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SOME RESULTS AND PROBLEMS IN PROBABILISTIC NUMBER THEORY

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Abstract. We formulate some results, open problems and conjectures in probabilistic number theory.

1. Introduction

Notation. Let, as usual, $\mathcal{P}$, $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ be the set of primes, positive integers, non-negative integers, integers, real and complex numbers, respectively. We say that $f : \mathbb{N} \to \mathbb{R}$ is an additive function, if $f(1) = 0$ and $f(mn) = f(m) + f(n)$ for all $(m, n) = 1$. Let $\mathcal{A}$ denote set of all additive functions. A function $g : \mathbb{N} \to \mathbb{C}$ is multiplicative, if $g(1) = 1$ and $g(mn) = g(m) \cdot g(n)$ for all $(m, n) = 1$. We denote by $\mathcal{M}$ the set of all multiplicative functions.

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Let $q \geq 2$, integer, $A_q = \{0, \dots, q-1\}$. Every $n \in \mathbb{N}_0$ can be written as

$$n = \sum_{j=0}^{k} \varepsilon_j(n) q^j, \qquad \varepsilon_k(n) \neq 0, \qquad \varepsilon_j(n) \in A_q,$$

and this expansion is unique.

In the following let $\mathcal{A}_q$ denote the set of q-additive functions, i.e. a function $f : \mathbb{N}_0 \to \mathbb{R}$ belongs to $\mathcal{A}_q$, if $f(0) = 0$, and $f(n) = \sum_{j=0}^{k} f(\varepsilon_j(n) q^j)$ $(\forall n \in \mathbb{N}_0)$.

Similarly, we say that $g : \mathbb{N}_0 \to \mathbb{C}$ is q-multiplicative, if $g(0) = 1$, and $g(n) = \prod_{j=0}^{k} g(\varepsilon_j(n) q^j)$ $(\forall n \in \mathbb{N})$. Let $\mathcal{M}_q$ be the set of q-multiplicative functions.

Let $\pi(x)$ be the number of the primes up to x, and $\pi(x, k, l)$ be the number of the primes $p \leq x$ satisfying $p \equiv l \pmod{k}$.

Examples.

- $\omega(n)$= number of prime factors of n, $\omega(n) \in \mathcal{A}$,
- $\tau(n)$ = number of divisors of n, $\tau(n) \in \mathcal{M}$,
- 1cm $f(n) = \log n \ \in \mathcal{A}$,
- 1cm $\varphi(n)$ = Euler's totient function, $\varphi(n) \in \mathcal{M}$,
- 1cm $\sigma(n)$ = sum of divisors of n, $\sigma(n) \in \mathcal{M}$.
- If $f(n) \in \mathcal{A}$, $\quad g(n) = z^{f(n)}$, $\quad z \in \mathbb{C}$, then $g \in \mathcal{M}$.
- If $\alpha(n) = \sum_{j=0}^{k} \varepsilon_j(n)$ = sum of digits functions, then $\alpha(n) \in \mathcal{A}_q$.
- $f(n) = n \quad \in \mathcal{A}_q$,
- $g(n) = z^n \quad \in \mathcal{M}_q$.

The following wellknown results can be found in [8].

Definition 1. *We say that $f \in \mathcal{A}$ has a limit distribution (on the set $\mathbb{N}$) if*

$$F_N(y) := \frac{1}{N} \#\{n \leq N \mid f(n) < y\}$$

has a limit $\lim_{N \to \infty} F_N(y) = F(y)$ for almost all y (or in all continuity points of F).

Theorem 1 (Erdős-Wintner). *$f \in \mathcal{A}$ has a limit distribution if and only if the next three series converge*

$$\sum_{|f(p)|>1} \frac{1}{p}, \tag{1.1}$$

$$\sum_{|f(p)|\le 1} \frac{f(p)}{p}, \tag{1.2}$$

$$\sum_{|f(p)|\le 1} \frac{f^2(p)}{p}. \tag{1.3}$$

Theorem 2. *$f \in \mathcal{A}$ has a limit distribution with a suitable centralization $c(N)$, i.e.*

$$\lim_{N\to\infty} \frac{1}{N} \#\{n \le N \mid f(n) - c(N) < y\} = F(y) \tag{1.4}$$

exists for almost all y, if and only if (1.1), (1.3) converge.

Let $a(N) := \sum_{\substack{|f(p)|\le 1\\ p<N}} \frac{f(p)}{p}$. If (1.4) holds, then $\lim(a(N) - c(N)) = \alpha$ exists, and α is finite.

Theorem 3 (Erdős-Kac). *Let $f \in \mathcal{A}$,*

$$A_N := \sum_{p\le N} \frac{f(p)}{p}, \quad B_N^2 = \sum_{p^\alpha \le N} \frac{f^2(p^\alpha)}{p^\alpha}. \tag{1.5}$$

Assume that $f(p) = \mathcal{O}(1) \quad (p \in \mathcal{P})$. Then

$$\lim_{N\to\infty} \frac{1}{N} \# \left\{ n \le N \;\middle|\; \frac{f(n) - A(N)}{B_N} < y \right\} = \Phi(y),$$

$\Phi =$ Gaussian law.

Turán-Kubilius inequality. *Let $f \in \mathcal{A}$. Then*

$$\sum_{n\le N} (f(n) - A_N)^2 \le cNB_N^2,$$

A_N, B_N are defined in (1.5).

2. Distribution of additive functions on some subsets of integers

Let $\mathcal{B} \subseteq \mathbb{N}$, $\quad B(x) = \#\{n \le x, n \in \mathcal{B}\}$.

We say that $f \in \mathcal{A}$ is distributed in limit with centralization a_N and normalization b_N on the set $\mathcal{B}$, if

$$\lim_{N\to\infty} \frac{1}{B(N)} \# \left\{ n \le N,\ n \in \mathcal{B} \;\middle|\; \frac{f(n) - a_N}{b_N} < y \right\} = F(y)$$

exists a.a.

Results:

1. $\mathcal{B} = \mathcal{P}_{+1} = \{p+1 \mid p \in \mathcal{P}\}$ = set of shifted primes, $a_N = 0,\ b_N = 0$. $f \in \mathcal{A}$ has a limit distribution on $\mathcal{P}_{+1}$ if the 3 series (in Erdős-Wintner theorem) are convergent ([6]).

   The proof is based on the method of characteristic functions: $g(n) = e^{i\tau f(n)}$.

   $$\text{(2.1)}\quad \lim_{x\to\infty} \frac{1}{\pi(x)} \sum_{p<x} g(p+1) = M_\tau(x) = \\ = \prod_p \left(1 - \frac{1}{p-1} + \frac{g(p)}{p} + \frac{g(p^2)}{p^2} + \cdots \right).$$

   To prove (2.1), the Siegel-Walfisz theorem was used, namely that

   $$\pi(x,k,l) = \sum_{\substack{p \le x \\ p \equiv l \pmod k}} 1 = \frac{\mathrm{li}\, x}{\varphi(k)} \left(1 + \mathcal{O}\left(e^{-c\sqrt{\log x}}\right)\right)$$

   uniformly as $(l,k) = 1, \quad k < (\log x)^{c_1}$, where

   $$\mathrm{li}\, x = \int_2^x \frac{du}{\log u},$$

   and the A. I. Vinogradov - E. Bombieri inequality, according to it

   $$\sum_{k \le \frac{\sqrt{x}}{(\log x)^A}} \max_{(l,k)=1} \max_{y \le x} \left| \pi(y,k,l) - \frac{li\ y}{\varphi(k)} \right| \le C \frac{x}{(\log x)^B},$$

$B = 2A + 5$, C is a suitable (non-effective) constant.

J. Kubilius and P. Erdős asked on the necessity of the convergence of the 3 series (1.1), (1.2), (1.3). Partial results were obtained by P.D.T.A. Elliott, I. Kátai, and N. Timofeev. Finally more than 20 years later it was proved by A. Hidebrand that they are necessary ([4]).

2. L. Germán proved the analog of Erdős-Wintner theorem on the set

$$\mathcal{B} = \{n+1 \mid \omega(n) \le \varepsilon(n)\sqrt{\log\log n}\},$$

where $\varepsilon(n) \to 0$. (See [2].)

3. On q-additive functions

Let $f \in \mathcal{A}_q$, ξ_j be independent random variables,

$$P(\xi_j = f(aq^j)) = \frac{1}{q} \quad (a \in A_q),$$

$$\eta_N = \xi_0 + \xi_1 + \ldots + \xi_{N-1},$$

$$m_j = E(\xi_j) = \frac{1}{q}\sum_{a\in A_q} f(aq^j), \quad \sigma_j^2 = \frac{1}{q}\sum_{a\in A_q} f^2(aq^j) - m_j^2.$$

By using the standard method of probability theory one can prove

Theorem 4. *Let $f \in \mathcal{A}_q$. Then*

$$\lim_{x\to\infty} \frac{1}{x}\#\{n < x \mid f(n) < y\} = F(y)$$

for almost all y exists, if and only if (3.1), (3.2) *are convergent:*

$$\sum_{j=0}^{\infty}\sum_{a\in A_q} f(aq^j), \tag{3.1}$$

$$\sum_{j=0}^{\infty}\sum_{a\in A_q} f^2(aq^j). \tag{3.2}$$

What is happening if we consider the distribution of f over a subset $\mathcal{B}$ in $\mathbb{N}_0$?

Let $1 \le j_1 < \ldots < j_h < N, \quad b_1, \ldots, b_h \in A_q$,

$$\mathcal{B}_N \begin{pmatrix} j_1, \ldots, j_h \\ b_1, \ldots, b_h \end{pmatrix} = \{m < q^N \mid m \in \mathcal{B}, \quad \varepsilon_{j_l}(m) = b_l, l = 1, \ldots, h\},$$

$$B_N \begin{pmatrix} j_1, \ldots, j_h \\ b_1, \ldots, b_h \end{pmatrix} = \#\{m < q^N \mid m \in \mathcal{B}, \quad \varepsilon_{j_l}(m) = b_l, l = 1, \ldots, h\},$$

$$B(x) = \#\{m < x \mid m \in \mathcal{B}\}.$$

Let $P(u) \in \mathbb{Z}[u]$ be a polynomial $r = \deg P, \quad P(u) = a_r u^r + \ldots + a_1 u + a_0, \quad a_r > 0$.

Let

$$\Sigma_1 := A\left(x \middle| \begin{matrix} l_1, \ldots, l_h \\ b_1, \ldots, b_h \end{matrix}\right) = \#\{n \le x \mid \varepsilon_{l_j}(P(n)) = b_j, j = 1, \ldots, h\},$$

$$\Sigma_2 := \prod\left(x \middle| \begin{matrix} l_1, \ldots, l_h \\ b_1, \ldots, b_h \end{matrix}\right) = \#\{p \le x \mid \varepsilon_{l_j}(P(p)) = b_j, j = 1, \ldots, h\}.$$

Lemma 1. *Let $q^N \le x < q^{N+1}$. Let h be fixed, λ be an arbitrary constant,*

$$N^{\frac{1}{3}} \le l_1 < \ldots < l_h \le rN - N^{\frac{1}{3}}. \tag{3.3}$$

Then

$$\Sigma_1 = \frac{x}{q^h} + \mathcal{O}\left(\frac{x}{(\log x)^\lambda}\right),$$
$$\Sigma_2 = \frac{\pi(x)}{q^h} + \mathcal{O}\left(\frac{x}{(\log x)^\lambda}\right)$$

uniformly as $l_1, \ldots, l_h$ in (3.3), *$b_1, \ldots, b_h \in A_q$.*

The proof depends on the next Lemmas 2 and 3. Let $e(u) := e^{2\pi i u}$.

Lemma 2 (Hua Loo Keng)**.** *Let $0 < Q \le c_1(k)(\log x)^{\tau_1}$ and*

$$S = \sum_{\substack{p \le x \\ p \equiv t \pmod{Q}}} e(f(p))$$

in which

$$f(y) = \frac{h}{q} y^k + \alpha_1 y^{k-1} + \cdots + a_k, \quad (h, q) = 1.$$

Suppose that $(\log x)^{\tau} \le q \le x^k(\log x)^{-\tau}$. *For arbitrary* $\tau_0 > 0$, *when* $\tau > 2^{6k}(\tau_0+\tau_1+1)$, *we always have*

$$|S| \le c_2(k)x \cdot (\log x)^{-\tau_0} \cdot Q^{-1}.$$

$c_2(k)$ *depends only on* k.

Lemma 3 (I. M. Vinogradov)**.** *Let* f *be as in Lemma 2,*

$$S_1 = \sum_{n \le x} e(f(n)).$$

Let τ_0, τ_3, τ_4 *be arbitrary positive numbers,*

$$\tau \ge 2^k(\tau_0+\tau_3) + 2k\tau_4 + 2^{3(k-2)}.$$

Suppose

$$(\log x)^{\tau} < q \le x^k(\log x)^{-\tau}.$$

Then

$$S_1 \ll x(\log x)^{-\tau}.$$

The constant standing implicitly in $\ll$ *may depend on* τ_3, τ.

Theorem 5. ([1]) *Let* $f \in \mathcal{A}_q$, $\quad f(bq^j) = \mathcal{O}(1)$ *as* $j \to \infty$, $\quad b \in E$. *Furthermore let* $\dfrac{D(x)}{(\log x)^{\frac{1}{3}}} \to \infty$. *Here*

$$D^2(x) = \sum_{j=0}^{N} \sigma_j^2 \qquad (q^N \le x < q^{N+1}).$$

Let $P(u) \in \mathbb{Z}[u]$, $\quad \deg P = r$, $\quad P(u) \to \infty \quad (u \to \infty)$. *Then*

$$\lim \frac{1}{x} \# \left\{ n < x \;\middle|\; \frac{f(P(n)) - M(x^r)}{D(x^r)} < y \right\} = \Phi(y), \tag{3.4}$$

$$\lim \frac{1}{\pi(x)} \# \left\{ n < x \;\middle|\; \frac{f(P(p)) - M(x^r)}{D(x^r)} < y \right\} = \Phi(y), \tag{3.5}$$

$$c_k(x) = \frac{1}{x^r} \sum_{n \le x^r} \left(\frac{f_1(n) - M_1(x^r)}{D_1(x^r)} \right)^k.$$

From Lemma 1 one can deduce that

$$\lim a_k(x) = \lim b_k(x) = \lim c_k(x).$$

It is clear that (3.4) is true if $P(u) = u$, furthermore

$$\lim_{x\to\infty} c_k(x) = \int u^k d\phi(u) = \mu_k.$$

Consequently $\lim a_k(x) = \mu_k$, $\quad \lim b_k(x) = \mu_k$, and the Frechet-Shohat theorem implies the fulfilment of (3.4) and (3.5).

How to prove Lemma 1?

Let

$$\varphi_b(x) = \begin{cases} 1 & \text{if} \quad x \in \left[\frac{b}{q}, \frac{b+1}{q}\right), \\ 0 & \text{otherwise in } [0,1). \end{cases}$$

Let $\varphi_b(x+n) = \varphi_b(x) \quad (n \in \mathbb{Z})$.

Let $b_1, \dots, b_h \in A_q$, $\quad (1 \le) l_1 < \dots < l_h$ be arbitrary integers. Then

$$F(x_1, \dots, x_h) = \varphi_{b_1}(x_1) \dots \varphi_{b_h}(x_h),$$

$$\varepsilon_j(n) = b \quad \iff \quad \left\{\frac{n}{q^{j+1}}\right\} \in \left[\frac{b}{q}, \frac{b+1}{q}\right).$$

Let $t(y) = F\left(\frac{y}{q^{l_1+1}}, \dots, \frac{y}{q^{l_h+1}}\right)$.

Then

$$t(m) = \begin{cases} 1 & \text{if} \quad \varepsilon_{l_j}(m) = b_j, j = 1, \dots, h, \\ 0 & \text{otherwise.} \end{cases}$$

Let $0 < \Delta < \frac{1}{2q}$, $\quad f_b(x) = \frac{1}{2\Delta}\int\limits_{-\Delta}^{\Delta} \varphi_b(x+u)du,$

$$\tilde{t}(y) = f_{b_1}\left(\frac{y}{q^{l_1+1}}\right) \dots f_{b_h}\left(\frac{y}{q^{l_h+1}}\right).$$

We can prove that

$$\sum_{n\le x} |t(P(n)) - \tilde{t}(P(n))|$$

is small, furthermore

$$\tilde{t}(y) = \sum_M T_M e(MVy),$$

$$M = [m_1, \dots, m_h], \quad V = \left[\frac{1}{q^{l_1+1}}, \dots, \frac{1}{q^{l_h+1}}\right],$$

$$\sum |T_M| < \infty.$$

Thus
$$\sum \tilde{t}(P(n)) = \sum T_M \sum_{n\le x} e\left(\frac{A_M}{H_M}P(n)\right),$$
$$\sum \tilde{t}(P(p)) = \sum T_M \sum_{p\le x} e\left(\frac{A_M}{H_M}P(p)\right),$$
$VM = \frac{A_M}{H_M}, \quad (H_M, A_M) = 1.$

We can apply Lemma 1 and 2. This completes the proof.

Theorem 6. *Let $f \in \mathcal{A}_q$. Assume that $\lim M(x)$ exists and is finite, $\lim D^2(x) < \infty$. Then there exist suitable distribution functions $F_1(y), F_2(y)$ such that*

$$\lim \frac{1}{x}\#\left\{n < x \;\middle|\; f(P(n)) < y\right\} = F_1(y) \quad a.a.\ y, \tag{3.6}$$

$$\lim \frac{1}{\pi(x)}\#\left\{p < x \;\middle|\; f(P(p)) < y\right\} = F_2(y) \quad a.a.\ y. \tag{3.7}$$

Theorem 7. *If $P(n) = n$, and (3.7) holds true, then $\lim M(x)$ exists and is finite, furthermore $\lim\limits_{x\to\infty} D(x)$ is finite.*

4. Linear combinations of q - additive functions

Here we mention some theorems without proof.

In this section λ is the Lebesgue measure defined in $\mathbb{R}$.
Let

$$f_1, \ldots, f_k \in \mathcal{A}_q; \quad (1 \le) a_1 < \ldots < a_k (< q), \quad (a_i, q) = 1, \quad (a_i, a_j) = 1 \ \text{ if } \ i \ne j,$$
$$l(n) = f_1(a_1 n) + \ldots + f_k(a_k n).$$

Definition 2. *We say that $l(n)$ is a tight sequence if $\exists\, A_N$ such that*

$$C(K) := \limsup \frac{1}{q^N}\#\{n < q^N \mid |l(n) - A_N| > K\} \to 0$$

as $K \to \infty$.

Theorem 8. (K.-H. Indlekofer and I. Kátai, [5])

1. *$l(n)$ is tight if and only if $\exists\ \gamma_1, \dots, \gamma_k$ for which $a_1\gamma_1 + \dots + a_k\gamma_k = 0$ and for $\psi_l(n) := f_l(n) - \gamma_l(n)$ we have*

$$(4.1)_l \qquad \sum_{j=0}^{\infty}\sum_{a=0}^{q-1}\psi_l(aq^j)^2 < \infty \quad (l = 1, \dots, k).$$

2. *If the conditions of 1) hold, then*

$$\lim \frac{1}{q^N}\#\{n < q^N \mid l(n) - E_N < y\} = F(y) \quad a.a.\ y,$$

where F is a suitable distribution function,

$$A_N^{(l)} := \frac{1}{q}\sum_{j=0}^{N}\sum_{a\in A_q}\psi_l(aq^j),$$

$$E_N = \sum_{l=1}^{k} A_N^{(l)}.$$

3. *$l(n)$ has a limit distribution, if and only if the conditions of 1) are satisfied, and if E_N has a finite limit.*

Theorem 9. *1. The sequence $l(p)$ $(p \in \mathcal{P})$ is tight if it is tight on the whole set $\mathbb{N}$.*

2. *If $((4.1)_l)$ hold, then*

$$\lim_{N\to\infty}\frac{1}{\pi(q^N)}\#\{p < q^N \mid l(p) - E_N < y\} = F^*(y) \quad a.a.\ y.$$

3. *$l(p)$ has a limit distribution if and only if $l(p)$ is tight and if E_N has a finite limit.*

Let $\mu_l(u) = \frac{1}{q}\sum_{c\in A_q} f_l(cq^u), \quad p(u) = \sum_{l=1}^{k}\mu_l(u),$

$$F(N) := \sum_{u=0}^{N-1} p(u),$$

$$\pi_u(c_1, \dots, c_k) := \frac{1}{q^{u+1}}\#\{n < q^{u+1} \mid \varepsilon_u(a_j n) = c_j\},$$

$$\pi(c_1,\ldots,c_k)=\lambda\left(\left\{x\in[0,1]\mid\{a_jx\}\in\left[\frac{c_j}{q},\frac{c_j+1}{q}\right),j=1,\ldots,k\right\}\right).$$

We have $\pi_u(c_1,\ldots,c_k)=\pi(c_1,\ldots,c_k)+\mathcal{O}\left(\frac{1}{q^u}\right)\quad(u\to\infty)$.

Let

$$\tau_u:=\sum_{c_1,\ldots,c_k\in A_q}(f_1(c_1q^u)+\ldots+f_k(c_kq^u)-p(u))^2\pi_u(c_1,\ldots,c_k),$$

$$\varrho_N^2=\sum_{u=0}^{N-1}\tau_u.$$

Theorem 10. *Assume that*

$$\frac{|f_l(cq^M)|}{\varrho_M}\to 0\qquad as\quad M\to\infty$$

for $l=1,\ldots,k$. Then

$$\lim_{N\to\infty}\frac{1}{q^N}\#\left\{n<q^N\;\middle|\;\frac{l(n)-F_N}{\varrho_N}<y\right\}=\phi(y).$$

Theorem 11. *Assume that*

$$\sup_M\max_{c\in A_q}\max_l|f_l(cq^M)|<\infty,$$

and that

$$\varrho_{K_N}^2/\varrho_N^2\to 0,\quad(\varrho_N^2-\varrho_{N-K_N}^2)/\varrho^2N\to 0$$

as $K_N=[\log N]$, $\quad N\to\infty$. Then

$$\lim_{N\to\infty}\frac{1}{\pi(q^N)}\#\left\{p<q^N\;\middle|\;\frac{l(p)-F_N}{\varrho_N}<y\right\}=\phi(y).$$

5. On a theorem of G. Harman and I. Kátai ([3])

Let $0\le j_1<j_2<\ldots<j_r\le N-1,\quad x>\exp(q^2),\ q^{N-1}\le x\le q^N-1$. Let $\underline{b}=(b_1,\ldots,b_r),\ \underline{j}=(j_1,\ldots,j_r)$,

$$\prod\left(x\middle|\begin{matrix}\underline{j}\\ \underline{b}\end{matrix}\right):=\#\{p\le x\mid\varepsilon_{l_j}(p)=b_j,j=1,\ldots,r\},$$

$$A\left(x\middle|\begin{matrix}\underline{j}\\ \underline{b}\end{matrix}\right):=\#\{n\le x\mid\varepsilon_{l_j}(n)=b_j,j=1,\ldots,r\}.$$

Theorem 12. *Suppose that* $1 \leq r < C\sqrt{N}/\log N$. *Then*

$$\prod \left(x \middle| \begin{matrix} \underline{j} \\ \underline{b} \end{matrix} \right) = \frac{q^r f(\underline{b}, \underline{j})}{\log x} A \left(x \middle| \begin{matrix} \underline{j} \\ \underline{b} \end{matrix} \right) + O\left(\frac{x \log\log x}{\varphi(q) q^{r-1} (\log x)^2} \right),$$

where

$$f(b,j) = \begin{cases} q^{-r} & \textit{if} \;\; j_1 > 0, \\ 0 & \textit{if} \;\; j_1 = 0, (b_1, q) > 1, \\ q^{1-r}\varphi(q)^{-1} & \textit{if} \;\; j_1 = 0, (b_1, q) = 1. \end{cases}$$

A similar theorem is claimed in the paper ([7]) of I. Kátai, but there were some mistakes in the proof.

Our guess: Theorem 12 remains valid up to $r < \frac{1}{3}N$.

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