)$ by the same formula

$$\mu * f(x) = \int f(x-y)\, d\mu(y)$$

for each μ in $\mathbb{C}G$, f in $\mathcal{C}(G)$ and x in G.

Translation with the element y in G is the operator mapping the function f in $\mathcal{C}(G)$ to its *translate* $\tau_y f$ defined by $\tau_y f(x) = f(x+y)$ for each x in G. Clearly, $\tau_y f = \delta_{-y} * f$ holds for each function f in $\mathcal{C}(G)$ and y in G. A subset of $\mathcal{C}(G)$ is called *translation invariant*, if it contains all translates of its elements. A closed linear subspace of $\mathcal{C}(G)$ is called a *variety* on G, if it is translation invariant. For each subset in G the smallest variety containing this subset is called the *variety generated* by this subset. In particular, for each function f the variety generated by f is called the *variety of* f and is denoted by $\tau(f)$. It is the intersection of all varieties containing f.

Obviously, $\mathcal{C}(G)$ can be considered as a $\mathbb{C}G$-module with the natural action of $\mathbb{C}G$ on $\mathcal{C}(G)$ defined by $(\mu, f) \mapsto \mu * f$, whenever μ is in $\mathbb{C}G$ and f is in $\mathcal{C}(G)$. Submodules of this module are exactly the translation invariant subspaces and the closed submodules are exactly the varieties. For all subsets J in $\mathbb{C}G$ and H in $\mathcal{C}(G)$ we use the notation JH for the set of all functions $\mu * f$ with μ in J and f in H.

It is very easy to check that the annihilator of each variety on G is an ideal in $\mathbb{C}G$, which is proper if and only if the variety is nonzero. The annihilator of the variety V will be denoted by $V^\perp$. Analogously, for each ideal I in $\mathbb{C}G$ its annihilator is defined by

$$I^\perp = \{f : \, f \in \mathcal{C}(G), \mu(f) = 0 \ \text{ for each } \ \mu \in I\}\,.$$

It follows from [14, Theorem 11.12] and [14, Theorem 11.13], pp. 146–147, that $I^\perp$ is a variety on G, which is nonzero if and only if I is proper.

2. Spectral synthesis and function classes

Spectral analysis and spectral synthesis deals with the description of varieties. It turns out that the basic building bricks of this description are the exponential monomials.

Let G be an Abelian group. The function $m : G \to \mathbb{C}$ is called an *exponential*, if it is a homomorphism into the multiplicative group of nonzero complex numbers. Such functions sometimes are called *generalized characters*. The function $a : G \to \mathbb{C}$ is called *additive*, if if it is a homomorphism into the additive group of complex numbers. The set of all additive functions is a linear space and it is denoted by $\mathrm{Hom}(G, \mathbb{C})$. It is known (see [9]) that the dimension of this linear space is equal to the torsion free rank of G in the sense that either both are infinite, or both are finite and in this case they are equal. Here the *torsion free rank* of an Abelian group is the largest cardinal κ for which G has a free subgroup of rank κ.

Additive functions are special cases of polynomials. The function $p : G \to \mathbb{C}$ is called a *polynomial*, if it has the form

$$p(x) = P\big(a_1(x), a_2(x), \ldots, a_N(x)\big)$$

for each x in G, where N is a positive integer, $P : \mathbb{C}^N \to \mathbb{C}$ is an ordinary polynomial in N variables, and $a_1, a_2, \ldots, a_N : G \to \mathbb{C}$ are additive functions. We note that here we can always suppose that these additive functions are linearly independent.

Using Taylor's Formula it is easy to check that every polynomial $p : G \to \mathbb{C}$ satisfies *Fréchet's Functional Equation*

$$\Delta_{y_1, y_2, \ldots, y_{n+1}} * p = 0 \tag{2.1}$$

for each $y_1, y_2, \ldots, y_{n+1}$ in G with some natural number n. Here the *difference* $\Delta_{y_1, y_2, \ldots, y_{n+1}}$ is the element of $\mathbb{C}G$ defined by

$$\Delta_{y_1, y_2, \ldots, y_{n+1}} = \Pi_{k=1}^{n+1}(\delta_{-y_k} - \delta_0)\,,$$

where the multiplication means convolution. Nevertheless, equation (2.1) does not characterize polynomials – it turns out that, in general, there are functions satisfying (2.1), which are not polynomials. A function $f : G \to \mathbb{C}$ is called a *generalized polynomial*, if it satisfies (2.1) with f in place of p for some natural number n and for each $y_1, y_2, \ldots, y_{n+1}$ in G.

Functions of the form $f = p \cdot m$ are called *exponential monomials*, resp. *generalized exponential monomials*, if m is an exponential and p is a polynomial, resp. generalized polynomial. Linear combinations of exponential monomials, resp. generalized exponential monomials are called *exponential polynomials*, resp. *generalized exponential polynomials*.

We say that *spectral analysis* holds for a variety, if every nonzero subvariety of it contains an exponential. We say that *spectral analysis holds for G*, if spectral analysis holds for every variety on G. We say that a variety is *synthesizable*, if the exponential monomials in this variety span a dense subspace. We say that *spectral synthesis* holds for a variety, if every subvariety of it is synthesizable. We say that *spectral synthesis holds for G*, if every variety on G is synthesizable. Obviously, spectral synthesis for a variety implies spectral analysis for this variety, but the converse is not true.

Abelian groups, for which spectral analysis, resp. spectral synthesis holds have been characterized by their torsion free rank: for spectral analysis on the group it is necessary and sufficient that the torsion free rank is less than the continuum, and for spectral synthesis on the group it is necessary and sufficient that the torsion free rank is finite (see [8], [11]). However, much less is known about spectral analysis and synthesis on particular varieties. A classical result in this direction is due to L. Ehrenpreis in [2], the Principal Ideal Theorem, which states that in the space of infinitely differentiable functions on $\mathbb{R}^n$ spectral synthesis holds for a variety if its annihilator is a principal ideal. A special case of this result is due to B. Malgrange in [1] in the context of the solution space of linear partial differential operators with constant coefficients. This result has been generalized by R. J. Elliott for locally compact Abelian groups in [3]. Another well-known result in another direction is that spectral synthesis holds for finite dimensional varieties. In fact, finite dimensional varieties consist of exponential polynomials (see e.g. [4]). Our main result below can be considered as a generalization of this result, as the residue ring with respect to the annihilator of a finite dimensional variety obviously satisfies the conditions of Theorem 3.2 below. In addition, a negative result in this respect in [15] gives a necessary condition for a variety to have spectral synthesis. In this paper we prove a sufficient condition of this type. Our approach depends on the annihilator technique developed in [16], [14]. For more about polynomials, exponential polynomials and spectral synthesis on Abelian groups the reader is referred to [7], [10], [13], [12], [14].

3. The results

A maximal ideal M in a commutative ring R with unit is called an *exponential* maximal ideal, if R/M is isomorphic to the complex field. It is known that in the case of $R=\mathbb{C}G$ this is the case if and only if M is the annihilator of a variety generated by an exponential. We have the following result (see [16, Theorem 5]).

Theorem 3.1. *Let G be an Abelian group. Then spectral analysis holds for a nonzero variety on G if and only if every maximal ideal including the annihilator of this variety is exponential.*

It is easy to see (see e.g. [14, Theorem 12.5], p. 156.) that for the exponential maximal ideal M there is a unique exponential m such that $M=\tau(m)^{\perp}$, in this case we use the notation $M=M_m$.

We recall that a commutative ring with unit is called *local*, if it has exactly one maximal ideal, and it is called *semi-local*, if it has finitely many maximal ideals (see [6]). Our main result follows.

Theorem 3.2. *Let G be an Abelian group and V a variety on G such that $\mathbb{C}G/V^{\perp}$ is a semi-local ring. If $\mathbb{C}G/V^{\perp}$ is a Noetherian ring with exponential maximal ideals, then V is synthesizable.*

Proof. Suppose that $\mathbb{C}G/V^{\perp}$ is a semi-local Noetherian ring with exponential maximal ideals $M_1,M_2,\dots,M_N$. Let $\Phi:\mathbb{C}G\to\mathbb{C}G/V^{\perp}$ denote the natural homomorphism. Then we have $M_k=\Phi(M_{m_k})$ for some exponentials m_k on G $(k=1,2,\dots,N)$. It follows $V^{\perp}\subseteq M_{m_k}$ for $k=1,2,\dots,N$, which implies that m_k belongs to V, in fact, $m_1,m_2,\dots,m_N$ are the only exponentials in V. By Krull's Intersection Theorem (see e.g. [5, Theorem 7.23], the zero ideal is the intersection of the positive powers of the Jacobson radical of $\mathbb{C}G/V^{\perp}$:

$$\bigcap_{n=0}^{\infty} J^{n+1}=0\,.$$

However, as powers of different maximal ideals are co-prime, we have

$$J=\bigcap_{k=1}^{N} M_k=\prod_{k=1}^{N} M_k\,,$$

consequently, we infer

$$J^{n+1}=\Bigl(\prod_{k=1}^{N} M_k\Bigr)^{n+1}=\prod_{k=1}^{N} M_k^{n+1}=\bigcap_{k=1}^{N} M_k^{n+1}\,,$$

whence

$$V^{\perp}=\bigcap_{n=0}^{\infty}\bigcap_{k=1}^{N} M_{m_k}^{n+1}\,,$$

which means that

$$V = \sum_{n=0}^{\infty}\sum_{k=1}^{N}\big(M_{m_k}^{n+1}\big)^{\perp},$$

by [14, Theorem 11.12] and [14, Theorem 11.15], pp. 146 and 148. As $\big(M_{m_k}^{n+1}\big)^{\perp}$ consists of generalized exponential monomials corresponding to the exponential m_k, it follows that the generalized exponential monomials corresponding to the exponentials m_k for $k = 1, 2, \dots, N$ span a dense subspace in the variety V. Let φ be one of them. Then we have $\tau(\varphi) \subseteq V$, hence $V^{\perp} \subseteq \tau(\varphi)^{\perp}$. The mapping $F : \mathbb{C}G/V^{\perp} \to \mathbb{C}G/\tau(\varphi)^{\perp}$ defined by

$$F(\mu + V^{\perp}) = \mu + \tau(\varphi)^{\perp}$$

for each μ in $\mathbb{C}G$ is well-defined. Indeed, if $\mu + V^{\perp} = \nu + V^{\perp}$, then $\mu - \nu$ is in $V^{\perp}$, hence it is in $\tau(\varphi)^{\perp}$, which implies $\mu + \tau(\varphi)^{\perp} = \nu + \tau(\varphi)^{\perp}$. On the other hand, it is obvious, that F is a surjective homomorphism. It follows that $\mathbb{C}G/\tau(\varphi)^{\perp}$ is a Noetherian ring with exponential maximal ideal. Obviously, $\mathbb{C}G/\tau(\varphi)^{\perp}$ is a local ring, as $\tau(\varphi)$ contains exactly one exponential. This implies that $\tau(f)^{\perp}$ is included in exactly one maximal ideal of the form M_m, where m is one of the exponentials m_k $(k = 1, 2, \dots, N)$. On the other hand, as φ is a generalized exponential monomial corresponding to the exponential m it follows that $\tau(\varphi)$ is annihilated by some power of M_m. In other words, the maximal ideal of the ring $\mathbb{C}G/\tau(\varphi)^{\perp}$, which corresponds to M_m is nilpotent. By Theorem 7.15 in [5], Vol. II. on pp. 426–427. it follows, that $\mathbb{C}G/\tau(\varphi)^{\perp}$ is an Artinian ring, hence, by [14, Theorem 12.11] and [14, Theorem 12.28], pp. 162. and 178, φ is an exponential monomial. It follows that spectral synthesis holds for V and our theorem is proved.

We note that the condition of the theorem is not necessary, even if $\mathbb{C}G/V^{\perp}$ is a local ring. This is shown by the following theorem.

Theorem 3.3. *Let G be an Abelian group with infinite torsion free rank. If V denotes the linear span of all additive functions and the constant functions on G, then V is a variety, spectral synthesis holds for V, and $\mathbb{C}G/V^{\perp}$ is not a Noetherian ring.*

Proof. Let M_1 be the annihilator in $\mathbb{C}G$ of the variety consisting of all constant functions on G. Then, obviously, M_1 is an exponential maximal ideal. Let f be in $(M_1^2)^{\perp}$, then we have for each x, y, z in G:

$$0 = \Delta_{y,z} * f(x) = f(x + y + z) - f(x + y) - f(x + z) + f(x)\,. \tag{3.1}$$

Substituting $x = 0$ we have

$$f(y + z) + f(0) = f(y) + f(z)\,,$$

which implies that $f - f(0)$ is additive, that is, f is in V. Conversely, it is easy to check that every constant function and every additive function satisfies (3.1). It follows that $V = (M_1^2)^\perp$, in particular, V is a variety. As every element of V is a sum of an additive function and a constant, hence spectral synthesis holds for V. Let $\Phi : \mathbb{C}G \to \mathbb{C}G/V^\perp$ denote the natural homomorphism, then $\Phi(M_1)$ is a nilpotent maximal ideal in $\mathbb{C}G/V^\perp$, by the above computation, which means that $\Phi(M_1)^2 = 0$. It follows that $\Phi(M_1)^2$ is the only maximal ideal in $\mathbb{C}G/V^\perp$, that is, $\mathbb{C}G/V^\perp$ is a local ring with maximal ideal. Suppose that it is Noetherian. Then, by [5, Theorem 7.15] cited above, $\mathbb{C}G/V^\perp$ is an Artinian ring. As the torsion free rank of G is infinite, it follows, by [9, Theorem 2], that there are infinitely many linearly independent additive functions on G. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of linearly independent additive functions and let V_n denote the linear span of $1, a_0, a_1, \ldots, a_n$ for each natural number n. Obviously, V_n is a variety, and we have $V_n \subset V_{n+1}$ for each n, where the inclusion is proper. By [14, Theorem 11.12] and [14, Theorem 11.13], pp. 146–147, the chain of ideals $V_0^\perp \supset V_1^\perp \supset \cdots \supset V_n^\perp \supset \ldots$ is strictly descending, which generates a strictly descending chain of ideals $\Phi(V_n^\perp)$ in the ring $\mathbb{C}G/V^\perp$. This contradicts the Artinian property and our theorem is proved.

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