HAPPY BIRTHDAY FERI*, HAPPY BIRTHDAY PÉTER*

Imre Kátai (Budapest, Hungary)

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Abstract. In this short paper I would like to mention such a result which might be interesting for the people dealing with number theory, and those working on Walsh, Vilenkin orthogonal systems.

1. Introduction

Notation: We write $e(\alpha) := e^{2\pi i\alpha}$. Let q be an integer, $q \geq 2$.

1. $\mathcal{P}$ = set of prime numbers.

2. $\mathcal{M}$ = set of multiplicative functions, $\mathcal{M}_q$ = set of $q-$multiplicative functions.

We say that $f : \mathbb{N} \to \mathbb{C}$ belongs to $\mathcal{M}$, if $f(1) = 1$ and $f(mn) = f(m) \cdot f(n)$, whenever $GCD\ (m, n) = 1$.

We say that $g : \mathbb{N}_0 (= \mathbb{N} \cup \{0\}) \to \mathbb{C}$ belongs to $\mathcal{M}_q$, if $g(0) = 1$ and $g(n) = \prod_{j=0}^{t} g\left(\varepsilon_j(n) q^j\right)$, where $\varepsilon_j(n)$ are the q-ary digits of n, i.e.

$$n = \sum_{j=0}^{t} \varepsilon_j(n) q^j, \qquad \varepsilon_j(n) \in A_q := \{0, \ldots, q-1\}.$$

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*This paper is dedicated to professor Ferenc Schipp (nickname Walsh II.) on his 75th, and to professor Péter Simon (nickname Vilenkin II.) on his 65th anniversary.

3. $\mathcal{M}^{(1)} := \{f \in \mathcal{M} \mid |f(n)| \le 1, n \in \mathbb{N}\}$,
   $\mathcal{M}_q^{(1)} := \{g \in \mathcal{M}_q \mid |g(n)| = 1, n \in \mathbb{N}_0\}$.
4. A function $f : \mathbb{N}_0 \to \mathbb{C}$ is uniformly summable ($= US$) if

$$C(K) := \sup_{x \ge 1} \frac{1}{x} \sum_{\substack{n \le x \\ |f(n)| \ge K}} |f(n)| \to 0 \quad \text{as} \quad K \to \infty.$$

The set of US functions is denoted as $\mathcal{L}^*$. It was studied by K.-H. Indlekofer [3]. Let $\mathcal{M}^{(US)} = \mathcal{M} \cap \mathcal{L}^*$.

We say that $\alpha \in \mathbb{R}$ belongs to the Bohr–Fourier spectrum of f, if

$$\limsup_{x \to \infty} \frac{1}{x} \Big| \sum_{n \le x} f(n)\, e(-n\alpha) \Big| > 0. \tag{1.1}$$

Preliminary results.

5. A nice theorem of H. Daboussi [1, 2] asserts that for every irrational α:

$$\sup_{f \in \mathcal{M}_1} \frac{1}{x} \Big| \sum_{n \le x} f(n)\, e(n\alpha) \Big| \to 0 \quad \text{as} \quad x \to \infty.$$

6. It is a consequence of the following

Theorem A. ([4]) *Let $t : \mathbb{N} \to \mathbb{R}$. Assume that for every $K > 0$ there exists a finite set $\mathcal{P}_K$ of primes $p_1 < \dots < p_R$ such that*

$$E_K := \sum_{i=1}^{R} \frac{1}{p_i} > K$$

and that for the sequences

$$\eta_{i,j}(m) = t(p_i m) - t(p_j m)$$

the relation

$$\lim_{x \to \infty} \frac{1}{x} \sum_{m=1}^{[x]} e\left(\eta_{i,j}(m)\right) = 0$$

holds whenever $i \ne j$, $i, j \in \{1, \dots, R\}$.

Then, there exists a function ϱ_x, which tends to zero as $x \to \infty$ and such that

$$\sup_{f \in \mathcal{M}_1} \frac{1}{x} \Big| \sum_{n \le x} f(n)\, e\left(t(n)\right) \Big| \le \varrho_x.$$

Similar method has been used by some other mathematicians [9–17]. Now it has the name: Kátai–Bourgain–Sarnak–Ziegler orthogonality criterion.

7. In a paper written jointly with K.-H. Indlekofer [5] we proved

Lemma 1. *Let* $1 \le a < b$, $(a,b) = 1$, $(ab,q) = 1$, $g \in \mathcal{M}_q^{(1)}$. *If*

$$\limsup_{x\to\infty} \Big| \frac{1}{x} \sum_{n\le x} g(an)\ g(bn) \Big| > 0, \tag{1.2}$$

then there exists such an integer r, *for which*

$$\sum_{j=0}^{\infty} \sum_{c\in A_q} \mathrm{Re} \left(1 - e\left(-\frac{rcq^j}{b-a} \right) g(cq^j) \right) < \infty. \tag{1.3}$$

Hence, and from Theorem A we deduced

Theorem B. ([6]) *Let us suppose that* $f \in \mathcal{M}^{(US)}$, $g \in \mathcal{M}_q^{(1)}$ *and that*

$$\limsup_{x} \frac{1}{x} \Big| \sum_{n\le x} f(n) g(n) \Big| > 0. \tag{1.4}$$

Then $g(n) = e\big(\frac{r^* n}{D}\big) h(n)$, $(r^*, D) = 1$ *with such* $h \in \mathcal{M}_q^{(1)}$ *for which*

$$\sum_{j=0}^{\infty} \sum_{c\in A_q} \mathrm{Re} \left(1 - h(cq^j) \right) < \infty \tag{1.5}$$

holds.

If the Bohr–Fourier spectrum of f *is empty, then*

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} f(n) g(n) = 0$$

for every $g \in \mathcal{M}_q^{(1)}$.

Remark 1. Assume that Theorem B holds for $g \in \mathcal{M}_q^{(1)}$ for which $g^q(n) = 1$ $(n \in \mathbb{N})$. Since $e\big(\frac{r^* n}{D}\big)$ runs over D distinct values, the values of $h(n)$ belong to a finite set. (1.5) implies that $h\left(cq^j\right) = 1$ if $j \ge j_0$, $c \in A_q$.

2. On multiplicative functions

Lemma 2. (G. Halász [7]) *If* $h \in \mathcal{M}_1$ *and*

$$\limsup_x \frac{1}{x}\Big|\sum_{n\le x} h(n)\Big| > 0,$$

then there is a $\tau \in \mathbb{R}$ *such that*

$$\sum_{p\in\mathcal{P}} \frac{\mathrm{Re}\,\big(1-h(p)p^{-i\tau}\big)}{p} < \infty. \tag{2.1}$$

Lemma 3. *If* $f \in \mathcal{M}_1$ *and for some* $k, R,$ $(k,R)=1$

$$\limsup \frac{1}{x}\Big|\sum_{\substack{n\le x\\ n\equiv k \pmod R}} f(n)\Big| > 0, \tag{2.2}$$

then there exists a Dirichlet character $\chi \bmod R$ *for which*

$$\limsup \frac{1}{x}\Big|\sum_{n\le x} f(n)\chi(n)\Big| > 0, \tag{2.3}$$

and a $\tau \in \mathbb{R}$ *such that*

$$\sum_{p\in\mathcal{P}} \frac{\mathrm{Re}\,\big(1-\chi(p)p^{i\tau}f(p)\big)}{p} < \infty. \tag{2.4}$$

Proof. Since

$$\frac{1}{\varphi(R)} \sum_{\chi \,(\mathrm{mod}\, R)} \overline{\chi}(l)\chi(n) = \begin{cases} 1, & \text{if } n \equiv l \pmod R; \\ 0, & \text{otherwise,} \end{cases}$$

(2.3) clearly holds. (2.4) follows from Lemma 2. ■

Remark 2. The Möbius function $\mu(n)$ and the Liouville function $\lambda(n)$ are defined on prime powers p^α such that

$$\mu(p^\alpha) = \begin{cases} -1, & \text{if } \alpha = 1 \\ 0, & \text{if } \alpha > 1 \end{cases} \quad ; \qquad \lambda(p^\alpha) = (-1)^\alpha.$$

They are multiplicative. It is easy to observe that (2.4) does not hold if $f=\mu$, or λ for any χ.

3. The Vilenkin–Chrestenson systems

Let the q-ary expansion of x in $[0,1)$ be

$$x = \sum_{i=0}^{\infty} \frac{x_i}{q^{i+1}}, \qquad x_i \in A_q.$$

Let $r_k(x) = e\big(\frac{x_k}{q}\big)$, and for $n = n_0 + n_1 q + \cdots + n_t q^t$ let

$$\psi_n(x) = \prod_{k=0}^{t} r_k^{n_k}(x) = e\Big(\frac{1}{q}\sum_{k=0}^{t} n_k x_k\Big).$$

Then $g(n) := \psi_n(x)$ belongs to $\mathcal{M}_q^{(1)}$ for every x, furthermore $g^q(n) = 1$ $(n \in \mathbb{N})$. From Theorem B and Remark 1 we obtain

Theorem 1. *Let $f \in \mathcal{M}^{(1)}$. If*

$$\limsup_{N} \frac{1}{N}\Big|\sum_{n\le N} f(n)\psi_n(x)\Big| > 0, \tag{3.1}$$

then there exists a j for which $\psi_n(x) = \psi_{n_0}(x)$ if $n \equiv n_0 \pmod{q^{j+1}}$. Consequently $x_{j+1} = x_{j+2} = \cdots = 0$.

Proof. Since for $g(n) = \psi_n(x)$ we have $1 = g(n)^q$, and $h(mq^j) = 1$ if j is large enough, we obtain that $1 = g^q(nq^j) = e\big(\frac{mq^{j+1}}{D}\big)$, and so $\psi_{n_0+q^{j+1}m}(x) = \psi_{n_0}(x)$. ■

Lemma 4. *Let $f \in \mathcal{M}^{(1)}$, $g \in \mathcal{M}_q^{(1)}$, for some $j \in \mathbb{N}$ let $g(mq^{j+1}) = 1$ $(m \in \mathbb{N})$, furthermore $g(n)^q = 1$ and*

$$\limsup \frac{1}{N}\Big|\sum_{n\le N} f(n)g(n)\Big| > 0. \tag{3.2}$$

Then there exists at least one k $\bmod q^{j+1}$ such that $(k,q) = 1$ and

$$\limsup_{N\to\infty} \frac{1}{N}\Big|\sum_{\substack{n\le N \\ n\equiv k \pmod{q^{j+1}}}} f(n)\Big| > 0. \tag{3.3}$$

Proof. Since $g(n)$ is periodic $\bmod q^{j+1}$, therefore (3.3) holds for some l, i.e.

$$\limsup_{N\to\infty} \frac{1}{N} \Big| \sum_{\substack{n\le N \\ n\equiv l \pmod{q^{j+1}}}} f(n) \Big| > 0. \tag{3.4}$$

Let $\mathcal{B}_q$ be the multiplicative semigroup generated by the prime factors of q, and $\mathcal{D}_q = \{m \mid (m,q)=1\}$.

Let us write the integers n in (3.4) as $n = \nu m \equiv l \pmod{q^{j+1}}$, where $\nu \in \mathcal{B}_q$, $m \in \mathcal{D}_q$. From (3.4), and from

$$\sum_{\nu\in\mathcal{B}_q} \frac{1}{\nu} < \infty$$

we obtain that there exists such a $\nu = \nu_0$ for which

$$\limsup_{N\to\infty} \frac{1}{N} \Big| \sum_{\substack{\nu_0 m\le N \\ \nu_0 m\equiv l \pmod{q^{j+1}}}} f(n) \Big| > 0. \tag{3.5}$$

Let $k_1, \dots, k_T$ be those residues $\bmod q^{j+1}$ for which $\nu_0 k_h \equiv l \pmod{q^{j+1}}$ occurs. Due to (3.5) $T \ge 1$. Furthermore $(k_h, q) = 1$, and

$$\limsup \max_{h=1,\dots,T} \frac{1}{N} \Big| \sum_{\substack{m\le N/\nu_0 \\ m\equiv k_h \pmod{q^j}}} f(m) \Big| > 0,$$

and so the assertion is true. ∎

Theorem 2. *Let $f \in \mathcal{M}^{(1)}$ and assume that*

$$\limsup_{N\to\infty} \Big| \frac{1}{N} \sum_{n\le N} f(n)\psi_n(x) \Big| > 0. \tag{3.6}$$

Then there exists a j_0 such that $\psi(x) = \psi_{n_0}(x)$ if $n \equiv n_0 \pmod{q^{j_0+1}}$, and $x = \frac{x_0}{q} + \dots + \frac{x_{j_0}}{q^{j_0+1}}$, and so $x_m = 0$ if $m > j_0$.

Furthermore, there exists a Dirichlet character $\bmod q^{j_0+1}$ and a real τ for which

$$\sum_{p\in\mathcal{P}} \frac{\mathrm{Re}\,\big(1 - \chi(p)p^{i\tau} f(p)\big)}{p} < \infty.$$

Proof. This follows from Theorem 1, Lemma 3 and Lemma 4. ∎

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I. Kátai
Department of Computer Algebra
Faculty of Informatics
Eötvös Loránd University
H-1117 Budapest, Pázmány Péter sétány 1/C
Hungary
katai@compalg.inf.elte.hu