Annales Univ. Sci. Budapest., Sect. Comp. 43 (2014) 217–237

MAXIMAL ORDER OF A CLASS OF MULTIPLICATIVE FUNCTIONS

Titus Hilberdink (Reading, UK)

Communicated by Pavel Varbanets

(Received May 15, 2013; accepted June 25, 2013)

Abstract. In this paper we obtain the maximal order of the multiplicative function given at the prime powers by $f(p^k) = \exp\{h(k)l(p)\}$ where $h(\cdot)$ and $l(\cdot)$ are increasing and decreasing functions respectively with $l(p)$ regularly varying of index $-\alpha$ ($0 \leq \alpha < 1$). For example, we show that under appropriate conditions

$$\max_{n\leq N} \log f(n) \sim \left(\sum_{n=1}^{\infty} \Delta h(n)^{1/\alpha} \right)^{\alpha} L(\log N)$$

where $L(x) = \sum_{p\leq x} l(p)$ and $\Delta h(n) = h(n) - h(n-1)$.

1. Introduction

We consider a class of multiplicative functions $f(n)$ which at the prime powers are given by

$$f(p^k) = e^{h(k)l(p)} \qquad p \in \mathbb{P},\ k \in \mathbb{N}_0. \tag{1.1}$$

In particular, we are interested in the maximal order of such functions*. If $l(p)$ is constant, then f is a *prime-independent* multiplicative function and the

Key words and phrases: Arithmetical functions, maximal order.
2010 Mathematics Subject Classification: 11N37, 11N56.

*More accurately, the maximal order of $\log f$; here the maximal order of F is loosely defined to be any real positive function G such that $\limsup_{n\to\infty} \frac{F(n)}{G(n)} = 1$. In practise, one chooses the simplest possible G.

maximal order has been discussed by various authors (see for example, [7], [8], [9] and references therein). Thus, for example, Shui [8] has proven that (using our notation) if $f(p^k) = e^{h(k)}$ where $0 \le h(k) \le Ak^\beta$ with $0 < \beta < 1$ and some A, then

$$\limsup_{n\to\infty} \frac{\log f(n) \log\log n}{\log n} = \max_{k\ge 1} \frac{h(k)}{k}.$$

In this case, the maximal order occurs, roughly, for n of the form $(\prod_{p\le P} p)^m$ where m maximises $h(k)/k$. Results such as the above were then applied to find the maximal order of divisor-like functions.

For non prime-independent multiplicative functions not much work appears to have been done. In [10], Tóth and Wirsing consider a class of multiplicative functions which are at most of order $\log\log n$ including $\frac{n}{\varphi(n)}$, but their results do not overlap with ours.

For the function $\sigma_{-\alpha}(n) = \sum_{d|n} d^{-\alpha}$, Gronwall [3] showed 100 years ago that for $0 < \alpha < 1$, the maximal order is given by

$$\exp\left\{ \frac{1+o(1)}{1-\alpha} \cdot \frac{(\log n)^{1-\alpha}}{\log\log n} \right\}.$$

Notice that in this case

$$\sigma_{-\alpha}(p^k) = 1 + \frac{1}{p^\alpha} + \cdots + \frac{1}{p^{k\alpha}} = \exp\left\{ \frac{1+o(1)}{p^\alpha} \right\}$$

which is of the form (1.1) in an asymptotic sense, with $h(k)$ constant and $l(p) = p^{-\alpha}$. In fact, the maximum order occurs for n of the form $\prod_{p\le P} p$, and to find this maximum is then relatively easy, using the prime number theorem. More generally, if f is multiplicative and given by (1.1) and both h and l are decreasing (and non-negative), then the maximum order of $f(n)$ again occurs for n of the form $\prod_{p\le P} p$, since $f(p^k) \le f(p)$ and $f(q) \le f(p)$ for primes p, q with $p < q$. As such, $\log n = \theta(P) \sim P$ by the prime number theorem and multiplicativity of $f(n)$ gives

$$\log f(n) = h(1) \sum_{p\le P} l(p) = h(1)L(P),$$

where $L(x) = \sum_{p\le x} l(p)$. If now we assume that $L(y) \sim L(x)$ whenever $y \sim x$, then $\log f(n) \sim h(1)L(\log n)$ (for such n) and this represents the maximal order.

In this article, we consider the less trivial (and perhaps more interesting) case is where h is increasing, while keeping l decreasing. As such we shall see that the maximal order occurs for $n = \prod_{p\le P} p^{a_p}$ with a_p decreasing. The problem then reduces to finding the optimal a_p which maximises $f(n)$. A simple lower bound for the maximal order can be found by taking $a_p = 1$ for all $p \le P$, giving (under some mild conditions on L)

$$\limsup_{n\to\infty} \frac{\log f(n)}{L(\log n)} \ge h(1).$$

With some extra conditions, we also have $\log f(n) \ll L(\log n)$ and the question reduces to finding this limsup. First, we require some bound on the growth of h with respect to L if we want $\log f(n) \ll L(\log n)$. For if $n = 2^k$, then

$$\log f(n) = \log f(2^k) = l(2)h(k) = l(2)h\Big(\frac{\log n}{\log 2}\Big),$$

so $h(k) = o(L(k))$ is necessary. A futher natural condition is that L should be *regularly varying* (see §1. for the definition). In fact, for our main results we shall assume that L is regularly varying of index $1-\alpha$ for some $\alpha \in [0,1)$, while

$$h(k) \ll k^\beta \qquad \text{for some } \beta < 1-\alpha.$$

As such, $L(y) \sim L(x)$ whenever $y \sim x$ and $L(x) = x^{1-\alpha+o(1)}$.

Finally, we prove a slightly stronger result in that we find an asymptotic formula for $\max_{n\le N} \log f(n)$.

Let $\Delta h(n) = h(n) - h(n-1)$ for $n \in \mathbb{N}$. Note that $h(0) = 0$ (by definition) and so $\Delta h(1) = h(1)$. Our main result is:

Theorem 1. *Let f be multiplicative and given at the prime powers by* (1.1), *where we assume that h is increasing and l is decreasing. Further suppose that $L(x) = \sum_{p\le x} l(p)$ is regularly varying of index $1-\alpha$, where $0 \le \alpha < 1$, and $h(n) \ll n^\beta$ for some $\beta < 1-\alpha$. Then*

$$\max_{n\le N} \log f(n) \sim R_\alpha L(\log N)$$

where

$$R_\alpha = \sup_{a_n \searrow 0} \frac{\sum_{n=1}^\infty \Delta h(n) a_n^{1-\alpha}}{(\sum_{n=1}^\infty a_n)^{1-\alpha}} = \sup_{\substack{a_n\searrow 0\\ \sum_{n=1}^\infty a_n = 1}} \sum_{n=1}^\infty \Delta h(n) a_n^{1-\alpha}. \tag{1.2}$$

The supremum here is over all decreasing sequences a_n, not identically zero, for which $\sum_1^\infty a_n$ converges. In various cases we can evaluate R_α more explicitly. In particular we note that by Hölder's inequality

$$\sum_{n=1}^\infty \Delta h(n) a_n^{1-\alpha} \le \Big(\sum_{n=1}^\infty \Delta h(n)^{1/\alpha}\Big)^\alpha \Big(\sum_{n=1}^\infty a_n\Big)^{1-\alpha} \tag{1.3}$$

and $R_\alpha \le (\sum_{n=1}^\infty \Delta h(n)^{1/\alpha})^\alpha$ always. The case of equality leads to:

Theorem 2. *Let f be as in Theorem 1 and suppose further that $\Delta h(n)$ decreases with n. Then*

$$\max_{n\le N}\log f(n)\sim\left(\sum_{n=1}^{\infty}\Delta h(n)^{1/\alpha}\right)^{\alpha}L(\log N).$$

Note that the series $\sum_{n=1}^{\infty}\Delta h(n)^{1/\alpha}$ converges if $\Delta h(n)$ decreases as $\Delta h(n)\le \le\frac{h(n)}{n}$, so $\Delta h(n)^{1/\alpha}\ll n^{-\gamma}$ where $\gamma=\frac{1-\beta}{\alpha}>1$.

In the case $\alpha=0$, R_α can be evaluated and gives:

Theorem 3. *Let f be multiplicative and given at the prime powers by* (1.1), *where h is increasing and l is decreasing and L is regularly varying of index* 1. *Suppose that $h(n)\ll n^{\beta}$ for some $\beta<1$. Then*

$$\max_{n\le N}\log f(n)\sim\left(\max_{n\in\mathbb{N}}\frac{h(n)}{n}\right)L(\log N).$$

The form (1.1) (with h increasing and l decreasing) may seem restrictive, but actually the results apply to cases where (1.1) holds in an asymptotic sense. We illustrate this in Example 2, in Section 6. Indeed, the example

$$f(n)=\frac{1}{d(n)}\sum_{d|n}\sigma_{-\alpha}(d)^2$$

for which $\log f(p^k)=\frac{2k}{(k+1)p^{\alpha}}(1+O(\frac{1}{p^{\alpha}}))$, motivated the present results.

The rest of the paper is organised as follows. First we recall the notion of regular variation, then in Section 3 we find lower bounds for $\log f(n)$, to be followed in Section 4 by upper bounds and the proofs of the results.

In Section 5, we show how to evaluate R_α in case $\Delta h(n)$ is not decreasing and $\alpha\ne 0$. Finally, we present some examples.

2. Some preliminaries

Notations: We write $f\ll g$ to mean $f=O(g)$; i.e. $|f(x)|\le Ag(x)$ for some constant A and all x sufficiently large. We write $f\lesssim g$ to mean $f(x)\le(1+o(1))g(x)$, and similarly for $f\gtrsim g$. Finally, $f\prec g$ means $f(x)=o(g(x))$, while $f\succ g$ is the same as $g\prec f$.

Regular Variation: A function $\ell : [A, \infty) \to \mathbb{R}$ is *regularly varying of index* ρ if it is measurable, eventually positive, and

$$\ell(\lambda x) \sim \lambda^\rho \ell(x) \quad \text{as } x \to \infty \text{ for every } \lambda > 0 \tag{2.1}$$

(see [2] for a detailed treatise on the subject). We shall sometimes denote this by $\ell \in \mathcal{R}_\rho$. If $\rho = 0$, then ℓ is said to be *slowly varying*. For example, $x^\rho (\log x)^\tau$ is regularly varying of index ρ for any τ. Trivially, if $\ell_1 \in \mathcal{R}_\rho$ and $\ell_2 \in \mathcal{R}_\sigma$, then $\ell_1 \ell_2 \in \mathcal{R}_{\rho+\sigma}$, while $\ell_1^\lambda \in \mathcal{R}_{\rho\lambda}$.

The Uniform Convergence Theorem says that (2.1) is automatically *uniform* for λ in compact subsets of $(0, \infty)$. In particular, $\ell(x) \sim \ell(y)$ whenever $x \sim y$. We shall make use of *Karamata's Theorem: for ℓ regularly varying of index ρ,*

$$\int_A^x \ell \sim \frac{x\ell(x)}{\rho+1} \quad \textit{if } \rho > -1, \qquad \int_x^\infty \ell \sim -\frac{x\ell(x)}{\rho+1} \quad \textit{if } \rho < -1,$$

while if $\rho = -1$, $\int^x \ell$ is slowly varying and $\int^x \ell \succ x\ell(x)$.

We shall also make use of *Potter's bounds* (see [2], p.25): *if ℓ is regularly varying of index ρ then for any chosen $A > 1$ and $\delta > 0$, there exists $X = = X(A, \delta)$ such that*

$$\frac{\ell(y)}{\ell(x)} \le A \max\Big\{ \Big(\frac{y}{x}\Big)^{\rho+\delta}, \Big(\frac{y}{x}\Big)^{\rho-\delta} \Big\} \qquad \text{for } x, y \ge X.$$

The notion of regular variation extends to sequences ([2], p.52). For l defined on $\mathbb{P}$ — the set of primes, we say l is *regularly varying of index* ρ if there exists a $\tilde{l} \in \mathcal{R}_\rho$, defined on $[2, \infty)$ such that $\tilde{l}(p) = l(p)$. As such, we can always take $\tilde{l}$ to be the step function defined by $\tilde{l}(x) = l(p)$ for $p \le x < p'$ where p and p' are consecutive primes, which we shall do from now on, and we denote this extension by l.

We note that if l is decreasing, regular variation of l (of index > -1) is equivalent to regular variation of L, where $L(x) = \sum_{p \le x} l(p)$. Indeed, by the Prime Number Theorem and Karamata's Theorem, if l is regularly varying of index $-\alpha$ and $\alpha < 1$, then

$$L(x) = \int_{2-}^x l(t)\, d\pi(t) \sim \int_2^x \frac{l(t)}{\log t}\, dt \sim \frac{x l(x)}{(1-\alpha)\log x} \tag{2.2}$$

which is regularly varying of index $1 - \alpha$. Conversely, if $L \in \mathcal{R}_{1-\alpha}$ for some $\alpha < 1$ and l is decreasing, then for every $\lambda > 1$

$$l(\lambda x)(\pi(\lambda x) - \pi(x)) \le L(\lambda x) - L(x) = \sum_{x < p \le \lambda x} l(p) \le l(x)(\pi(\lambda x) - \pi(x)).$$

Using $L \in \mathcal{R}_{1-\alpha}$ and $\pi \in \mathcal{R}_1$ and dividing by $L(x)$ gives

$$\frac{\lambda^{1-\alpha}-1}{\lambda-1} \lesssim \frac{l(x)\pi(x)}{L(x)} \lesssim \frac{\lambda^{1-\alpha}-1}{\lambda-1}\lambda^{\alpha},$$

and on letting $\lambda \to 1$, (2.2) follows again, so that $l \in \mathcal{R}_{-\alpha}$.

3. Lower bounds for $\log f(n)$

Proposition 3.1. *Let f be multiplicative with $f(p^k) = \exp\{h(k)l(p)\}$. Put $n = \prod_{p\le P} p^{[g(P/p)]}$, where $g : [1,\infty) \to \mathbb{R}$ is continuous, strictly increasing without bound, and $g(1) = 1$. Then*

$$\log n = \sum_{r\ge1} \theta\Big(\frac{P}{g^{-1}(r)}\Big) \tag{3.1}$$

$$\log f(n) = \sum_{r\ge1} \Delta h(r) L\Big(\frac{P}{g^{-1}(r)}\Big) \tag{3.2}$$

where $\theta(x) = \sum_{p\le x} \log p$ and $L(x) = \sum_{p\le x} l(p)$.

Of course the series are finite, ending when $g^{-1}(r) > P/2$.

Proof. We have

$$\log n = \sum_{p\le P} \Big[g\Big(\frac{P}{p}\Big)\Big] \log p = \sum_{r\ge1} r \sum_{\substack{p\le P\\ \text{st } [g(P/p)]=r}} \log p.$$

But $[g(P/p)] = r \iff \frac{P}{g^{-1}(r+1)} < p \le \frac{P}{g^{-1}(r)}$, so

$$\log n = \sum_{r\ge1} r\bigg(\theta\Big(\frac{P}{g^{-1}(r)}\Big) - \theta\Big(\frac{P}{g^{-1}(r+1)}\Big)\bigg) = \sum_{r\ge1} \theta\Big(\frac{P}{g^{-1}(r)}\Big).$$

For (3.2), we have

$$\log f(n) = \sum_{p\le P} h\Big(\Big[g\Big(\frac{P}{p}\Big)\Big]\Big) l(p) = \sum_{r\ge1} h(r) \sum_{\substack{p\le P\\ \text{st } [g(P/p)]=r}} l(p) =$$

$$= \sum_{r\ge1} h(r)\bigg(L\Big(\frac{P}{g^{-1}(r)}\Big) - L\Big(\frac{P}{g^{-1}(r+1)}\Big)\bigg) = \sum_{r\ge1} (h(r)-h(r-1)) L\Big(\frac{P}{g^{-1}(r)}\Big),$$

as required. ∎

Proposition 3.2. *Let $g : [1,\infty) \to \mathbb{R}$ be continuous, strictly increasing without bound, and $g(1) = 1$. Suppose further that $\sum_1^\infty 1/g^{-1}(n)$ converges. Let l be regularly varying of index $-\alpha$, with $\alpha \in (0,1)$, and h increasing such that $h(k) = O(k^\beta)$ for some $\beta < 1-\alpha$. Then*

$$(1)\qquad \sum_{p\le x}\Big[g\Big(\frac{x}{p}\Big)\Big]\log p \sim \Big(\sum_{n=1}^\infty \frac{1}{g^{-1}(n)}\Big)x,$$

$$(2)\qquad \sum_{p\le x} h\Big(\Big[g\Big(\frac{x}{p}\Big)\Big]\Big)l(p) \sim \Big(\sum_{n=1}^\infty \frac{\Delta h(n)}{g^{-1}(n)^{1-\alpha}}\Big)L(x),$$

where $L(x) = \sum_{p\le x} l(p)$.

Proof. (1) Let $G(x)$ denote the sum on the left in (1). Then from the proof of (3.1), we see that

$$G(x) = \sum_{n\le g(x)} \theta\Big(\frac{x}{g^{-1}(n)}\Big).$$

By the Prime Number Theorem, we can write $\theta(x) = x + \eta(x)$, where $\eta(x) = o(x)$. Let $\lambda = \sum_1^\infty 1/g^{-1}(n)$. The term involving x is

$$x \sum_{n\le g(x)} \frac{1}{g^{-1}(n)} \sim \lambda x.$$

Now, given $\varepsilon > 0$, there exists x_0 such that $|\eta(x)| \le \varepsilon x$ for $x \ge x_0$. Note that $x/g^{-1}(n) \ge x_0$ for $n \le g(x/x_0)$. Hence

$$\Big|\sum_{n\le g(x/x_0)} \eta\Big(\frac{x}{g^{-1}(n)}\Big)\Big| \le \varepsilon \sum_{n\le g(x/x_0)} \frac{x}{g^{-1}(n)} < \varepsilon\lambda x.$$

For the remaining range $g(x/x_0) < n \le g(x)$, the terms are $O(1)$ and so the sum is $O(g(x))$. But $g^{-1}(n) \succ n$ (since $\frac{n/2}{g^{-1}(n)} \le \sum_{n/2}^n \frac{1}{g^{-1}(n)} \to 0$) so that $g(x) = o(x)$. Thus $G(x) \sim \lambda x$ follows.

(2) Let $H(x)$ denote the LHS of (2). From the proof of (3.2) we see that

$$(3.3)\qquad \begin{aligned} H(x) &= \sum_{n\le g(x)} h(n)\Big\{L\Big(\frac{x}{g^{-1}(n)}\Big) - L\Big(\frac{x}{g^{-1}(n+1)}\Big)\Big\} = \\ &= \sum_{n\le g(x)} \Delta h(n) L\Big(\frac{x}{g^{-1}(n)}\Big). \end{aligned}$$

Since h is increasing,

$$H(x) \ge \sum_{n\le N} \Delta h(n) L\Big(\frac{x}{g^{-1}(n)}\Big) \sim \sum_{n\le N} \frac{\Delta h(n)}{g^{-1}(n)^{1-\alpha}} L(x).$$

for every $N \in \mathbb{N}$, by regular variation of L. Note that by Hölder's inequality

$$\sum_{n\le N} \frac{\Delta h(n)}{g^{-1}(n)^{1-\alpha}} \le A\left(\sum_{n\le N} \frac{1}{n^{\frac{1-\beta}{\alpha}}}\right)^{\alpha}\left(\sum_{n\le N} \frac{1}{g^{-1}(n)}\right)^{1-\alpha} < \infty.$$

Hence† $\sum_{n\ge 1} \frac{\Delta h(n)}{g^{-1}(n)^{1-\alpha}} < \infty$ and $H(x)/L(x) \gtrsim \sum_{n=1}^{\infty} \frac{\Delta h(n)}{g^{-1}(n)^{1-\alpha}}$.

For the range $n > N$, we use the bound $h(n) \le An^{\beta}$ in the middle expression of (3.3) and Potter's bounds on L

$$\frac{L(\frac{x}{g^{-1}(n)})}{L(x)} \le \frac{A_1}{g^{-1}(n)^{1-\alpha-\delta}}$$

for every $\delta > 0$ (some A_1). But with δ sufficiently small,

$$\sum_{n>N} \frac{1}{n^{1-\beta} g^{-1}(n)^{1-\alpha-\delta}} \le \left(\sum_{n>N} \frac{1}{n^{\frac{1-\beta}{\alpha+\delta}}}\right)^{\alpha+\delta}\left(\sum_{n>N} \frac{1}{g^{-1}(n)}\right)^{1-\alpha-\delta}.$$

Both sums converge, and so tend to zero as $N \to \infty$. Thus the result follows.■

Proposition 3.3. *Let f be multiplicative and given at the prime powers by* (1.1)*, and assume that h and l satisfy the conditions of Proposition* 3.2*. Then, with R_α given by* (1.2)*,*

$$\max_{n\le N} \log f(n) \gtrsim R_\alpha L(\log N).$$

Proof. It is clear that in the definition of R_α we may range over *strictly* decreasing a_n rather than just decreasing. Thus, given $\varepsilon > 0$, there exists a strictly decreasing a_n for which $\sum a_n < \infty$ and

$$\frac{\sum_{n=1}^{\infty} \Delta h(n) a_n^{1-\alpha}}{(\sum_{n=1}^{\infty} a_n)^{1-\alpha}} > R_\alpha - \varepsilon.$$

Without loss of generality we may assume $a_1 = 1$, as we may replace a_n by a_n/a_1. Let g be an increasing bijection on $[1, \infty)$ such that $g(1/a_n) = n$. Then $a_n = 1/g^{-1}(n)$ so that $\sum \frac{1}{g^{-1}(n)} < \infty$. Take n of the form

$$n = \prod_{p\le P} p^{[g(P/p)]} \tag{3.4}$$

As such, Proposition 3.2 implies

$$\log n \sim \left(\sum_{r=1}^{\infty} a_r\right) P \quad \text{and} \quad \log f(n) \sim \left(\sum_{r=1}^{\infty} \Delta h(r) a_r^{1-\alpha}\right) L(P)$$

†This incidentally shows that R_α is finite.

as $P \to \infty$ through the primes. Using the fact that L is regularly varying of index $1-\alpha$,

$$\frac{\log f(n)}{L(\log n)} \sim \frac{\sum_{r=1}^{\infty} \Delta h(r) a_r^{1-\alpha}}{(\sum_{r=1}^{\infty} a_r)^{1-\alpha}} > R_\alpha - \varepsilon. \tag{3.5}$$

Now note that if n and n' are consecutive numbers of the form (3.4) (i.e. $n' = \prod_{p \le P'} p^{[g(P'/p)]}$ where P' is the prime after P) then, with $\lambda = \sum_{n\ge 1} a_n$,

$$\log n' \sim \lambda P' \sim \lambda P \sim \log n.$$

Hence, with $\tilde{N}$ denoting the largest number of the form (3.4) below N,

$$\max_{n \le N} \log f(n) \ge \log f(\tilde{N}) \gtrsim (R_\alpha - \varepsilon) L(\log \tilde{N}) \sim (R_\alpha - \varepsilon) L(\log N).$$

This holds for every $\varepsilon > 0$, hence it must also hold for $\varepsilon = 0$. ■

4. Upper bounds and proofs of Theorems 1-3

The lower bound obtained in Proposition 3.3 already gives the maximum order of $\log f(n)$ for n of the form $\prod_{p \le P} p^{[g(P/p)]}$ with g an increasing bijection on $[1,\infty)$ such that $\sum g^{-1}(n)^{-1}$ converges. We have to show that no other n gives still larger values of $\log f(n)$.

Lemma 4.1. *Let f be multiplicative with $f(p^k) = e^{h(k)l(p)}$ for $p \in \mathbb{P}, k \in \mathbb{N}_0$, where h is increasing and l is decreasing. Then the maximal size of $f(n)$ occurs when n is of the form*

$$n = \prod_{p \le P} p^{a_p} \tag{4.1}$$

with a_p decreasing with p. More precisely, if n is as in (3.1) and $a_{p_i} < a_{p_j}$ for some $i < j$ (where p_i is the i^{th}-prime) then there exists $n' < n$ such that $f(n') \ge f(n)$.

Proof. Let n be as in (4.1) with $a_{p_i} < a_{p_j}$ for some $i < j$ and put $n' = \prod_{p \le P} p^{a'_p}$ where

$$a'_p = a_p \quad \text{if } p \ne p_i, p_j, \text{ and } a'_{p_i} = a_{p_j}, a'_{p_j} = a_{p_i}.$$

Then $n'/n = (p_i/p_j)^{a_{p_j} - a_{p_i}} < 1$, while

$$\log \frac{f(n')}{f(n)} = \Big(h(a_{p_j}) - h(a_{p_i})\Big)\Big(l(p_i) - l(p_j)\Big) \ge 0.$$ ■

Proof of Theorem 1. By Lemma 4.1, we need only consider n of the form (4.1) with a_p decreasing. Suppose, without loss of generality, that $a_P \geq 1$. Then

$$\log n = \sum_{p\leq P} a_p \log p \geq \sum_{p\leq P} \log p = \theta(P),$$

while $\log f(n) = \sum_{p\leq P} h(a_p)l(p)$. Consider $\sum_{p\leq\delta\log n} h(a_p)l(p)$ for $\delta > 0$ (small). Using $h(k) \ll k^\beta$, we have

$$\sum_{p\leq\delta\log n} h(a_p)l(p) \ll \sum_{p\leq\delta\log n} a_p^\beta l(p) = \sum_{p\leq\delta\log n} (a_p\log p)^\beta \Big(\frac{l(p)^{\frac{1}{1-\beta}}}{(\log p)^{\frac{\beta}{1-\beta}}}\Big)^{1-\beta} \leq$$

$$\leq (\log n)^\beta \Bigg(\sum_{p\leq\delta\log n} \frac{l(p)^{\frac{1}{1-\beta}}}{(\log p)^{\frac{\beta}{1-\beta}}}\Bigg)^{1-\beta}, \tag{4.2}$$

by Hölder's inequality. Now $\frac{l(p)^{\frac{1}{1-\beta}}}{(\log p)^{\frac{\beta}{1-\beta}}}$ is regularly varying of index $-\frac{\alpha}{1-\beta}$, which is greater than -1. Thus by Karamata's Theorem and the prime number theorem,

$$\sum_{p\leq x} \frac{l(p)^{\frac{1}{1-\beta}}}{(\log p)^{\frac{\beta}{1-\beta}}} = \int_{2-}^{x} \frac{l(t)^{\frac{1}{1-\beta}}}{(\log t)^{\frac{\beta}{1-\beta}}}\, d\pi(t) \sim \frac{xl(x)^{\frac{1}{1-\beta}}}{(1-\frac{\alpha}{1-\beta})(\log x)^{\frac{1}{1-\beta}}}. \tag{4.3}$$

Hence (4.2) gives

$$\sum_{p\leq\delta\log n} h(a_p)l(p) \ll (\log n)^\beta \frac{(\delta\log n)^{1-\beta} l(\delta\log n)}{(1-\frac{\alpha}{1-\beta})^{1-\beta}\log\log n} \sim \frac{\delta^\eta(1-\alpha)L(\log n)}{(1-\frac{\alpha}{1-\beta})^{1-\beta}}$$

where $\eta = 1-(\alpha+\beta) > 0$. Let $\varepsilon > 0$. Thus we can find $\delta > 0$ such that $\sum_{p\leq\delta\log n} h(a_p)l(p) < \varepsilon L(\log n)$. As such

$$\log f(n) < \sum_{\delta\log n<p\leq P} h(a_p)l(p) + \varepsilon L(\log n). \tag{4.4}$$

From (4.4) and the fact that $\log f(n)$ is sometimes as large as $cL(\log n)$, it follows that for the maximal order we must have $P > \delta\log n$ for δ sufficiently small. Now for every prime p,

$$\log n \geq a_p \sum_{q\leq p} \log q = a_p\theta(p)$$

(here q runs over the primes $\leq p$). So, for the range of p under consideration (i.e. $\delta\log n < p \leq P$) and using $\theta(x) \geq a_0 x$ for some absolute constant a_0,

$$a_p \leq \frac{\log n}{\theta(p)} \leq \frac{1}{a_0\delta}. \tag{4.5}$$

The bound is independent of n, only depending on α, β and ε, and so a_p takes only finitely many values, say $a_p \in \{1, \ldots, M\}$. Let

$$T_r = \sum_{\substack{\delta \log n < p \leq P \\ a_p \geq r}} l(p).$$

Then

$$\begin{aligned}\sum_{\delta \log n < p \leq P} h(a_p) l(p) &= \sum_{r=1}^{M} h(r) \sum_{\substack{\delta \log n < p \leq P \\ a_p = r}} l(p) = \\ &= \sum_{r=1}^{M} h(r)(T_r - T_{r+1}) = \sum_{r=1}^{M} \Delta h(r) T_r\end{aligned} \tag{4.6}$$

Since a_p decreases with p, we have $a_p \geq r \Leftrightarrow p \leq q_r$, for some q_r (depending on r and P), decreasing with r. Thus $q_r \leq q_1 = P$. For a non-zero contribution, we require $q_r > \delta \log n \geq \delta\theta(P)$, so that $a_0 \delta < \frac{q_r}{P} \leq 1$. By the uniform convergence theorem for regular variation, $L(q_r) = L(\frac{q_r}{P} \cdot P) \sim (\frac{q_r}{P})^{1-\alpha} L(P)$ and

$$\sum_{\delta \log n < p \leq P} h(a_p) l(p) \leq \sum_{r=1}^{M} \Delta h(r) L(q_r) \sim \left(\sum_{r=1}^{M} \Delta h(r) \Big(\frac{q_r}{P} \Big)^{1-\alpha} \right) L(P) \tag{4.7}$$

Also

$$\begin{aligned}\log n = \sum_{p \leq P} a_p \log p &\geq \sum_{r=1}^{M} r \sum_{\substack{p \leq P \\ a_p = r}} \log p \geq \\ &\geq \sum_{r=1}^{M} \sum_{\substack{p \leq P \\ a_p \geq r}} \log p = \sum_{r=1}^{M} \theta(q_r) \sim \left(\sum_{r=1}^{M} \frac{q_r}{P} \right) P\end{aligned}$$

by the Prime Number Theorem, so

$$L(\log n) \gtrsim \left(\sum_{r=1}^{M} \frac{q_r}{P} \right)^{1-\alpha} L(P). \tag{4.8}$$

Finally (4.4), (4.7) and (4.8) give

$$\limsup_{n \to \infty} \frac{\log f(n)}{L(\log n)} \leq \frac{\sum_{r=1}^{M} \Delta h(r) (\frac{q_r}{P})^{1-\alpha}}{(\sum_{r=1}^{M} \frac{q_r}{P})^{1-\alpha}} + \varepsilon \leq R_\alpha + \varepsilon.$$

This holds for all $\varepsilon > 0$, so the above holds with $\varepsilon = 0$. Combining with Proposition 3.3 concludes the proof of Theorem 1. ■

Proof of Theorem 2. We already noted in the introduction that $R_\alpha \le S_\alpha$ where

$$S_\alpha = \left(\sum_{n=1}^{\infty} \Delta h(n)^{1/\alpha}\right)^{\alpha}.$$

But equality holds in (1.3) if $a_n = c\Delta h(n)^{1/\alpha}$ for some constant c. So we choose a_n as such (with $c > 0$) which is valid as $\Delta h(n)^{1/\alpha}$ is decreasing and summable. Thus $R_\alpha = S_\alpha$ in this case. ■

Proof of Theorem 3. Consider $\alpha = 0$. For $M \in \mathbb{N}$, let

$$R_0(M) = \sup_{0 \le a_M \le \ldots \le a_1} \frac{\sum_{n=1}^{M} \Delta h(n) a_n}{\sum_{n=1}^{M} a_n},$$

the supremum being over all $a_1, \ldots, a_M$ satisfying $a_1 \ge \ldots \ge a_M \ge 0$. It is clear that $R_0(M) \to R_0$ as $M \to \infty$. We show that

$$R_0(M) = \max_{n \le M} \frac{h(n)}{n}. \tag{4.9}$$

Let $a_1 \ge \ldots a_M \ge 0$ and put $b_n = a_n - a_{n+1}$ $(n = 1, \ldots, M)$ with $a_{M+1} = 0$. So $a_n = \sum_{r=n}^{M} b_r$. Then

$$\sum_{n=1}^{M} \Delta h(n) a_n = \sum_{n=1}^{M} \Delta h(n) \sum_{r=n}^{M} b_r = \sum_{r=1}^{M} b_r \sum_{n=1}^{r} \Delta h(n) = \sum_{r=1}^{M} b_r h(r),$$

while $\sum_{n=1}^{M} a_n = \sum_{r=1}^{M} r b_r$. Thus

$$R_0(M) = \sup_{b_1, \ldots, b_M \ge 0} \frac{\sum_{n=1}^{M} h(n) b_n}{\sum_{n=1}^{M} n b_n} = \sup_{c_1, \ldots, c_M \ge 0} \frac{\sum_{n=1}^{M} \frac{h(n)}{n} c_n}{\sum_{n=1}^{M} c_n}$$

on putting $nb_n = c_n$. The expression on the right is $\le \max_{n \le M} \frac{h(n)}{n}$ while, choosing $c_k = 1$ and $c_n = 0$ for $n \ne k$ (k any fixed integer from $1, \ldots, M$), we find $R_0(M) \ge \frac{h(k)}{k}$. Thus (4.9), and hence, Theorem 3 follows. Note that the supremum is a maximum since $h(n)/n \to 0$. ■

5. On the value of R_α

The evaluation of R_α is an intriguing optimization problem in its own right. In the case $\alpha = 0$ and the case where $\Delta h(n)$ is decreasing one obtains simple explicit formulas for R_α. In general, one can still evaluate R_α but there does not appear to be an elegant formula.

We can turn it into a finite-dimensional problem by defining, for $M \in \mathbb{N}$,

$$R_\alpha(M) = \sup_{\substack{a_1 \geq \dots \geq a_M \geq 0 \\ \sum_{n=1}^M a_n = 1}} \sum_{n=1}^{M} \Delta h(n) a_n^{1-\alpha}.$$

We first prove that where $\Delta h(n)$ is increasing, we must take a_n constant. In fact, we prove this for a slightly more general problem:

Lemma 5.1. *Let $\alpha \in (0,1)$, $\ell = (l_1, \dots, l_M) \in \mathbb{N}^M$ and $\Lambda = (\lambda_1, \dots, \lambda_M) \in \in \mathbb{R}^M$ with each $\lambda_i \geq 0$ and consider*

$$R_\alpha(\Lambda, \ell; M) = \max_{\substack{a_1 \geq \dots \geq a_M \geq 0 \\ \sum_{r=1}^M l_r a_r = 1}} \sum_{m=1}^{M} \lambda_m l_m a_m^{1-\alpha}.$$

(i) *Suppose that $\lambda_k < \lambda_{k+1}$ for some $k \in \{1, \dots, M-1\}$. Then for the above maximum, we must take $a_k = a_{k+1}$.*

(ii) *If $\lambda_k \geq \lambda_{k+1}$ for every k, then $R_\alpha(\Lambda, \ell; M) = (\sum_{m=1}^M \lambda_m^{1/\alpha} l_m)^\alpha$.*

Proof. (i) In any case $a_k \geq a_{k+1}$, so it suffices to show that if $a_k > a_{k+1}$ then there exists $a' = (a'_1, \dots, a'_M)$ with $a'_1 \geq \dots \geq a'_M \geq 0$ and $\sum_{m=1}^M l_m a'_m = 1$ for which

$$\sum_{m=1}^{M} \lambda_m l_m {a'_m}^{1-\alpha} > \sum_{m=1}^{M} \lambda_m l_m a_m^{1-\alpha}. \tag{5.1}$$

So, suppose $a_k > a_{k+1}$. Let $a'_n = a_n$ for $n \neq k, k+1$ and put

$$a'_k = a'_{k+1} = \frac{l_k a_k + l_{k+1} a_{k+1}}{l_k + l_{k+1}}.$$

As such, $a'_1 \geq \dots \geq a'_M \geq 0$ (since $a_{k+1} < a'_k < a_k$) and $\sum_{m=1}^M l_m a'_m = 1$ (since $l_k a'_k + l_{k+1} a'_{k+1} = l_k a_k + l_{k+1} a_{k+1}$) while

$$\begin{aligned} &\sum_{m=1}^{M} \lambda_m l_m {a'_m}^{1-\alpha} - \sum_{m=1}^{M} \lambda_m l_m a_m^{1-\alpha} = \\ &= (\lambda_k l_k + \lambda_{k+1} l_{k+1})(a'_k)^{1-\alpha} - (\lambda_k l_k a_k^{1-\alpha} + \lambda_{k+1} l_{k+1} a_{k+1}^{1-\alpha}) = \\ &= \lambda_{k+1} a_k^{1-\alpha} \left\{ \frac{(l_k s + l_{k+1})}{(l_k + l_{k+1})^{1-\alpha}} (l_k + l_{k+1} t)^{1-\alpha} - (l_k s + l_{k+1} t^{1-\alpha}) \right\} \end{aligned} \tag{5.2}$$

where $s = \frac{\lambda_k}{\lambda_{k+1}}$ and $t = \frac{a_{k+1}}{a_k}$. Note that $0 \le s, t < 1$. Now put

$$F(x,y) = F_{m,n}(x,y) = \frac{(mx+n)(m+ny)^{1-\alpha}}{(m+n)^{1-\alpha}} - (mx+ny^{1-\alpha}) \qquad (m,n \in \mathbb{N}).$$

So the RHS of (5.2) is $\lambda_{k+1} a_k^{1-\alpha} F_{l_k, l_{k+1}}(s,t)$. We claim that for any $x, y \in [0,1]$, $F(x,y) \ge 0$ with equality if and only if $x = y = 1$. For

$$F(x,y) \ge 0 \quad \forall x,y \in [0,1] \iff$$

$$\iff \frac{(mx+n)(m+ny)^{1-\alpha}}{(m+n)^{1-\alpha}} \ge mx + ny^{1-\alpha} \quad \forall x,y \in [0,1] \iff$$

$$\iff mx\left\{1 - \left(\frac{m+ny}{m+n}\right)^{1-\alpha}\right\} \le n\left\{\left(\frac{m+ny}{m+n}\right)^{1-\alpha} - y^{1-\alpha}\right\} \forall x,y \in [0,1] \iff$$

$$\iff m\left\{1 - \left(\frac{m+ny}{m+n}\right)^{1-\alpha}\right\} \le n\left\{\left(\frac{m+ny}{m+n}\right)^{1-\alpha} - y^{1-\alpha}\right\} \quad \forall y \in [0,1]$$

since the LHS is largest when $x = 1$. Rearranging, we see that this holds if and only if $G(y) \ge 0$ for $0 \le y \le 1$ where

$$G(y) = (m+n)^\alpha (m+ny)^{1-\alpha} - m - ny^{1-\alpha}.$$

But $G'(y) = (1-\alpha)ny^{-\alpha}((\frac{my+ny}{m+ny})^\alpha - 1) < 0$ for $0 < y < 1$. Thus G is strictly decreasing in $[0,1]$. Since $G(1) = 0$ the result follows.

For the second part, note that by Hölder's inequality

$$\sum_{m=1}^{M} \lambda_m l_m a_m^{1-\alpha} = \sum_{m=1}^{M} \lambda_m l_m^\alpha (l_m a_m)^{1-\alpha} \le \Big(\sum_{m=1}^{M} \lambda_m^{1/\alpha} l_m\Big)^\alpha.$$

Equality holds if $\lambda_m^{1/\alpha} = c a_m$ for some constant c, which is feasible if λ_m is decreasing. ■

Determining R_α. Thus, in the evaluation of $R_\alpha(\Lambda, \ell; M)$, for the optimal solution we need to take a_n constant on intervals where λ_k is strictly increasing. Partition $\{1, \dots, M\}$ into consecutive intervals[‡] $\mathcal{L}_1, \dots, \mathcal{L}_{M'}$ and λ_n is strictly increasing on each $\mathcal{L}_r$. Thus we can write $\mathcal{L}_r = \{L_{r-1}+1, \dots, L_r\}$ for $r = 1, \dots, M'$ where L_r is a strictly increasing sequence of integers with $L_0 = 0$ and $L_{M'} = M$, and $\lambda_{n+1} > \lambda_n$ for $L_{r-1} < n < L_r$, while $\lambda_{n+1} \le \lambda_n$ for $n = L_r$ $(1 \le r < M')$. (If λ_k is decreasing, we must take $\mathcal{L}_r = \{r\}$.) As such, we take a_n constant on each $\mathcal{L}_r$. Writing

[‡]That is; sets of the form $\{k, k+1, k+2, \dots, l\}$ where $k, l \in \mathbb{N}$.

$$l'_r = \sum_{n \in \mathcal{L}_r} l_n \quad \text{and} \quad b_r = a_{L_r},$$

gives $\sum_{n=1}^{M} l_n a_n = \sum_{r=1}^{M'} (\sum_{n \in \mathcal{L}_r} l_n) a_{L_r} = \sum_{r=1}^{M'} l'_r b_r = 1$, while

$$\sum_{n=1}^{M} \lambda_n l_n a_n^{1-\alpha} = \sum_{r=1}^{M'} \Big(\sum_{n \in \mathcal{L}_r} \lambda_n l_n \Big) b_r^{1-\alpha} = \sum_{r=1}^{M'} \lambda'_r l'_r b_r^{1-\alpha}$$

where $\lambda'_r = \frac{1}{l'_r} \sum_{n \in \mathcal{L}_r} \lambda_n l_n$. Thus

$$R_\alpha(\Lambda, \ell; M) = \max_{\substack{b_1 \geq \cdots \geq b_{M'} \geq 0 \\ \sum_{r=1}^{M'} l'_r b_r = 1}} \sum_{m=1}^{M'} \lambda'_m l'_m b_m^{1-\alpha} = R_\alpha(\Lambda', \ell'; M')$$

where $\Lambda' = (\lambda'_1, \ldots, \lambda'_{M'})$ and $\ell' = (l'_1, \ldots, l'_{M'})$. Note that $M' < M$, unless λ_n is decreasing, in which case $R_\alpha(M)$ can be evaluated. Now apply Lemma 5.1 to this optimization problem and continue the process repeatedly. Thus

$$R_\alpha(\Lambda, \ell; M) = R_\alpha(\Lambda', \ell'; M') = \cdots = R_\alpha(\Lambda^*, \ell^*; M^*)$$

where the process stops when Λ^* is a decreasing set. This is guaranteed to happen when $M^* = 1$, but could happen earlier. Notice that at each stage, the forms for Λ and l are the same. Consider for example the second stage, where we have partitioned $\{1, \ldots, M'\}$ into consecutive intervals $\mathcal{L}'_1, \ldots, \mathcal{L}'_{M''}$ with corresponding ℓ'' and Λ''. Then

$$l''_k = \sum_{n \in \mathcal{L}'_k} l'_n = \sum_{n \in \mathcal{L}'_k} \sum_{m \in \mathcal{L}_k} l_m = \sum_{n \in \mathcal{L}} l_n$$

for some consecutive set $\mathcal{L}$ (dependent on k). Likewise

$$\lambda''_k l''_k = \sum_{n \in \mathcal{L}'_k} \lambda'_n l'_n = \sum_{n \in \mathcal{L}'_k} \sum_{m \in \mathcal{L}_k} \lambda_m l_m = \sum_{n \in \mathcal{L}} \lambda_n l_n$$

In particular, this holds for ℓ^* and Λ^*. Rewriting, the above shows that the optimal solution always has the form[§]

$$R_\alpha(\Lambda, \ell; M) = \left(\sum_{k=1}^{K} (q(m_k) - q(m_{k-1})) \Big(\frac{s(m_k) - s(m_{k-1})}{q(m_k) - q(m_{k-1})} \Big)^{1/\alpha} \right)^{\alpha}$$

[§]Another way to see this is to realise that at each stage more consecutive a_ns are equated until the corresponding λ'_ns (or λ''_ns etc.) are decreasing.

where $q(r) = l_1 + \cdots + l_r$ and $s(r) = \lambda_1 l_1 + \cdots + \lambda_r l_r$, for some sequence of integers m_k satisfying $0 = m_0 < m_1 < \cdots < m_K = M$. Being optimal, this requires that

$$\frac{s(m_k) - s(m_{k-1})}{q(m_k) - q(m_{k-1})}$$

is decreasing.

For the special case $l_k \equiv 1$ and $\lambda_k = \Delta h(k)$, $q(r) = r$ and $s(r) = h(r)$. Thus

$$R_\alpha(M) = \left(\sum_{k=1}^{K}(m_k - m_{k-1})\Big(\frac{h(m_k) - h(m_{k-1})}{m_k - m_{k-1}}\Big)^{1/\alpha}\right)^{\alpha}$$

for some such sequence m_k for which $\frac{h(m_k)-h(m_{k-1})}{m_k-m_{k-1}}$ decreases.

6. Examples and final comments

Now we illustrate our results with a few examples.

Example 1. Let f be multiplicative with $f(p^k) = \exp\{k^\beta p^{-\alpha}\}$ where $0 < \alpha < 1$ and $0 < \beta < 1 - \alpha$ for prime powers p^k. Thus $h(k) = k^\beta$, which is increasing and $\Delta h(k)$ is stricly decreasing as can be readily verified. In this case $L(x) \sim \frac{x^{1-\alpha}}{(1-\alpha)\log x}$. Thus, by Theorem 2,

$$\max_{n \le N} \log f(n) \sim \left(\sum_{n=1}^{\infty}(n^\beta - (n-1)^\beta)^{1/\alpha}\right)^{\alpha} \frac{(\log N)^{1-\alpha}}{(1-\alpha)\log\log N}.$$

(For $\alpha = 0$ the RHS is $\frac{\log N}{\log\log N}$.) In some cases the constant can be evaluated in terms of ζ-values. For example, taking $\beta = \frac{1}{2}$ and $\alpha = \frac{1}{3}$,

$$\sum_{n=1}^{N}(\sqrt{n} - \sqrt{n-1})^3 = 4N^{3/2} + 3\sqrt{N} - 6\sum_{n=1}^{N}\sqrt{n} \to -6\zeta\Big(-\frac{1}{2}\Big),$$

after suitable manipulations. By the functional equation for $\zeta(s)$ this equals $\frac{3}{2\pi}\zeta(\frac{3}{2})$. That is, the maximal order of the multiplicative function with $f(p^k) = \exp\{\sqrt{k}/\sqrt[3]{p}\}$ is

$$\exp\left\{\left(\frac{3}{2}\sqrt[3]{\frac{3}{2\pi}\zeta\Big(\frac{3}{2}\Big)} + o(1)\right)\frac{(\log N)^{2/3}}{\log\log N}\right\}.$$

Example 2. Theorem 2 can also be used in cases where $\log f(p^k)$ is not the form $h(k)l(p)$, but only asymptotically of this form. In [5], the maximal order of the function

$$\eta_{\alpha,\gamma}(n) = \frac{1}{d(n)} \sum_{d|n} \sigma_{-\alpha}(d)^\gamma$$

was required, where $\sigma_{-\alpha}(n) = \sum_{d|n} d^{-\alpha}$ and $d(n) = \sigma_0(n)$. It was shown that for $\alpha \in (0,1)$ and any $\gamma > 0$

$$\max_{n \le N} \log \eta_{\alpha,\gamma}(n) \asymp \frac{(\log N)^{1-\alpha}}{(1-\alpha)\log\log N}$$

but the true maximal order was left open. With Theorem 2, this can now be established.

Note that $\eta_{\alpha,\gamma}(n)$ is multiplicative with

$$\eta_{\alpha,\gamma}(p^k) = \frac{1}{k+1}\sum_{r=0}^{k} \sigma_{-\alpha}(p^r)^\gamma = \frac{1}{k+1}\left(1 + \sum_{r=1}^{k}\Big(1 + \frac{1}{p^\alpha} + O\Big(\frac{1}{p^{2\alpha}}\Big)\Big)^\gamma\right) =$$

$$= 1 + \frac{\gamma k}{(k+1)p^\alpha} + O\Big(\frac{1}{p^{2\alpha}}\Big) = \exp\Big\{\frac{\gamma k}{(k+1)p^\alpha} + O\Big(\frac{1}{p^{2\alpha}}\Big)\Big\},$$

the implied constants being independent of k (and p). Let $s(n)$ denote the multiplicative function with $s(p^k) = \exp\{\frac{\gamma k}{(k+1)p^\alpha}\}$. Then $\eta_{\alpha,\gamma}(n) = s(n)t(n)$ and from the above, $\sigma_{-2\alpha}(n)^{-\kappa} \le t(n) \le \sigma_{-2\alpha}(n)^\kappa$ for some $\kappa > 0$. It follows that $\log t(n) \ll (\log n)^{1-2\alpha+\varepsilon}$ for every $\varepsilon > 0$. Thus the maximal order of $\log \eta_{\alpha,\gamma}(n)$ is the same as for $\log s(n)$, which can be found from Theorem 2. In this case $h(k) = \frac{\gamma k}{k+1}$ which is increasing and $\Delta h(k) = \frac{\gamma}{k(k+1)}$ which is decreasing, while $l(p) = p^{-\alpha}$. Theorem 2 now gives

$$\max_{n\le N} \log \eta_{\alpha,\gamma}(n) \sim \max_{n \le N} \log s(n) \sim \gamma\left(\sum_{n=1}^{\infty}\Big(\frac{1}{n(n+1)}\Big)^{1/\alpha}\right)^\alpha \frac{(\log N)^{1-\alpha}}{(1-\alpha)\log\log N}.$$

For particular values of α the constant may be evaluated. Take, say, $\alpha = \frac{1}{2}$. Then the sum above becomes

$$\sum_{n=1}^{\infty}\Big(\frac{1}{n} - \frac{1}{n+1}\Big)^2 = \sum_{n=1}^{\infty}\Big(\frac{1}{n^2} + \frac{1}{(n+1)^2} - \frac{2}{n(n+1)}\Big) = 2\zeta(2) - 3.$$

Hence, with say $\gamma = 2$,

$$\max_{n \le N} \log \eta_{\frac{1}{2},2}(n) \sim 4\sqrt{\frac{\pi^2}{3} - 3}\,\frac{\sqrt{\log N}}{\log\log N}.$$

Example 3. Let f be multiplicative with $\log f(p^k) = h(k)l(p)$ where $h(k) = [\sqrt{k}]$. This time $h(k)$ is increasing but $\Delta h(k)$ is not, as $\Delta h(k) = 1$ for k a square and zero otherwise. Note that to apply Theorem 1, we require $\alpha < \frac{1}{2}$. To calculate R_α we use the method in Section 5. Thus

$$R_\alpha = \sup_{\substack{a_n \searrow 0 \\ \sum_{n=1}^\infty a_n = 1}} \sum_{n=1}^\infty \Delta h(n) a_n^{1-\alpha} = \sup_{\substack{a_n \searrow 0 \\ \sum_{n=1}^\infty a_n = 1}} \sum_{m=1}^\infty a_{m^2}^{1-\alpha}.$$

Putting $b_1 = a_1$, $b_2 = a_2 = a_3 = a_4$, $b_3 = a_5 = \cdots = a_9$ etc. for the optimal solution gives

$$R_\alpha = \sup_{\substack{b_n \searrow 0 \\ \sum_{n=1}^\infty (2n-1) b_n = 1}} \sum_{n=1}^\infty b_n^{1-\alpha} = \left(\sum_{n=1}^\infty \frac{1}{(2n-1)^{\frac{1-\alpha}{\alpha}}} \right)^\alpha,$$

by taking the optimal choice $b_n = c(2n-1)^{-1/\alpha}$ for some $c > 0$. Thus, if l is decreasing and regularly varying of index $-\alpha$ with $0 < \alpha < \frac{1}{2}$ then

$$\max_{n \le N} \log f(n) \sim (1 - 2^{1-\frac{1}{\alpha}})^\alpha \zeta\Big(\frac{1}{\alpha} - 1\Big)^\alpha \sum_{p \le \log N} l(p).$$

Final comments

The constant appearing in the asymptotic formula in the theorems has the form of an l^p-norm. For $a = (a_n)$ the l^p- *norm* is defined for $1 \le p < \infty$ and $p = \infty$ respectively by

$$\|a\|_p = \left(\sum_{n=1}^\infty |a_n|^p \right)^{1/p}, \qquad \|a\|_\infty = \sup_{n \in \mathbb{N}} |a_n|.$$

Writing $\alpha = 1/p$ ($p > 1$) we therefore see that, given the conditions of Theorem 2,

$$\max_{n \le N} \log f(n) \sim \|\Delta h\|_p L(\log N),$$

while for Theorem 3, with $\alpha = 0$ corresponding to $p = \infty$

$$\max_{n \le N} \log f(n) \sim \|h_1\|_\infty L(\log N),$$

where $h_1(n) = h(n)/n$.

This type of formula is strangely similar to an asymptotic formula found for the following 'quasi'-norm of an arithmetical operator (see [6]). Let

$$M_f(T) = \sup_{\substack{g \in \mathcal{M}^2 \\ \|g\|_2 = T}} \frac{\|f * g\|_2}{\|g\|_2}$$

where $\mathcal{M}^2$ is the set of square-summable multiplicative functions and $*$ is Dirichlet convolution. Taking $f \in \mathcal{M}^2$ to be completely multiplicative such that $f(p)$ is regularly varying with index $-\alpha$, it was proven in [6] that for $\frac{1}{2} < \alpha < 1$

$$\log M_f(T) \sim \left(\tfrac{1}{2}B(\tfrac{1}{\alpha}, 1 - \tfrac{1}{2\alpha})\right)^{\alpha} F(\log T \log\log T)$$

where $F(x) = \sum_{p \le x} f(p)$. Here $B(x,y)$ is the beta-function. Writing $p = 1/\alpha$, the constant can be rewritten as $\|h'\|_p$ where $h(x) = \sqrt{1 - e^{-2x}}$. With some heurstic reasoning, it was further suggested in the case where $f(n) = n^{-\alpha}$ that $M_f(T)$ represents the maximal order of $\zeta(\alpha + it)$ up to height T; i.e.

$$\max_{|t| \le T} \log|\zeta(\alpha + it)| \sim \|h'\|_p \frac{(\log T)^{1-\alpha}}{(1-\alpha)(\log\log T)^{\alpha}}$$

where $\|h'\|_p = (\int_0^\infty |h'|^p)^{1/p}$ is now the L_p-norm. The similarity of form between these 'discrete' and 'continuous' cases is rather striking, and suggests that there might be a more general framework which combines these formulae.

7. Appendix

To put the results into a broader context, we consider a few classes of multiplicative functions of the form (1.1) where h and l satisfy slightly altered assumptions.

1. Case where h is increasing, l decreasing and such that $\sum_p l(p) < \infty$. In this case we find the maximal order of $\log f(n)$ is of size $h([\frac{\log n}{\log 2}])$. More precisely, with $\lambda = \sum_p l(p)$

$$l(2)h\Big(\Big[\frac{\log n}{\log 2}\Big]\Big) \le \log f(n) \le \lambda h\Big(\Big[\frac{\log n}{\log 2}\Big]\Big),$$

where the RHS inequality holds for all n and the LHS for infinitely many n, namely, $n = 2^k$.

Proof. Let $n = \prod_{p \le P} p^{a_p}$ where a_p can be taken to be decreasing after Lemma 4.1. Thus $\log n = \sum_{p \le P} a_p \log p \ge a_2 \log 2$ and

$$\log f(n) = \sum_{p \le P} h(a_p)l(p) \le h(a_2)\sum_p l(p) = \lambda h(a_2) \le \lambda h\Big(\Big[\frac{\log n}{\log 2}\Big]\Big).$$

On the other hand, with $n = 2^k$, $\log f(n) = l(2)h(k) = l(2)h(\frac{\log n}{\log 2})$. ■

2. Case where h and Δh are increasing, and l decreasing. Now the maximum for f occurs when $n = 2^k$ and

$$\max_{n\le N} f(n) = \exp\left\{ l(2)h\left(\left[\frac{\log n}{\log 2}\right]\right)\right\}.$$

To see this, suppose $p|n$ where p is an odd prime, so $n = 2^k \dots p^l$ for some $k, l \in \mathbb{N}$. After Lemma 4.1 we can take $k \ge l$. Then, with $n' = \frac{2}{p}n$,

$$\frac{f(n')}{f(n)} = \frac{f(2^{k+1})f(p^{l-1})}{f(2^k)f(p^l)} = \exp\{l(2)\Delta h(k+1) - l(p)\Delta h(l)\} \ge 1.$$

Thus, with K such that $2^K \le N < 2^{K+1}$

$$\max_{n\le N} f(n) = f(2^K) = e^{l(2)h(K)} = \exp\left\{ l(2)h\left(\left[\frac{\log n}{\log 2}\right]\right)\right\}.$$

References

[1] **Apostol, T.M.,** *Introduction to Analytic Number Theory*, Springer, 1976.

[2] **Bingham, N.H., C.M. Goldie, and J.L. Teugels,** *Regular Variation*, Cambridge University Press, 1987.

[3] **Gronwall, T.H.,** Asymptotic expressions in the theory of numbers, *Trans. Amer. Math. Society,* **14** (1913), 113–122.

[4] **Hardy, G.H. and E.M. Wright,** *An Introduction to the Theory of Numbers*, 4th edition, Oxford University Press, 1979.

[5] **Hilberdink, T.W.,** An arithmetical mapping and applications to Ω-results for the Riemann zeta function, *Acta Arith.*, **139** (2009), 341–367.

[6] **Hilberdink, T.W.,** "Quasi"-norm of an arithmetical convolution operator and the order of the Riemann zeta function, *Functiones et Approximatio Commentarii Mathematici*, **49** (2013), 201–220.

[7] **Knopfmacher, J.,** *Abstract Analytic Number Theory*, Dover Edition, 1990.

[8] **Shui, P.,** The maximum orders of multiplicative functions, *Quart. J. Math. Oxford*, **31** (1980), 247–252.

[9] **Suryanarayana, D. and R. Sita Rama Chandra Rao,** On the true maximal order of a class of arithmetical functions, *Math. Jour. Okoyama Univ.,* **17** (1975), 95–101.

[10] **Tóth, L. and E. Wirsing,** The maximal order of a class of multiplicative arithmetical functions, *Annales Univ. Sci. Budapest., Sec. Comp.*, **22** (2003), 353–364.

T. Hilberdink
Department of Mathematics
University of Reading
PO Box 220
Reading, RG6 6AX
UK
t.w.hilberdink@reading.ac.uk