Annales Univ. Sci. Budapest., Sect. Comp. 43 (2014) 245–252

ON THE ITERATES OF SOME MULTIPLICATIVE FUNCTIONS

Hoang Thi Lan Giao (Hue, Vietnam)

Imre Kátai and Bui Minh Phong (Budapest, Hungary)

Communicated by J.-M. DeKoninck

(Received July 30, 2013; accepted February 3, 2014)

Abstract. Let $f_k(n)$ be the k-th iterates of a function $f(n)$, i.e. $f_0(n) := n$, $f_1(n) := f(n), \dots, f_{k+1}(n) := f(f_k(n))$ $(k = 1, 2, \cdots)$. We prove that if $n \in \mathbb{N}$ and the function f is defined by $f(p) = p$ and $f(p^\alpha) = p + p^2$ for all primes p, $\alpha \geq 2$, then for some $k \in \mathbb{N}$ there are an

$$u \in \{1,\ 2,\ 3,\ 2.3,\ 2^3.3^2, 2^2.3, 2.3^2, 2^3.3\}$$

and a square-free $\mathcal{D} \in \mathbb{N}$, $(\mathcal{D}, 6) = 1$ such that $f_k(n) = u.\mathcal{D}$.

1. Let, as usual, $\mathcal{P}$, $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ be the set of primes, positive integers, non-negative integers, integers, real and complex numbers, respectively. (n, m) denotes the greatest common divisor of n and m. We say that $f : \mathbb{N} \to \mathbb{R}$ is an additive function, if $f(mn) = f(m) + f(n)$ for all $(m, n) = 1$. Let $\mathcal{A}$ denote the set of all additive functions. A function $g : \mathbb{N} \to \mathbb{C}$ is multiplicative, if $g(1) = 1$ and $g(mn) = g(m) \cdot g(n)$ for all $(m, n) = 1$. We denote by $\mathcal{M}$ the set of all multiplicative functions.

Key words and phrases: multiplicative functions, additive functions, iterates of function.
2010 Mathematics Subject Classification: 11A07, 11A25, 11N25, 11N64.
This work was completed with the support of the Hungarian and Vietnamese TET (grant agreement no. TET 10-1-2011-0645).

For some n let $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ be its prime decomposition. We say that n is square-free, if $\alpha_1 = \cdots = \alpha_r = 1$, and n is square-full if $\min \alpha_j \geq 2$. The Möbius function $\mu(n)$ is defined as follows:

$$\mu(n) = \begin{cases} 1 & \text{if } \ n = 1, \\ 0 & \text{if } \ \max \alpha_j \geq 2, \\ (-1)^r & \text{if } \ \alpha_1 = \cdots = \alpha_r = 1. \end{cases}$$

It is clear that $\mu \in \mathcal{M}$, $|\mu(n)| = 1$ if and only if n is square free.

We are interested in such multiplicative functions f for which $f(n) \in \mathbb{N}$ ($n \in \mathbb{N}$), and $f(p^\alpha) = Q_\alpha(p)$ for all primes p, $Q_\alpha(z) \in \mathbb{Z}[z]$, $Q_\alpha(z) > 0$ for every $z \in \mathbb{N}$. Many of interesting multiplicative functions is of that type, e.g.

$\sigma(n) =$ sum of divisors function,

$\sigma^\star(n) =$ sum of unitary divisors function,

$\sigma^{(e)}(n) =$ sum of exponential divisors function,

$\varphi(n) =$ Euler's totient function.

We have

$$\varphi(p^\alpha) = p^{\alpha-1}(p-1), \quad \sigma(p^\alpha) = \frac{p^{\alpha+1}-1}{p-1},$$
$$\sigma^\star(p^\alpha) = 1 + p^\alpha, \quad \sigma^{(e)}(p^\alpha) = \sum_{\beta | \alpha} p^\beta.$$

The corresponding $Q_\alpha(0) \in \{-1, 1\}$ in the cases $f = \varphi, \sigma, \sigma^\star$, while in the case $f = \sigma^{(e)}$ we have $Q_\alpha(0) = 0$.

Let

$$f_0(n) := n, \ f_1(n) := f(n), \ldots, \ f_{k+1}(n) := f(f_k(n)), \quad (k = 1, 2, \cdots).$$

Some functions of $f_k(n)$, especially $\omega(f_k(n)), \omega(f_k(p+1))$ have been investigated in several papers. Here $\omega(n)$ is the number of prime factors of n. For the function $f(n) = \sigma^{(e)}(n)$ is was proved that

$$\lim_{x\to\infty} \frac{1}{x} \natural \left\{ n \leq x \ \Big| \ \frac{f_j(n)}{f_{j-1}(n)} < \alpha_j, \ j = 1, 2, \cdots, k \right\} = \\ = F_k(\alpha_1, \cdots, \alpha_k)$$

exists, F_k is strictly monotonic in each variables in $(\alpha_1, \cdots, \alpha_k) \in (1, \infty)^k$. (See [1] and the previous papers [2], [3], [4]).

If $f = \sigma^{(e)}$, then $f(n) > n$ holds for every non square-free n, and $f(n) = n$ if n is square-free ($n = 1$ is included).

Conjecture 1. *For every $n \in \mathbb{N}$ there exists a constant K_n such that*

$$p^2 \nmid f_k(n) \quad (k = 0, 1 \cdots) \quad \textit{if} \quad p > K_n.$$

Conjecture 2. *There exists an absolute constant R such that for every n there is a $k_0(n)$ such that*

$$p^2 \nmid f_k(n) \qquad \textit{if} \quad p > R, \; k > k_0(n).$$

We are unable to prove our conjectures. We shall prove such theorem in the case when

$$f(p^\alpha) = \begin{cases} p & \text{if } \alpha = 1, \\ p + p^2 & \text{if } \alpha \geq 2. \end{cases}$$

2. Let $Q_\alpha(z) \in \mathbb{Z}[z]$ be such a sequence of polynomials ($\alpha = 1, 2, \cdots$) for which

$$Q_1(z) = z, \quad Q_\alpha(z) = z + \sum_{\ell=2}^{t(\alpha)} a_\ell(\alpha) z^\ell,$$

where

$$t(\alpha) \leq C\alpha, \; |a_\ell(\alpha)| \leq C \quad (\ell = 2, \cdots, t(\alpha), \alpha \geq 2),$$

C is a suitable constant, $Q_\alpha(n) \geq n$.

Let $f \in \mathcal{M}$ be defined by

$$f(p^\alpha) = Q_\alpha(p) \quad (p \in \mathcal{P}).$$

Let h be a fixed positive number. Let $n = f_0(n)$, $n_j = f_j(n)$ $(j = 1, \cdots, h)$. We can write n as $n = Km$, where K is square-full, m is square-free, $(K, m) = 1$. Let m_1 be the largest divisor of m which is coprime to $f(K)$. Let $m = m_1\nu_1$. Then $\nu_1 \mid f(K)$. Write $K_1 := f(K)\nu_1$. We have

$$n_1 = f(n) = f(Km) = f(K)m = f(K)m_1\nu_1 = K_1 m_1.$$

Then clearly $(K_1, m_1) = 1$ and m_1 is square-free.

We can continue this procedure. We have

$$n_2 = f(K_1)m_1 = f(K_1)\nu_2 m_2,$$

where

$$\nu_2|f(K_1),\ (m_2, f(K_1)) = 1,\quad m_h = \nu_2 m_2,\quad K_2 = f(K_1)\nu_2.$$

In general we have

$$n_j = f(K_{j-1})m_{j-1} = K_j m_j,\ K_j = f(K_{j-1})\nu_j$$

and

$$m_{j-1} = \nu_j m_j,\ \nu_j|f(K_{j-1}),\ (m_j, K_j) = 1$$

for $j = 1, \cdots, k$.

Assume now that K is a fixed square-full integer, $K_1,\ \nu_1, \cdots, K_h,\ \nu_h$ are defined as follows:

(1) $(\nu_1, K) = 1$, ν_1 is square-free, $\nu_1|f(K)$, $K_1 = f(K)\nu_1$,

(2) $(\nu_2, KK_1) = 1$, ν_2 is square-free, $\nu_2|f(K_1)$, $K_2 = f(K_1)\nu_2$,

$\vdots$

(h) $(\nu_h, KK_1 \cdots K_{h-1}) = 1$, ν_h is square-free, $\nu_h|f(K_{h-1})$, $K_h = f(K_{h-1})\nu_h$.

Let us consider those integers n which can be written as $n = KTM$, where $T = \nu_1 \cdots \nu_h$, $(M, KK_1 \cdots K_h) = 1$, M is square-free.

For such an n we have

$$n = KTM,\quad n_1 = f(K)TM,\quad \frac{n_1}{n} = \frac{f(K)}{K} = \frac{K_1}{\nu_1 K},$$

$$n_2 = f\big(f(K)\nu_1\big)\nu_2 \cdots \nu_h M,\quad n_1 = f(K)\nu_1\nu_2 \cdots \nu_h M,$$

thus

$$\frac{n_2}{n_1} = \frac{f(K_1)}{K_1} = \frac{K_2}{\nu_2 K_1},$$

and in general

$$\frac{n_{j+1}}{n_j} = \frac{K_{j+1}}{\nu_{j+1}K_j} \qquad (j = 1, \cdots, h-1).$$

Let A be an arbitrary positive integer and

$$M\big(x|A\big) = \sum_{\substack{n \le x \\ (n,A)=1}} |\mu(n)|.$$

Let $\kappa(p) = \frac{1}{1+\frac{1}{p}}$ $(p \in \mathcal{P})$, κ be strongly multiplicative.

It can be proved easily that

$$M\big(x|A\big) = (1 + o_x(1))\frac{6}{\pi^2}\kappa(A)x.$$

Consequently

$$\natural\big\{n = KTM \le x, \quad M \text{ is square-free}, \quad (M,\ KT) = 1\big\} = \\ = (1 + o_x(1))\frac{6}{\pi^2}\frac{\kappa(KT)}{KT}x$$

for every fixed K and T.

Hence we obtain easily

Theorem 1. *There exists a sequence*

$$\Big(y_1^{(m)}, \cdots, y_h^{(m)}\Big) \in (0, \infty)^h \quad (m = 1, 2 \cdots)$$

such that

$$\frac{1}{x}\natural\Big\{n \le x \ \Big|\ \frac{f_j(n)}{f_{j-1}(n)} = y_j^{(m)}, \quad j = 1, \cdots, h\Big\} \to d_j \quad (x \to \infty),$$

and

$$\sum_{j=1}^{\infty} d_j = 1.$$

Let

$$(*) \qquad R\big(x|A\big) = \sum_{\substack{p \le x \\ (p+1, A) = 1}} |\mu(p + 1)|.$$

By using the prime number theorem for the arithmetical progression, we obtain that

$$R\big(x|A\big) = (1 + o_x(1))\frac{6}{\pi^2}\mathcal{D}(A)\mathrm{li}x,$$

where

$$\mathcal{D}(A) = \prod_{p|A}\Big(1 - \frac{1}{p}\Big)\cdot\prod_{p\nmid A}\Big(1 - \frac{1}{p(p-1)}\Big) = \prod_{p|A}\Big(\frac{p^2 - 2p}{p^2 - p - 1}\Big)E$$

and

$$E := \prod_{p \in \mathcal{P}}\Big(1 - \frac{1}{p(p-1)}\Big).$$

Hence we can deduce

Theorem 2. *There exists a sequence*

$$\left(z_1^{(n)},\cdots,z_h^{(n)}\right)\in(0,\infty)^h\quad(n=1,2\cdots)$$

such that

$$\lim_{x\to\infty}\frac{1}{\pi(x)}\natural\left\{p\le x\;\middle|\;\frac{f_j(p+1)}{f_{j-1}(p+1)}=z_j^{(n)},\quad j=1,\cdots,h\right\}\to\mathcal{D}_j\quad(x\to\infty)$$

and

$$\sum_{j=1}^{\infty}\mathcal{D}_j=1.$$

Remark. The relation $(*)$ is even uniform as $1\le A\le C_1(\log x)^{C_2}$, where $C_1,\;\; C_2$ are arbitrary constants. This easily follows from the Siegel-Walfisz theorem.

3. Let

$$f(p^\alpha)=\begin{cases}p & \text{if}\;\;\alpha=1,\\ p+p^2 & \text{if}\;\;\alpha\ge 2.\end{cases}$$

Theorem 3. *Let $n\in\mathbb{N}$. Then for some $k\in\mathbb{N}$ there are an*

$$u\in\{1,\;2,\;3,\;2.3,\;2^3.3^2,2^2.3,2.3^2,2^3.3\}$$

and a square-free $\mathcal{D}\in\mathbb{N}$, $(\mathcal{D},6)=1$ such that

$$f_k(n)=u.\mathcal{D}.$$

Furthermore, the following possibilities can be occur:

a) If $u\in\{1,\;2,\;3,\;2.3,\;2^3.3^2\}$, then

$$f_{k+\ell}(n)=u.\mathcal{D}\quad\textit{for all}\quad\ell=0,1,2,\cdots.$$

b) If $u\in\{2^2.3,\;2.3^2,\;2^3.3\}$, then

$$f_{k+\ell}(n)\in\{2^2.3.\mathcal{D},\;2.3^2.\mathcal{D},\;2^3.3.\mathcal{D}\}\quad\textit{for all}\quad\ell=0,1,2,\cdots.$$

Proof of Theorem 3. For each $n \in \mathbb{N}, n > 1$ let

$$\kappa(n) := \max_{\substack{p^\alpha \| n \\ \alpha \geq 2}} \{ p \in \mathcal{P} \}.$$

Let us observe that if $\kappa(n) \geq 5$, then $\kappa(f(n)) < \kappa(n)$. This is obvious. Let

$$n = p_1^{\alpha_1} \cdots p_j^{\alpha_j} p_{j+1} \cdots p_r,$$

where $p_1 < \cdots < p_j,\ \alpha_1 \geq 2, \cdots, \alpha_j \geq 2$ and $(p_1 \cdots p_j,\ p_{j+1} \cdots p_r) = 1$. Since $\kappa(n) = p_j \geq 5$, $\alpha_j \geq 2$, therefore $f(p_j^{\alpha_j}) = p_j(1 + p_j)$, $1 + p_j$ is even, thus the largest prime factor of $1+p_j$ is less than p_j. Consequently $\kappa(f(n)) < p_j = \kappa(n)$.

Hence it follows that for every n in the sequence $f_0(n), f_1(n), \cdots$, there is a k for which $p^2 \mid f_k(n)$ implies that $p = 2$ or $p = 3$. Then $f_k(n) = 2^\alpha 3^\beta \mathcal{D}$, $(\mathcal{D}, 6) = 1$, $\mathcal{D}$ is square-free. It is clear that

$$\begin{aligned} f(2^\alpha 3^\beta) \in \{1, f(2), f(3), f(2^2), f(3^2), f(2.3), f(2^2.3), f(2.3^2), f(2^2.3^2)\} = \\ = \{1,\ 2,\ 3,\ 2.3,\ 2^2.3,\ 2.3^2,\ 2^3.3,\ 2^3.3,\ 2^3.3^2\} := \mathcal{U}. \end{aligned}$$

Let

$$\mathcal{U}_1 = \{1,\ 2,\ 3,\ 2.3,\ 2^3.3^2\} \quad \text{and} \quad \mathcal{U}_2 = \{2^2.3, 2.3^2, 2^3.3\}.$$

Let us consider the graph

$$2^\alpha.3^\beta \to 2^{\alpha_1}.3^{\beta_1} \quad \text{if} \quad f(2^\alpha.3^\beta) = 2^{\alpha_1}.3^{\beta_1}.$$

It is clear that

$$a \to a \quad \text{for all} \quad a \in \mathcal{U}_1.$$

Furthermore

$$2^2.3 \to\ 2.3^2 \to\ 2^3.3 \to\ 2.3^2.$$

Theorem 3 is proved.

References

[1] **Kátai, I. and M.V. Subbarao,** On the iterates of the sum of exponential divisors, *Math. Pannon.,* **10** (2) (1999), 153–158.

[2] **Lucht, L.,** On the sum of exponential divisors and its iterates, *Arch. Math. (Basel),* **27** (4) (1976), 383–386.

[3] **Straus, E.G. and M.V. Subbarao,** On exponential divisors, *Duke Math. J.,* **41** (1974), 465–471.

[4] **Subbarao, M.V.,** *On some arithmetic convolutions. The theory of arithmetic functions,* Lecture Notes in Math. **251**, Springer, Berlin, 1972.

Hoang Thi Lan Giao
Hue University
77 Nguyen Hue
Hue City, Vietnam
htlgao@hueuni.edu.vn

Imre Kátai and Bui Minh Phong
Eötvös Loránd University
Pázmány Péter sét. 1/C
H-1117 Budapest, Hungary
katai@compalg.inf.elte.hu
bui@compalg.inf.elte.hu