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THE FUNCTIONAL EQUATION $f(\mathcal{A}+\mathcal{B})=g(\mathcal{A})+h(\mathcal{B})$

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Dedicated to Dr. László Szili on his 60th anniversary

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Abstract. We shall investigate the functional equation $f(a+b)=g(a)+$ $+h(b)$ for all $a\in\mathcal{A}, b\in\mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$ are subsets of natural numbers satisfying some condition.

1. Introduction

Let $\mathcal{P},\mathbb{N}$ and $\mathbb{C}$ be the set of primes, positive integers and complex numbers, respectively. For the sets $\mathcal{A},\ \mathcal{B}\subseteq\mathbb{N}$ we define $\mathcal{A}+\mathcal{B},\mathcal{A}-\mathcal{B}$ as follows:

$$\mathcal{A}+\mathcal{B}:=\{a+b\mid a\in\mathcal{A},b\in\mathcal{B}\}\quad\text{and}\quad\mathcal{A}-\mathcal{B}:=\{a-b\ \mid a\in\mathcal{A},b\in\mathcal{B},a>b\}.$$

In this paper we shall investigate those subsets $\mathcal{A},\ \mathcal{B}\subseteq\mathbb{N}$ for which if the functions

$$f:\mathcal{A}+\mathcal{B}\to\mathbb{C},\quad g:\mathcal{A}\to\mathbb{C}\quad\text{and}\quad h:\mathcal{B}\to\mathbb{C}$$

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satisfy the condition

$$f(a+b) = g(a) + h(b) \quad \text{for all} \quad a \in \mathcal{A}, b \in \mathcal{B},$$

then there is a complex number A such that

$$f(a+b) - A(a+b) = O(1),\ g(a) - Aa = O(1) \quad \text{and} \quad h(b) - Ab = O(1)$$

for all $a \in \mathcal{A}, b \in \mathcal{B}$.

Many results concerning this topics are known for multiplicative functions, for example C. Spiro proved that if f is a multiplicative, $f(p_0) \neq 0$ for some prime p_0 and $f(p+q) = f(p) + f(q)$ for all primes p, q, then $f(n) = n$ for all $n \in \mathbb{N}$. For some similar results, we refer the works of P.V. Chung [2], K.K. Chen and Y.G. Chen [3], P.V. Chung and B.M. Phong [4], J-H. Fang [5], K-H. Indlekofer and B.M. Phong [6], J.M. De Koninck, I. Kátai and B.M. Phong [8], B.M. Phong [13]–[18], C. Spiro [19].

Bojan Basic [1] investigated a function $f : \mathbb{N} \to \mathbb{C}$ such that $f(n^2 + m^2) = f(n)^2 + f(m)^2$ is satisfied for all n, m. Let

$$\mathfrak{M} := \{p_1 + p_2 + p_3 \mid \ p_1, p_2, p_3 \in \mathcal{P}\}.$$

Recently, by using result of H. A. Helfgott concerning the ternary Goldbach conjecture, I. Kátai and B. M. Phong [7] proved that if the function $f : \mathfrak{M} \to \mathbb{C}$, $g : \mathcal{P} \to \mathbb{C}$ satisfy the condition

$$f\left(p_1 + p_2 + p_3\right) = g(p_1) + g(p_2) + g(p_3)$$

for every $p_1, p_2, p_3 \in \mathcal{P}$, then there exist suitable constants $A, B \in \mathbb{C}$ such that

$$f(n) = An + 3B \quad \text{and} \quad g(p) = Ap + B \quad \text{for all} \quad n \in \mathfrak{M}, p \in \mathcal{P}.$$

Our purpose of this paper is to prove the following result:

Theorem 1. *Assume that the sets*

$$\mathcal{A} = \{a_1 < a_2 < \cdots\} \subseteq \mathbb{N}, \quad \mathcal{B} := \{m^2 \mid m \in \mathbb{N}\}$$

and the arithmetical functions $f : \mathcal{A} + \mathcal{B} \to \mathbb{C}$, $g : \mathcal{A} \to \mathbb{C}$ *and* $h : \mathcal{B} \to \mathbb{C}$ *satisfy the equation*

$$f(a + n^2) = g(a) + h(n^2) \quad \textit{for all} \quad a \in \mathcal{A}, n \in \mathbb{N}.$$

If

$$8\mathbb{N} \subseteq \mathcal{A} - \mathcal{A},$$

then there is a complex number A such that

$$g(a)=Aa+\widetilde{g}(a),\ \ h(n^2)=An^2+\widetilde{h}(n)\quad \textit{and}\quad f(a+n^2)=A(a+n^2)+\widetilde{g}(a)+\widetilde{h}(n)$$

hold for all $a\in\mathcal{A}, n\in\mathbb{N}$, furthermore

$$\widetilde{g}(a)=\widetilde{g}(b)\quad \textit{if}\quad a\equiv b \pmod{120},\quad (a,b\in\mathcal{A})$$

and

$$\widetilde{h}(n)=\widetilde{h}(m)\quad \textit{if}\quad n\equiv m \pmod{60},\quad (n,m\in\mathbb{N}).$$

Maillet [10] formulated in 1905 the following

Conjecture 1. *Every even number is the difference of two primes.*

J. Pintz [4] proved that a positive proportion of even numbers in an interval of type $[x,x+(\log x)^C]$ can be written as the difference of two primes if $C>C_0$ and $x>x_0$.

Conjecture 1 is clearly weaker than

Conjecture 2. (Kronecker [9], 1901) *Every even number can be expressed in infinitely many ways as the difference of two primes,*

or

Conjecture 3. (Polignac [12], 1912) *Every even number can be written in infinitely many ways as the difference of two consecutive primes.*

A weaker form of Conjecture 1 is

Conjecture 4. *Every positive number of the form $8n$ is the difference of two primes.*

We obtain from Theorem 1 the following

Theorem 2. *Assume that the sets*

$$\mathcal{A}=\mathcal{P},\quad \mathcal{B}:=\{m^2\mid m\in\mathbb{N}\}$$

and the arithmetical functions $F:\mathcal{P}+\mathcal{B}\to\mathbb{C}$, $G:\mathcal{P}\to\mathbb{C}$ and $H:\mathcal{B}\to\mathbb{C}$ satisfy the equation

$$F(p+n^2)=G(p)+H(n^2)\quad \textit{for all}\quad p\in\mathcal{P}, n\in\mathbb{N}.$$

For each odd prime p let $\overline{p}\in\{1,3\}$ such that $p\equiv\overline{p} \pmod 4$.

If Conjecture 4 holds, then there are complex numbers A, A_2, D *such that*

$$G(p) = Ap + G(\overline{p}) - A\overline{p}, G(2) = A + G(1) + A_2,$$

$$H(n^2) = An^2 + A_2\chi_2(n) + D,$$

$$F(p+n^2) = A(p+n^2) + G(\overline{p}) - A\overline{p} + A_2\chi_2(n) + D$$

for all $p \in \mathcal{P} \setminus \{2\}, n \in \mathbb{N}$*, where* $\chi_2(n)$ *is the Dirichlet character* (mod 2)*, that is* $\chi_2(0) = 0, \chi_2(1) = 1$.

2. Lemmata

In this section we denote by $\mathcal{A}, \mathcal{B}$ the following sets:

$$\mathcal{A} = \{a_1 < a_2 < \cdots\} \subseteq \mathbb{N} \quad \text{and} \quad \mathcal{B} := \{m^2 \mid m \in \mathbb{N}\}.$$

We consider those arithmetical functions

$$U : \mathcal{A} \to \mathbb{C} \quad \text{and} \quad V : \mathcal{B} \to \mathbb{C},$$

for which

(1) $U(a) + V(n^2) = U(b) + V(m^2) \quad \text{if} \quad a + n^2 = b + m^2,\ a, b \in \mathcal{A}, n, m \in \mathbb{N}.$

Let $S_m := V(m^2)\ \ (m \in \mathbb{N})$.

Lemma 1. *Assume that the arithmetical functions* U, V *satisfy* (1)*. If*

$$8\mathbb{N} \subseteq \mathcal{A} - \mathcal{A} \tag{2}$$

is satisfied, then

$$S_{n+12} = S_{n+9} + S_{n+8} + S_{n+7} - S_{n+5} - S_{n+4} - S_{n+3} + S_n \tag{3}$$

holds for all $n \in \mathbb{N}$ *and*

$$\begin{cases} S_7 &= 2S_5 - S_1 \\ S_8 &= 2S_5 + S_4 - 2S_1 \\ S_9 &= S_6 + 2S_5 - S_2 - S_1 \\ S_{10} &= S_6 + 3S_5 - S_3 - 2S_1 \\ S_{11} &= S_6 + 4S_5 - S_3 - S_2 - 2S_1 \\ S_{12} &= S_6 + 4S_5 + S_4 - S_2 - 4S_1. \end{cases} \tag{4}$$

Proof. The proof of this result is similar to the proof of Lemma 1 in [18].

First we infer from (1) and (2) that for each $m \in \mathbb{N}, m > 7$ we have

(5) $S_{2m+1} - S_{2m-1} = S_{m+2} - S_{m-2}$ and $S_{2m+1} - S_{2m-5} = S_{m+5} - S_{m-7}$.

Indeed, for $m \in \mathbb{N}, m > 7$ we have

$$\begin{cases} 8m & = (2m+1)^2 - (2m-1)^2 = (m+2)^2 - (m-2)^2 \\ 24m & = (2m+1)^2 - (2m-5)^2 = (m+5)^2 - (m-7)^2, \end{cases}$$

and so (2) implies that there are $x, x', y, y' \in \mathcal{A}, x > x', y > y'$ such that

$$x - x' = (2m+1)^2 - (2m-1)^2 = (m+2)^2 - (m-2)^2,$$

and

$$y - y' = (2m+1)^2 - (2m-5)^2 = (m+5)^2 - (m-7)^2.$$

These with (1) imply that

$$U(x) - U(x') = S_{2m+1} - S_{2m-1} = S_{m+2} - S_{m-2},$$

and

$$U(y) - U(y') = S_{2m+1} - S_{2m-5} = S_{m+5} - S_{m-7}.$$

Hence (5) is proved.

Finally, we infer from (5) that

$$\begin{aligned} & S_{m+5} - S_{m-7} = S_{2m+1} - S_{2m-5} = \\ & = \Big(S_{2m+1} - S_{2m-1}\Big) + \Big(S_{2m-1} - S_{2m-3}\Big) + \Big(S_{2m-3} - S_{2m-5}\Big) = \\ & = \Big(S_{m+2} - S_{m-2}\Big) + \Big(S_{m+1} - S_{m-3}\Big) + \Big(S_{m} - S_{m-4}\Big). \end{aligned}$$

This with $m := n+7$ proves that (3) holds for all $n \in \mathbb{N}$.

Now we prove (4). Indeed, by using (5), we have

$$S_7 = S_{2.3+1} = 2S_5 - S_1,$$

$$S_9 = S_{2.4+1} = S_7 + S_6 - S_2 = S_6 + 2S_5 - S_2 - S_1$$

and

$$S_{11} = S_{2.5+1} = S_9 + S_7 - S_3 = S_6 + 4S_5 - S_3 - S_2 - 2S_1.$$

Since the following three numbers

$$8^2 - 4^2 = 7^2 - 1^2, \quad 10^2 - 2^2 = 11^2 - 5^2, \quad 12^2 - 8^2 = 9^2 - 1^2$$

are in $\mathcal{A}-\mathcal{A}$, therefore from (1) and (2) we have

$$S_8 = S_7 + S_4 - S_1 = 2S_5 + S_4 - 2S_1,$$

$$S_{10} = S_{11} + S_2 - S_5 = S_6 + 3S_5 - S_3 - 2S_1$$

and

$$S_{12} = S_9 + S_8 - S_1 = S_6 + 4S_5 + S_4 - S_2 - 4S_1.$$

which completes the proof (4).

Lemma 1 is proved. ■

Lemma 2. *Assume that the arithmetical functions* U, V *satisfy* (1). *Let*

$$\text{(6)}\qquad \begin{cases} A := \frac{1}{120}(S_6 + 4S_5 - S_3 - S_2 - 3S_1), \\ A_2 := \frac{-1}{8}(S_6 - 4S_5 + 4S_4 - S_3 - 3S_1), \\ A_3 := \frac{-1}{3}(S_6 - 2S_5 + 2S_3 - S_2), \\ A_4 := \frac{1}{4}(S_6 - 2S_4 - S_3 + S_2 + S_1), \\ A_5 := \frac{1}{5}(S_6 - S_5 - S_3 - S_2 + 2S_1), \\ D := \frac{1}{4}(S_6 - 4S_5 + 2S_4 + 3S_3 + S_2 + S_1), \\ B_k := A_2\chi_2(k) + A_3\chi_3(k) + A_4\chi_4(k-1) + A_5\chi_5(k) + D, \end{cases}$$

where $\chi_2(k) \pmod 2$, $\chi_3(k) \pmod 3$ *are the principal Dirichlet characters and* $\chi_4(k) \pmod 4$, $\chi_5(k) \pmod 5$ *are the real, non-principal Dirichlet characters, i.e.*

$$\chi_2(0) = 0, \chi_2(1) = 1, \chi_3(0) = 0, \chi_3(1) = \chi_3(2) = 1,$$

$$\chi_4(0) = \chi_4(2) = 0, \chi_4(1) = 1, \chi_4(3) = -1,$$

$$\chi_5(0) = 0, \chi_5(2) = \chi_5(3) = -1, \chi_5(1) = \chi_5(4) = 1.$$

If (2) *holds, then*

$$\text{(7)}\qquad S_k = Ak^2 + B_k \quad \textit{for all} \quad k \in \mathbb{N}.$$

Proof. By using (3) and (4), the proof Lemma 2 is similar to the proof of Lemma 2 in [18]. For the sake of completeness, we give the proof here.

From the definition of B_k, using Maple program, we may compute the values of B_k for $k = 1, 2, \cdots, 12$. We have

$$B_1 = -\frac{1}{120}S_6 - \frac{1}{30}S_5 + \frac{1}{120}S_3 + \frac{1}{120}S_2 + \frac{41}{40}S_1,$$

$$B_2 = -\tfrac{1}{30}S_6 - \tfrac{2}{15}S_5 + \tfrac{1}{30}S_3 + \tfrac{31}{30}S_2 + \tfrac{1}{10}S_1,$$

$$B_3 = -\tfrac{3}{30}S_6 - \tfrac{3}{10}S_5 + \tfrac{43}{40}S_3 + \tfrac{3}{40}S_2 + \tfrac{9}{40}S_1,$$

$$B_4 = -\tfrac{2}{15}S_6 - \tfrac{8}{15}S_5 + S_4 + \tfrac{2}{15}S_3 + \tfrac{2}{15}S_2 + \tfrac{2}{5}S_1,$$

$$B_5 = -\tfrac{5}{24}S_6 + \tfrac{1}{6}S_5 + \tfrac{5}{24}S_3 + \tfrac{5}{24}S_2 + \tfrac{5}{8}S_1,$$

$$B_6 = \tfrac{7}{10}S_6 - \tfrac{6}{5}S_5 + \tfrac{3}{10}S_3 + \tfrac{3}{10}S_2 + \tfrac{9}{10}S_1,$$

$$B_7 = -\tfrac{49}{120}S_6 + \tfrac{11}{30}S_5 + \tfrac{49}{120}S_3 + \tfrac{49}{120}S_2 + \tfrac{9}{40}S_1,$$

$$B_8 = -\tfrac{8}{15}S_6 - \tfrac{2}{15}S_5 + S_4 + \tfrac{8}{15}S_3 + \tfrac{8}{15}S_2 - \tfrac{2}{5}S_1,$$

$$B_9 = \tfrac{13}{40}S_6 - \tfrac{7}{10}S_5 + \tfrac{27}{40}S_3 - \tfrac{13}{40}S_2 + \tfrac{41}{40}S_1,$$

$$B_{10} = \tfrac{1}{6}S_6 - \tfrac{1}{3}S_5 - \tfrac{1}{6}S_3 + \tfrac{5}{6}S_2 + \tfrac{1}{2}S_1,$$

$$B_{11} = -\tfrac{1}{120}S_6 - \tfrac{1}{30}S_5 + \tfrac{1}{120}S_3 + \tfrac{1}{120}S_2 + \tfrac{41}{40}S_1,$$

$$B_{12} = -\tfrac{1}{5}S_6 - \tfrac{4}{5}S_5 + S_4 + \tfrac{6}{5}S_3 + \tfrac{1}{5}S_2 - \tfrac{2}{5}S_1.$$

Consequently, we obtain from (4) and $A = \frac{1}{120}(S_6+4S_5-S_3-S_2-3S_1)$ that

$$A \cdot k^2 + B_k = S_k \quad \text{for all} \quad 1 \le k \le 6,$$

$$A \cdot 7^2 + B_7 = 2S_5 - S_1 = S_7,$$

$$A \cdot 8^2 + B_8 = 2S_5 + S_4 - 2S_1 = S_8,$$

$$A \cdot 9^2 + B_9 = S_6 + 2S_5 - S_2 - S_1 = S_9,$$

$$A \cdot 10^2 + B_{10} = S_6 + 3S_5 - S_3 - 2S_1 = S_{10},$$

$$A \cdot 11^2 + B_{11} = S_6 + 4S_5 - S_3 - S_2 - 2S_1 = S_{11},$$

$$A \cdot 12^2 + B_{12} = S_6 + 4S_5 + S_4 - S_2 - 4S_1 = S_{12}.$$

Therefore, we proved that (7) holds for $1 \le k \le 12$.

Assume that $Ak^2 + B_k = S_k$ holds for $n \le k \le n+11$, where $n \ge 1$. Then we deduce from (4) that

$$S_{n+12} = S_{n+9} + S_{n+8} + S_{n+7} - S_{n+5} - S_{n+4} - S_{n+3} + S_n =$$
$$= A\Big[(n+9)^2 + (n+8)^2 + (n+7)^2 - (n+5)^2 - (n+4)^2 - (n+3)^2 + n^2\Big] +$$
$$+ \Big[B_{n+9} + B_{n+8} + B_{n+7} - B_{n+5} - B_{n+4} - B_{n+3} + B_n\Big] =$$
$$= A(n+12)^2 + B_{n+12},$$

which proves that (7) holds for $n+12$ and so it is true for all n. In the last relation, by using (6), we have

$$B_{n+9} + B_{n+8} + B_{n+7} - B_{n+5} - B_{n+4} - B_{n+3} + B_n =$$
$$= A_2\Big[\sum_{k=n+6}^{n+9} \chi_2(k) - \sum_{k=n+3}^{n+6} \chi_2(k) + \chi_2(n)\Big] +$$
$$+ A_3\Big[\sum_{k=n+7}^{n+9} \chi_3(k) - \sum_{k=n+3}^{n+5} \chi_3(k) + \chi_3(n)\Big] +$$
$$+ A_4\Big[\sum_{k=n+6}^{n+9} \chi_4(k-1) - \sum_{k=n+3}^{n+6} \chi_4(k-1) + \chi_4(n-1)\Big] +$$
$$+ A_5\Big[\sum_{k=n+6}^{n+10} \chi_5(k) - \sum_{k=n+2}^{n+6} \chi_5(k) - \chi_5(n+10) + \chi_5(n+2) + \chi_5(n)\Big] + D =$$
$$= A_2\chi_2(n) + A_3\chi_3(n) + A_4\chi_4(n-1) + A_5\chi_5(n+2) + D =$$
$$= A_2\chi_2(n+12) + A_3\chi_3(n+12) + A_4\chi_4(n+11) + A_5\chi_5(n+12) + D = B_{n+12}.$$

Lemma 2 is proved. ■

Lemma 3. *Let*

$$T(k) := B_{k+1} - B_{k-1},$$

where

$$B_m = A_2\chi_2(m) + A_3\chi_3(m) + A_4\chi_4(m-1) + A_5\chi_5(m) + D$$

and A_2, A_3, A_4, A_5, D *are defined in* (6) *of Lemma* 2. *Then*

$$T(k) = A_3\chi_3^*(k) + 2A_4\chi_4(k) + A_5E_5(k), \tag{8}$$

where $\chi_3^*(k)$ *and* $\chi_4(k)$ *are the real, non-principal Dirichlet characters, i.e.*

$$\chi_3^*(0) = 0, \quad \chi_3^*(1) = 1, \quad \chi_3^*(2) = -1,$$

$$\chi_4(0)=\chi_4(2)=0,\quad \chi_4(1)=1,\quad \chi_4(3)=-1$$

and

$$E_5(k):=r\in\{0,\pm1,\pm2\}\quad \textit{if}\quad k+r\equiv 0\pmod 5.$$

Proof. It is easy to check that

$$\chi_2(k+1)-\chi_2(k-1)=0,\quad \chi_3(k+1)-\chi_3(k-1)=\chi_3^*(k)$$

and

$$\chi_4(k)-\chi_4(k-2)=2\chi_4(k)$$

hold for all $k\in\mathbb{N}$. Finally, we infer from the definition of χ_5 that

$$\chi_5(k+1)-\chi_5(k-1)=\begin{cases} 1-1=0 & \text{if}\quad k\equiv 0\pmod 5\\ -1-0=-1 & \text{if}\quad k\equiv 1\pmod 5\\ -1-1=-2 & \text{if}\quad k\equiv 2\pmod 5\\ 1-(-1)=2 & \text{if}\quad k\equiv 3\pmod 5\\ 0-(-1)=1 & \text{if}\quad k\equiv 4\pmod 5),\end{cases}$$

consequently $\chi_5(k+1)-\chi_5(k-1)=E_5(k)$. Thus, we have

$$\begin{aligned} T(k)&=B_{k+1}-B_{k-1}=\\ &=A_2\Big(\chi_2(k+1)-\chi_2(k-1)\Big)+A_3\Big(\chi_3(k+1)-\chi_3(k-1)\Big)+\\ &+A_4\Big(\chi_4(k)-\chi_4(k-2)\Big)+A_5\Big(\chi_5(k+1)-\chi_5(k-1)\Big)=\\ &=A_3\chi_3^*(k)+2A_4\chi_4(k)+A_5E_5(k).\end{aligned}$$

Therefore, (8) and Lemma 3 is proved. ∎

In the following for each $a\in\mathcal{A}$ we denote by $\overline{a}$ the smallest element of $\mathcal{A}$ for which $a\equiv\overline{a}\pmod 4$.

Lemma 4. *Assume that the arithmetical functions U,V satisfy* (1). *If* (2) *holds, then*

(9) $$U(a)-Aa=U(\overline{a})-A\overline{a}+T\Big(\frac{a-\overline{a}}{4}\Big)\quad \textit{for all}\quad a\in\mathcal{A},$$

where $T(k)$ is defined in Lemma 3. *Consequently*

(10) $$U(a)-Aa=U(r)-Ar\quad \textit{if}\quad a,r\in\mathcal{A},\quad a\equiv r\pmod{120}.$$

Proof. For each $a \in \mathcal{A}$, we have

$$a + \Big(\frac{a-\overline{a}}{4} - 1\Big)^2 = \overline{a} + \Big(\frac{a-\overline{a}}{4} + 1\Big)^2,$$

which with (1) implies that

$$U(a) = U(\overline{a}) + S_{\frac{a-\overline{a}}{4}+1} - S_{\frac{a-\overline{a}}{4}-1}.$$

From (7) we have

$$S_{\frac{a-\overline{a}}{4}+1} - S_{\frac{a-\overline{a}}{4}-1} = A\Big[\Big(\frac{a-\overline{a}}{4} + 1\Big)^2 - \Big(\frac{a-\overline{a}}{4} - 1\Big)^2\Big] + \\ + B_{\frac{a-\overline{a}}{4}+1} - B_{\frac{a-\overline{a}}{4}-1} = Aa - A\overline{a} + T\Big(\frac{a-\overline{a}}{4}\Big),$$

which proves (9).

The proof of (10) is clear. Indeed, if $a \equiv r \pmod{120}$, then $\frac{a-r}{4} \equiv 0 \pmod{30}$, and so

$$T\Big(\frac{a-r}{4}\Big) = A_3\chi_3^*\Big(\frac{a-r}{4}\Big) + 2A_4\chi_4\Big(\frac{a-r}{4}\Big) + A_5E_5\Big(\frac{a-r}{4}\Big) = 0.$$

Thus, (10) and Lemma 4 is proved. ■

3. The proof of Theorem 1.

Assume that the sets

$$\mathcal{A} = \{a_1 < a_2 < \cdots\} \subseteq \mathbb{N}, \quad \mathcal{B} := \{m^2 \mid m \in \mathbb{N}\}$$

and the arithmetical functions $f : \mathcal{A}+\mathcal{B} \to \mathbb{C}$, $g : \mathcal{A} \to \mathbb{C}$ and $h : \mathcal{B} \to \mathbb{C}$ satisfy the equation

$$f(a+n^2) = g(a) + h(n^2) \quad \text{for all} \quad a \in \mathcal{A}, n \in \mathbb{N},$$

furthermore we assume that

$$8\mathbb{N} \subseteq \mathcal{A} - \mathcal{A}.$$

It is obvious that (1) is true for $U = g$ and $V = h$. In the notations of Lemma 2 for $U = g,\ V = h$ and $S_n = h(n^2)\ \ (n \in \mathbb{N})$, we put

$$\widetilde{g}(a) := g(a) - Aa \quad \text{and} \quad \widetilde{h}(n) := h(n^2) - An^2 \quad \text{for all} \quad a \in \mathcal{A}, n \in \mathbb{N}.$$

Then it follows from Lemma 3 and Lemma 4 that

$$\widetilde{h}(n)=A_2\chi_2(n)+A_3\chi_3(n)+A_4\chi_4(n-1)+A_5\chi_5(n)+D,$$

$$\widetilde{g}(a)=\widetilde{g}(\overline{a})+T\Big(\frac{a-\overline{a}}{4}\Big)$$

and

$$f(a+n^2)=A(a+n^2)+\widetilde{g}(a)+\widetilde{h}(n)$$

hold for all $a\in\mathcal{A}, n\in\mathbb{N}$, furthermore

$$\widetilde{g}(a)=\widetilde{g}(r)\quad\text{if}\quad a,r\in\mathcal{A}\quad\text{and}\quad a\equiv r\pmod{120}.$$

Thus the proof of Theorem 1 is completes. ■

4. The proof of Theorem 2.

Assume that the sets

$$\mathcal{A}=\mathcal{P},\quad \mathcal{B}:=\{m^2\mid m\in\mathbb{N}\}$$

and the arithmetical functions $F:\mathcal{P}+\mathcal{B}\to\mathbb{C}$, $G:\mathcal{P}\to\mathbb{C}$ and $H:\mathcal{B}\to\mathbb{C}$ satisfy the equation

$$F(p+n^2)=G(p)+H(n^2)\quad\text{for all}\quad p\in\mathcal{P}, n\in\mathbb{N}. \tag{11}$$

Let

$$S_m:=H(m^2)\quad\text{and}\quad S:=G(3)-G(2).$$

First we prove that

$$\begin{cases} S_3 &= 2S_2-S_1+2S,\\ S_4 &= 4S_2-3S_1+3S,\\ S_5 &= 6S_2-5S_1+6S,\\ S_6 &= 9S_2-8S_1+8S, \end{cases} \tag{12}$$

It follows from (11) that

$$G(p)-G(q)=S_m-S_n\quad\text{if}\quad p-q=m^2-n^2 \tag{13}$$

where $p,q\in\mathcal{P}$ and $n,m\in\mathbb{N}$.

From (13) we have $G(7) = S_3 - S_2 + G(2)$, $G(11) = S_3 - S_1 + G(3)$ and $G(19) = S_3 - S_1 + G(11) = 2S_3 - 2S_1 + G(3)$. Therefore, the relation $G(7) + S_4 = G(19) + S_2$ implies that

$$S_4 - S_3 - 2S_2 + 2S_1 - G(3) + G(2) = 0. \tag{14}$$

On the other hand, we also get from (13) that

$$S_5 = S_4 + G(11) - G(2) = S_4 + S_3 - S_1 + G(3) - G(2) \tag{15}$$

and so

$$S_5 + G(3) - (G(19) + S_3) = S_4 - 2S_3 + S_1 + G(3) - G(2) = 0. \tag{16}$$

Then (14) and (16) imply that

$$S_3 = 2S_2 - S_1 + 2G(3) - 2G(2) = 2S_2 - S_1 + 2S$$

and

$$S_4 = 4S_2 - 3S_1 + 3G(3) - 3G(2) = 4S_2 - 3S_1 + 3S,$$

which prove (12) for $m = 3, 4$. Finally, we infer from (15) that

$$S_5 = S_4 + S_3 - S_1 + G(3) - G(2) = 6S_2 - 5S_1 + 6G(3) - 6G(2) = 6S_2 - 5S_1 + 6S,$$

which proves (12) for $m = 5$.

Since $G(13) = S_3 - S_1 + G(5) = 3S_2 - 3S_1 + 2G(3) - G(2)$, we have

$$S_6 = G(13) + S_5 - G(2) = 9S_2 - 8S_1 + 8S,$$

which proves (12) for $m = 6$.

Assume that Conjecture 4 holds, then the condition (2) is true, i.e.

$$8\mathbb{N} \subseteq \mathcal{P} - \mathcal{P}.$$

Thus we infer from (6) and (12) that $A_3 = A_4 = A_5 = 0$ and

$$A = \frac{1}{4}\Big(S_2 - S_1 + S\Big), \quad A_2 = \frac{-1}{4}\Big(S_2 - S_1 - 3S\Big), D = S_1 - S,$$

which with Lemma 2 implies that

$$H(n^2) = An^2 + A_2\chi_2(n) + D \quad \text{for all} \quad n \in \mathbb{N}. \tag{17}$$

Finally, by using the fact $A_3 = A_4 = A_5 = 0$, we infer from the relations (9) and (10) of Lemma 4 that

$$G(p) - Ap = G(\overline{p}) - A\overline{p} \quad \text{for all} \quad p \in \mathcal{P} \setminus \{2\}, \tag{18}$$

where $\overline{p} \in \{3,5\}$ such that $p \equiv \overline{p} \pmod 4$.

For $p = 2$, we infer from (13), (17) and (18) that

$$G(2) = G(5) + H(1) - H(2^2) = \\ = \Big[5A + G(1) - A\Big] + \Big[A(1-4) + A_2\Big] = A + G(1) + A_2.$$

The proof of Theorem 2 is completed. ■

5. Remarks and corollaries

1. Theorem 1 and Theorem 2 remain valid if f, g, h and F, G, H maps into an arbitrary Abelian group.

2. Let $\mathbb{N}_k$ be the set of those integers n for which the number of prime power divisors of n $(= \Omega(n))$ is at most k. By using the method of Chen one can deduce that $\mathbb{N}_k - \mathbb{N}_k$ contains every even numbers if $k \geq 2$. Thus Theorem 1 is true for $\mathcal{A} = \mathbb{N}_k$, if $k \geq 2$.

3. Similar results are true if in Theorem 1 and Theorem 2 the set $\mathcal{B}$ is of the form $\{n^2 + M \mid n \in \mathbb{N}\}$, where $M \geq 0$ is a given integer.

4. Now we infer from Theorem 1 the following result of Bojan Basic [1]

Corollary. *Assume that* $f : \mathbb{N} \to \mathbb{C}$ *satisfies the relation*

$$f(n^2 + m^2) = f(n)^2 + f(m)^2 \quad \textit{for all} \quad n, m \in \mathbb{N}. \tag{19}$$

Then one of the following cases holds:

(a) $f(n) = 0$ *for all* $n \in \mathbb{N}$,

(b) $f(n)^2 = \frac{1}{4}$ *for all* $n \in \mathbb{N}$,

(c) $f(n)^2 = n^2$ *for all* $n \in \mathbb{N}$.

Proof. Let $\mathcal{A} = \{n^2 \mid n \in \mathbb{N}\}$. It is obvious that the condition (2) is satisfied. Let $S_k := f(k)^2$. From $2 = 1^2 + 1^2$, $5 = 1^2 + 2^2$, $8 = 2^2 + 2^2$, $10 = 1^2 + 3^2$ and $13 = 2^2 + 3^2$ we get from (19) that

$$S_2 = 4S_1^2,\ S_5 = (S_1 + S_2)^2,\ S_8 = 4S_2^2,\ S_{10} = (S_1 + S_3)^2 \quad \text{and} \quad S_{13} = (S_2 + S_3)^2.$$

Using Lemma 2, we obtain from $S_{13} = (S_2 + S_3)^2$ the equation

$$S_1(4S_1 - 1)(4S_1^2 + 4S_1 + 1 - S_3) = 0.$$

First assume that $S_1 = 0$. Then it follows from the fact $S_{20} = (S_2 + S_4)^2$ that $S_3 = 0$, consequently $f(n) = 0$ for all $n \in \mathbb{N}$.

If $S_1 = \frac{1}{4}$, then we get from the relations $S_{17} = (S_1 + S_4)^2$ and $S_{18} = 4S_3^2$ the following system of equations:

$$\begin{cases}(4S_3 + 3)(4S_3 - 1) &= 0\\ (4S_3 - 9)(4S_3 - 1) &= 0.\end{cases}$$

This implies that $S_3 = \frac{1}{4}$, which with Lemma 2 shows that $A = 0$ and $A_2 = A_3 = A_4 = A_5 = 0$ and $D = \frac{1}{4}$. Thus we have $f(n)^2 = \frac{1}{4}$ for all $n \in \mathbb{N}$.

Finally, assume that $S_3 = 4S_1^2 + 4S_1 + 1$, then we get from $S_{17} = (S_1 + S_4)^2$ and $S_{20} = (S_2 + S_4)^2$ that $S_1 = 1$ and $S_3 = 9$. In this case we have $A = 1$ and $A_2 = A_3 = A_4 = A_5 = D = 0$, and so $f(n)^2 = n^2$ for all $n \in \mathbb{N}$.

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