MAXIMAL ORDER OF A CLASS OF MULTIPLICATIVE FUNCTIONS

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Abstract. In this paper we obtain the maximal order of the multiplicative function given at the prime powers by \( f(p^k) = \exp\{h(k)l(p)\} \) where \( h(\cdot) \) and \( l(\cdot) \) are increasing and decreasing functions respectively with \( l(p) \) regularly varying of index \(-\alpha\) \((0 \leq \alpha < 1)\). For example, we show that under appropriate conditions
\[
\max_{n \leq N} \log f(n) \sim \left( \sum_{n=1}^{\infty} \Delta h(n)^{1/\alpha} \right)^{\alpha} L(\log N)
\]
where \( L(x) = \sum_{p \leq x} l(p) \) and \( \Delta h(n) = h(n) - h(n - 1) \).

1. Introduction

We consider a class of multiplicative functions \( f(n) \) which at the prime powers are given by
\[
(1.1) \quad f(p^k) = e^{h(k)l(p)} \quad p \in \mathbb{P}, \ k \in \mathbb{N}_0.
\]

In particular, we are interested in the maximal order of such functions\(^*\). If \( l(p) \) is constant, then \( f \) is a prime-independent multiplicative function and the

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\(^*\)More accurately, the maximal order of \( \log f \); here the maximal order of \( F \) is loosely defined to be any real positive function \( G \) such that \( \limsup_{n \to \infty} \frac{F(n)}{G(n)} = 1 \). In practise, one chooses the simplest possible \( G \).
maximal order has been discussed by various authors (see for example, [7], [8], [9] and references therein). Thus, for example, Shui [8] has proven that (using our notation) if $f(p^k) = e^{h(k)}$ where $0 \leq h(k) \leq Ak^\beta$ with $0 < \beta < 1$ and some $A$, then

$$\limsup_{n \to \infty} \frac{\log f(n) \log \log n}{\log n} = \max_{k \geq 1} \frac{h(k)}{k}.$$ 

In this case, the maximal order occurs, roughly, for $n$ of the form $(\prod_{p \leq P} p^m)$ where $m$ maximises $h(k)/k$. Results such as the above were then applied to find the maximal order of divisor-like functions.

For non prime-independent multiplicative functions not much work appears to have been done. In [10], Tóth and Wirsing consider a class of multiplicative functions which are at most of order $\log \log n$ including $\varphi(n)$, but their results do not overlap with ours.

For the function $\sigma_{-\alpha}(n) = \sum_{d|n} d^{-\alpha}$, Gronwall [3] showed 100 years ago that for $0 < \alpha < 1$, the maximal order is given by

$$\exp \left\{ \frac{1 + o(1)}{1 - \alpha} \cdot \frac{(\log n)^{1-\alpha}}{\log \log n} \right\}.$$ 

Notice that in this case

$$\sigma_{-\alpha}(p^k) = 1 + \frac{1}{p^\alpha} + \cdots + \frac{1}{p^{\alpha k}} = \exp \left\{ \frac{1 + o(1)}{p^\alpha} \right\}$$

which is of the form (1.1) in an asymptotic sense, with $h(k)$ constant and $l(p) = p^{-\alpha}$. In fact, the maximum order occurs for $n$ of the form $\prod_{p \leq P} p$, and to find this maximum is then relatively easy, using the prime number theorem. More generally, if $f$ is multiplicative and given by (1.1) and both $h$ and $l$ are decreasing (and non-negative), then the maximum order of $f(n)$ again occurs for $n$ of the form $\prod_{p \leq P} p$, since $f(p^k) \leq f(p)$ and $f(q) \leq f(p)$ for primes $p, q$ with $p < q$. As such, $\log n = \theta(P) \sim P$ by the prime number theorem and multiplicativity of $f(n)$ gives

$$\log f(n) = h(1) \sum_{p \leq P} l(p) = h(1) L(P),$$

where $L(x) = \sum_{p \leq x} l(p)$. If now we assume that $L(y) \sim L(x)$ whenever $y \sim x$, then $\log f(n) \sim h(1) L(\log n)$ (for such $n$) and this represents the maximal order.

In this article, we consider the less trivial (and perhaps more interesting) case is where $h$ is increasing, while keeping $l$ decreasing. As such we shall see that the maximal order occurs for $n = \prod_{p \leq P} p^{a_p}$ with $a_p$ decreasing. The problem then reduces to finding the optimal $a_p$ which maximises $f(n)$. A simple
lower bound for the maximal order can be found by taking \( a_p = 1 \) for all \( p \leq P \),
giving (under some mild conditions on \( L \))
\[
\limsup_{n \to \infty} \frac{\log f(n)}{L(\log n)} \geq h(1).
\]

With some extra conditions, we also have \( \log f(n) \ll L(\log n) \) and the
question reduces to finding this limsup. First, we require some bound on the
growth of \( h \) with respect to \( L \) if we want \( \log f(n) \ll L(\log n) \). For if \( n = 2^k \),
then
\[
\log f(n) = \log f(2^k) = l(2)h(k) = l(2)h\left(\frac{\log n}{\log 2}\right),
\]
so \( h(k) = o(L(k)) \) is necessary. A further natural condition is that \( L \) should
be regularly varying (see §1. for the definition). In fact, for our main results
we shall assume that \( L \) is regularly varying of index \( 1 - \alpha \) for some \( \alpha \in [0, 1) \),
while \( h(k) \ll k^\beta \) for some \( \beta < 1 - \alpha \).
As such, \( L(y) \sim L(x) \) whenever \( y \sim x \) and \( L(x) = x^{1-\alpha+o(1)} \).

Finally, we prove a slightly stronger result in that we find an asymptotic
formula for \( \max_{n \leq N} \log f(n) \).
Let \( \Delta h(n) = h(n) - h(n-1) \) for \( n \in \mathbb{N} \). Note that \( h(0) = 0 \) (by definition)
and so \( \Delta h(1) = h(1) \). Our main result is:

**Theorem 1.** Let \( f \) be multiplicative and given at the prime powers by (1.1),
where we assume that \( h \) is increasing and \( l \) is decreasing. Further suppose that
\( L(x) = \sum_{p \leq x} l(p) \) is regularly varying of index \( 1 - \alpha \), where \( 0 \leq \alpha < 1 \), and
\( h(n) \ll n^\beta \) for some \( \beta < 1 - \alpha \). Then
\[
\max_{n \leq N} \log f(n) \sim R_\alpha L(\log N)
\]
where
\[
(1.2) \quad R_\alpha = \sup_{\sum a_n \neq 0} \frac{\sum_{n=1}^\infty \Delta h(n) a_n^{1-\alpha}}{\left(\sum_{n=1}^\infty a_n\right)^{1-\alpha}} = \sup_{\sum a_n \neq 0} \frac{\sum_{n=1}^\infty \Delta h(n) a_n^{1-\alpha}}{\sum_{n=1}^\infty a_n}.
\]

The supremum here is over all decreasing sequences \( a_n \), not identically
zero, for which \( \sum_{n=1}^\infty a_n \) converges. In various cases we can evaluate \( R_\alpha \) more
explicitly. In particular we note that by Hölder’s inequality
\[
(1.3) \quad \sum_{n=1}^\infty \Delta h(n) a_n^{1-\alpha} \leq \left(\sum_{n=1}^\infty \Delta h(n)^{1/\alpha}\right)^\alpha \left(\sum_{n=1}^\infty a_n\right)^{1-\alpha}
\]
and \( R_\alpha \leq (\sum_{n=1}^\infty \Delta h(n)^{1/\alpha})^\alpha \) always. The case of equality leads to:
Theorem 2. Let $f$ be as in Theorem 1 and suppose further that $\Delta h(n)$ decreases with $n$. Then
\[
\max_{n \leq N} \log f(n) \sim \left( \sum_{n=1}^{\infty} \Delta h(n)^{1/\alpha} \right)^{\alpha} L(\log N).
\]
Note that the series $\sum_{n=1}^{\infty} \Delta h(n)^{1/\alpha}$ converges if $\Delta h(n)$ decreases as $\Delta h(n) \leq \frac{h(n)}{n}$, so $\Delta h(n)^{1/\alpha} \ll n^{-\gamma}$ where $\gamma = \frac{1-\beta}{\alpha} > 1$.

In the case $\alpha = 0$, $R_\alpha$ can be evaluated and gives:

Theorem 3. Let $f$ be multiplicative and given at the prime powers by (1.1), where $h$ is increasing and $l$ is decreasing and $L$ is regularly varying of index 1. Suppose that $h(n) \ll n^\beta$ for some $\beta < 1$. Then
\[
\max_{n \leq N} \log f(n) \sim \left( \max_{n \in \mathbb{N}} \frac{h(n)}{n} \right) L(\log N).
\]

The form (1.1) (with $h$ increasing and $l$ decreasing) may seem restrictive, but actually the results apply to cases where (1.1) holds in an asymptotic sense. We illustrate this in Example 2, in Section 6. Indeed, the example
\[
f(n) = \frac{1}{d(n)} \sum_{d|n} \sigma_\alpha(d)^2
\]

for which $\log f(p^k) = \frac{2k}{k+1} \log p + O\left(\frac{1}{p}\right)$, motivated the present results.

The rest of the paper is organised as follows. First we recall the notion of regular variation, then in Section 3 we find lower bounds for $\log f(n)$, to be followed in Section 4 by upper bounds and the proofs of the results.

In Section 5, we show how to evaluate $R_\alpha$ in case $\Delta h(n)$ is not decreasing and $\alpha \neq 0$. Finally, we present some examples.

2. Some preliminaries

Notations: We write $f \ll g$ to mean $f = O(g)$; i.e. $|f(x)| \leq A g(x)$ for some constant $A$ and all $x$ sufficiently large. We write $f \lesssim g$ to mean $f(x) \leq (1 + o(1))g(x)$, and similarly for $f \gtrsim g$. Finally, $f \prec g$ means $f(x) = o(g(x))$, while $f \succ g$ is the same as $g \ll f$. 


Regular Variation: A function $\ell : [A, \infty) \to \mathbb{R}$ is regularly varying of index $\rho$ if it is measurable, eventually positive, and

$$\ell(\lambda x) \sim \lambda^\rho \ell(x) \quad \text{as } x \to \infty \text{ for every } \lambda > 0$$

(2.1)

(see [2] for a detailed treatise on the subject). We shall sometimes denote this by $\ell \in \mathcal{R}_\rho$. If $\rho = 0$, then $\ell$ is said to be slowly varying. For example, $x^\rho (\log x)^\tau$ is regularly varying of index $\rho$ for any $\tau$. Trivially, if $\ell_1 \in \mathcal{R}_\rho$ and $\ell_2 \in \mathcal{R}_\sigma$, then $\ell_1 \ell_2 \in \mathcal{R}_{\rho + \sigma}$, while $\ell_1 \in \mathcal{R}_{\rho\lambda}$.

The Uniform Convergence Theorem says that (2.1) is automatically uniform for $\lambda$ in compact subsets of $(0, \infty)$. In particular, $\ell(x) \sim \ell(y)$ whenever $x \sim y$.

We shall make use of Karamata’s Theorem: for $\ell$ regularly varying of index $\rho$, $\int_A^x \ell(t) dt \sim \frac{x\ell(x)}{\rho + 1}$ if $\rho > -1$, $\int_x^\infty \ell(t) dt \sim -\frac{x\ell(x)}{\rho + 1}$ if $\rho < -1$,

while if $\rho = -1$, $\int^x \ell$ is slowly varying and $\int^x \ell \succ x\ell(x)$.

We shall also make use of Potter’s bounds (see [2], p. 25): if $\ell$ is regularly varying of index $\rho$ then for any chosen $A > 1$ and $\delta > 0$, there exists $X = X(A, \delta)$ such that

$$\frac{\ell(y)}{\ell(x)} \leq A \max \left\{ \left( \frac{y}{x} \right)^{\rho + \delta}, \left( \frac{y}{x} \right)^{\rho - \delta} \right\} \quad \text{for } x, y \geq X.$$ 

The notion of regular variation extends to sequences ([2], p. 52). For $l$ defined on $\mathbb{P}$ — the set of primes, we say $l$ is regularly varying of index $\rho$ if there exists a $\tilde{l} \in \mathcal{R}_\rho$, defined on $[2, \infty)$ such that $\tilde{l}(p) = l(p)$. As such, we can always take $\tilde{l}$ to be the step function defined by $\tilde{l}(x) = l(p)$ for $p \leq x < p'$ where $p$ and $p'$ are consecutive primes, which we shall do from now on, and we denote this extension by $l$.

We note that if $l$ is decreasing, regular variation of $l$ (of index $> -1$) is equivalent to regular variation of $L$, where $L(x) = \sum_{p \leq x} l(p)$. Indeed, by the Prime Number Theorem and Karamata’s Theorem, if $\ell$ is regularly varying of index $-\alpha$ and $\alpha < 1$, then

$$L(x) = \int_{2}^{x} l(t) \, d\pi(t) \sim \int_{2}^{x} \frac{l(t)}{\log t} \, dt \sim \frac{x\ell(x)}{(1 - \alpha) \log x}$$

which is regularly varying of index $1 - \alpha$. Conversely, if $L \in \mathcal{R}_{1-\alpha}$ for some $\alpha < 1$ and $l$ is decreasing, then for every $\lambda > 1$

$$l(\lambda x) (\pi(\lambda x) - \pi(x)) \leq L(\lambda x) - L(x) = \sum_{x < p \leq \lambda x} l(p) \leq l(x)(\pi(\lambda x) - \pi(x)).$$
Using \( L \in \mathcal{R}_{1-\alpha} \) and \( \pi \in \mathcal{R}_1 \) and dividing by \( L(x) \) gives
\[
\frac{\lambda^{1-\alpha} - 1}{\lambda - 1} \lesssim \frac{l(x)\pi(x)}{L(x)} \lesssim \frac{\lambda^{1-\alpha} - 1}{\lambda - 1} \lambda^\alpha,
\]
and on letting \( \lambda \to 1 \), (2.2) follows again, so that \( l \in \mathcal{R}_{-\alpha} \).

3. Lower bounds for \( \log f(n) \)

**Proposition 3.1.** Let \( f \) be multiplicative with \( f(p^k) = \exp\{h(k)l(p)\} \). Put 
\[
n = \prod_{p \leq P} p^{\lfloor g(P/p) \rfloor},
\]
where \( g : [1, \infty) \to \mathbb{R} \) is continuous, strictly increasing without bound, and \( g(1) = 1 \). Then
\[
\log n = \sum_{r \geq 1} r \sum_{p \leq P} \log p.
\]

But \( \lfloor g(P/p) \rfloor = r \iff \frac{P}{g^{-1}(r + 1)} < p \leq \frac{P}{g^{-1}(r)} \), so
\[
\log n = \sum_{r \geq 1} r \left( \theta\left(\frac{P}{g^{-1}(r)}\right) - \theta\left(\frac{P}{g^{-1}(r + 1)}\right)\right) = \sum_{r \geq 1} \theta\left(\frac{P}{g^{-1}(r)}\right).
\]

For (3.2), we have
\[
\log f(n) = \sum_{p \leq P} h\left(\left\lfloor \frac{P}{g(p)} \right\rfloor \right) l(p) = \sum_{r \geq 1} h(r) \sum_{p \leq P} l(p) = \sum_{r \geq 1} h(r) \left( L\left(\frac{P}{g^{-1}(r)}\right) - L\left(\frac{P}{g^{-1}(r + 1)}\right)\right) = \sum_{r \geq 1} (h(r) - h(r - 1)) L\left(\frac{P}{g^{-1}(r)}\right),
\]
as required. \( \blacksquare \)
Proposition 3.2. Let \( g : [1, \infty) \to \mathbb{R} \) be continuous, strictly increasing without bound, and \( g(1) = 1 \). Suppose further that \( \sum_{n=1}^{\infty} 1/g^{-1}(n) \) converges. Let \( l \) be regularly varying of index \(-\alpha\), with \( \alpha \in (0, 1) \), and \( h \) increasing such that \( h(k) = O(k^\beta) \) for some \( \beta < 1 - \alpha \). Then

\[
\begin{align*}
(1) \quad \sum_{p \leq x} \left[ g\left(\frac{x}{p}\right) \right] \log p & \sim \left(\sum_{n=1}^{\infty} \frac{1}{g^{-1}(n)}\right) x, \\
(2) \quad \sum_{p \leq x} h\left(\left[ g\left(\frac{x}{p}\right) \right]\right) l(p) & \sim \left(\sum_{n=1}^{\infty} \frac{\Delta h(n)}{g^{-1}(n)^{1-\alpha}}\right) L(x),
\end{align*}
\]

where \( L(x) = \sum_{p \leq x} l(p) \).

Proof. (1) Let \( G(x) \) denote the sum on the left in (1). Then from the proof of (3.1), we see that

\[ G(x) = \sum_{n \leq g(x)} \theta\left(\frac{x}{g^{-1}(n)}\right). \]

By the Prime Number Theorem, we can write \( \theta(x) = x + \eta(x) \), where \( \eta(x) = o(x) \). Let \( \lambda = \sum_{n=1}^{\infty} 1/g^{-1}(n) \). The term involving \( x \) is

\[ x \sum_{n \leq g(x)} \frac{1}{g^{-1}(n)} \sim \lambda x. \]

Now, given \( \varepsilon > 0 \), there exists \( x_0 \) such that \( |\eta(x)| \leq \varepsilon x \) for \( x \geq x_0 \). Note that \( x/g^{-1}(n) \geq x_0 \) for \( n \leq g(x/x_0) \). Hence

\[ \left| \sum_{n \leq g(x/x_0)} \eta\left(\frac{x}{g^{-1}(n)}\right) \right| \leq \varepsilon \sum_{n \leq g(x/x_0)} \frac{x}{g^{-1}(n)} < \varepsilon \lambda x. \]

For the remaining range \( g(x/x_0) < n \leq g(x) \), the terms are \( O(1) \) and so the sum is \( O(g(x)) \). But \( g^{-1}(n) \sim n \) (since \( \frac{n/2}{g^{-1}(n)} \leq \sum_{n/2}^{n} \frac{1}{g^{-1}(n)} \to 0 \)) so that \( g(x) = o(x) \). Thus \( G(x) \sim \lambda x \) follows.

(2) Let \( H(x) \) denote the LHS of (2). From the proof of (3.2) we see that

\[
(3.3) \quad H(x) = \sum_{n \leq g(x)} h(n) \left\{ L\left(\frac{x}{g^{-1}(n)}\right) - L\left(\frac{x}{g^{-1}(n+1)}\right)\right\} = \sum_{n \leq g(x)} \Delta h(n) L\left(\frac{x}{g^{-1}(n)}\right).
\]

Since \( h \) is increasing,

\[ H(x) \geq \sum_{n \leq N} \Delta h(n) L\left(\frac{x}{g^{-1}(n)}\right) \sim \sum_{n \leq N} \frac{\Delta h(n)}{g^{-1}(n)^{1-\alpha}} L(x). \]
for every \(N \in \mathbb{N}\), by regular variation of \(L\). Note that by Hölder’s inequality
\[
\sum_{n \leq N} \frac{\Delta h(n)}{g^{-1}(n)^{1-\alpha}} \leq A \left( \sum_{n \leq N} \frac{1}{n^{\frac{1-\alpha}{\alpha}}} \right)^\alpha \left( \sum_{n \leq N} \frac{1}{g^{-1}(n)} \right)^{1-\alpha} < \infty.
\]
Hence\(^\dagger\) \(\sum_{n \geq 1} \frac{\Delta h(n)}{g^{-1}(n)^{1-\alpha}} < \infty\) and \(H(x)/L(x) \gtrsim \sum_{n=1}^\infty \frac{\Delta h(n)}{g^{-1}(n)^{1-\alpha}}\).

For the range \(n > N\), we use the bound \(h(n) \leq A n^{\delta}\) in the middle expression of (3.3) and Potter’s bounds on \(L\)
\[
\frac{L\left(\frac{x}{g^{-1}(n)}\right)}{L(x)} \leq \frac{A_1}{g^{-1}(n)^{1-\alpha-\delta}}
\]
for every \(\delta > 0\) (some \(A_1\)). But with \(\delta\) sufficiently small,
\[
\sum_{n > N} \frac{1}{n^{1-\beta} g^{-1}(n)^{1-\alpha-\delta}} \leq \left( \sum_{n > N} \frac{1}{n^{\frac{1-\beta}{1-\alpha}}} \right)^{\alpha+\delta} \left( \sum_{n > N} \frac{1}{g^{-1}(n)} \right)^{1-\alpha-\delta}.
\]
Both sums converge, and so tend to zero as \(N \to \infty\). Thus the result follows.\(\blacksquare\)

**Proposition 3.3.** Let \(f\) be multiplicative and given at the prime powers by (1.1), and assume that \(h\) and \(l\) satisfy the conditions of Proposition 3.2. Then, with \(R_\alpha\) given by (1.2),
\[
\max_{n \leq N} \log f(n) \gtrsim R_\alpha L(\log N).
\]

**Proof.** It is clear that in the definition of \(R_\alpha\) we may range over strictly decreasing \(a_n\) rather than just decreasing. Thus, given \(\varepsilon > 0\), there exists a strictly decreasing \(a_n\) for which \(\sum a_n < \infty\) and
\[
\sum_{n=1}^\infty \frac{\Delta h(n)}{\sum_{n=1}^\infty a_n^{1-\alpha}} > R_\alpha - \varepsilon.
\]
Without loss of generality we may assume \(a_1 = 1\), as we may replace \(a_n\) by \(a_n/a_1\). Let \(g\) be an increasing bijection on \([1, \infty)\) such that \(g(1/a_n) = n\). Then \(a_n = 1/g^{-1}(n)\) so that \(\sum \frac{1}{g^{-1}(n)} < \infty\). Take \(n\) of the form
\[
(3.4) \quad n = \prod_{p \leq P} p^{[g(p)/p]}
\]
As such, Proposition 3.2 implies
\[
\log n \sim \left( \sum_{r=1}^\infty a_r \right) P \quad \text{and} \quad \log f(n) \sim \left( \sum_{r=1}^\infty \Delta h(r)a_r^{1-\alpha} \right) L(P)
\]
\(^\dagger\)This incidentally shows that \(R_\alpha\) is finite.
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as \( P \to \infty \) through the primes. Using the fact that \( L \) is regularly varying of index \( 1 - \alpha \),

\[
\log f(n) \sim \frac{\sum_{r=1}^{\infty} \Delta h(r)a_r^{1-\alpha}}{(\sum_{r=1}^{\infty} a_r)^{1-\alpha}} > R_\alpha - \varepsilon. \tag{3.5}
\]

Now note that if \( n \) and \( n' \) are consecutive numbers of the form (3.4) (i.e. \( n' = \prod_{p \leq P'} p^{g(P'/p)} \) where \( P' \) is the prime after \( P \) then, with \( \lambda = \sum_{n \geq 1} a_n \),

\[
\log n' \sim \lambda P' \sim \lambda P \sim \log n.
\]

Hence, with \( \tilde{N} \) denoting the largest number of the form (3.4) below \( N \),

\[
\max_{n \leq N} \log f(n) \geq \log f(\tilde{N}) \gtrsim (R_\alpha - \varepsilon)L(\log \tilde{N}) \sim (R_\alpha - \varepsilon)L(\log N).
\]

This holds for every \( \varepsilon > 0 \), hence it must also hold for \( \varepsilon = 0 \). \( \blacksquare \)

4. Upper bounds and proofs of Theorems 1-3

The lower bound obtained in Proposition 3.3 already gives the maximum order of \( \log f(n) \) for \( n \) of the form \( \prod_{p \leq P} p^{g(P/p)} \) with \( g \) an increasing bijection on \([1, \infty)\) such that \( \sum g^{-1}(n)^{-1} \) converges. We have to show that no other \( n \) gives still larger values of \( \log f(n) \).

**Lemma 4.1.** Let \( f \) be multiplicative with \( f(p^k) = e^{h(k)p^l(p)} \) for \( p \in P, k \in \mathbb{N}_0 \), where \( h \) is increasing and \( l \) is decreasing. Then the maximal size of \( f(n) \) occurs when \( n \) is of the form

\[
n = \prod_{p \leq P} p^{a_p}
\]

with \( a_p \) decreasing with \( p \). More precisely, if \( n \) is as in (3.1) and \( a_{p_i} < a_{p_j} \) for some \( i < j \) (where \( p_i \) is the \( i^{th} \)-prime) then there exists \( n' < n \) such that \( f(n') \geq f(n) \).

**Proof.** Let \( n \) be as in (4.1) with \( a_{p_i} < a_{p_j} \) for some \( i < j \) and put \( n' = \prod_{p \leq P} p^{a'_p} \) where

\[
a'_p = a_p \quad \text{if} \quad p \neq p_i, p_j, \quad \text{and} \quad a'_{p_i} = a_{p_j}, a'_{p_j} = a_{p_i}.
\]

Then \( n'/n = (p_i/p_j)^{a_{p_j} - a_{p_i}} < 1 \), while

\[
\log \frac{f(n')}{f(n)} = \left(h(a_{p_j}) - h(a_{p_i})\right)\left(l(p_i) - l(p_j)\right) \geq 0. \quad \blacksquare
\]
Proof of Theorem 1. By Lemma 4.1, we need only consider $n$ of the form (4.1) with $a_p$ decreasing. Suppose, without loss of generality, that $a_P \geq 1$. Then
\[ \log n = \sum_{p \leq P} a_p \log p \geq \sum_{p \leq P} \log p = \theta(P), \]
while $\log f(n) = \sum_{p \leq P} h(a_p)l(p)$. Consider $\sum_{p \leq \delta \log n} h(a_p)l(p)$ for $\delta > 0$ (small). Using $h(k) \ll k^\beta$, we have
\[ \sum_{p \leq \delta \log n} h(a_p)l(p) \ll \sum_{p \leq \delta \log n} a_p^\beta l(p) = \sum_{p \leq \delta \log n} (a_p \log p)^\beta \left( \frac{l(p)^{1/\beta}}{(\log p)^{1/\beta}} \right)^{1-\beta} \leq \]
\[ \leq (\log n)^\beta \left( \sum_{p \leq \delta \log n} \frac{l(p)^{1/\beta}}{(\log p)^{1/\beta}} \right)^{1-\beta}, \]
by Hölder’s inequality. Now $\frac{l(p)^{1/\beta}}{(\log p)^{1/\beta}}$ is regularly varying of index $-\frac{\alpha}{1-\beta}$, which is greater than $-1$. Thus by Karamata’s Theorem and the prime number theorem,
\[ \sum_{p \leq x} \frac{l(p)^{1/\beta}}{(\log p)^{1/\beta}} = \int_2^x \frac{l(t)^{1/\beta}}{(\log t)^{1/\beta}} d\pi(t) \sim \frac{x l(x)^{1/\beta}}{(1-\frac{\alpha}{1-\beta})(\log x)^{1/\beta}}. \]
Hence (4.2) gives
\[ \sum_{p \leq \delta \log n} h(a_p)l(p) \ll (\log n)^\beta \left( \frac{\delta \log n)^{1-\beta}l(\delta \log n)}{(1-\frac{\alpha}{1-\beta})^{1-\beta} \log n} \right) \sim \frac{\delta^\eta(1-\alpha)L(\log n)}{(1-\frac{\alpha}{1-\beta})^{1-\beta}}, \]
where $\eta = 1 - (\alpha + \beta) > 0$. Let $\varepsilon > 0$. Thus we can find $\delta > 0$ such that $\sum_{p \leq \delta \log n} h(a_p)l(p) < \varepsilon L(\log n)$. As such
\[ \log f(n) < \sum_{\delta \log n < p \leq P} h(a_p)l(p) + \varepsilon L(\log n). \]
From (4.4) and the fact that $\log f(n)$ is sometimes as large as $\varepsilon L(\log n)$, it follows that for the maximal order we must have $P > \delta \log n$ for $\delta$ sufficiently small. Now for every prime $p$,
\[ \log n \geq a_p \sum_{q \leq p} \log q = a_p \theta(p) \]
(here $q$ runs over the primes $\leq p$). So, for the range of $p$ under consideration (i.e. $\delta \log n < p \leq P$) and using $\theta(x) \geq a_0 x$ for some absolute constant $a_0$,
\[ a_p \leq \frac{\log n}{\theta(p)} \leq \frac{1}{a_0 \delta}. \]
Maximal order of a class of multiplicative functions

The bound is independent of \( n \), only depending on \( \alpha, \beta \) and \( \varepsilon \), and so \( a_p \) takes only finitely many values, say \( a_p \in \{1, \ldots, M\} \). Let

\[
T_r = \sum_{\delta \log n < p \leq P, \ a_p \geq r} l(p).
\]

Then

\[
\sum_{\delta \log n < p \leq P} h(a_p)l(p) = \sum_{r=1}^{M} h(r) \sum_{\delta \log n < p \leq P, \ a_p = r} l(p) = 
\]

\[
= \sum_{r=1}^{M} h(r)(T_r - T_{r+1}) = \sum_{r=1}^{M} \Delta h(r)T_r
\]

(4.6)

Since \( a_p \) decreases with \( p \), we have \( a_p \geq r \iff p \leq q_r \), for some \( q_r \) (depending on \( r \) and \( P \)), decreasing with \( r \). Thus \( q_r \leq q_1 = P \). For a non-zero contribution, we require \( q_r > \delta \log n \geq \delta \theta(P) \), so that \( a_0 \delta < \frac{q_r}{P} \leq 1 \). By the uniform convergence theorem for regular variation, \( L(q_r) = L\left(\frac{q_r}{P}\right) \sim \left(\frac{q_r}{P}\right)^{1-\alpha}L(P) \) and

\[
\sum_{\delta \log n < p \leq P} h(a_p)l(p) \leq \sum_{r=1}^{M} \Delta h(r)L(q_r) \sim \left(\sum_{r=1}^{M} \Delta h(r)\left(\frac{q_r}{P}\right)^{1-\alpha}\right)L(P)
\]

(4.7)

Also

\[
\log n = \sum_{p \leq P} a_p \log p \geq \sum_{r=1}^{M} r \sum_{p \leq P, \ a_p = r} \log p \geq 
\]

\[
\geq \sum_{r=1}^{M} \sum_{p \leq P, \ a_p \geq r} \log p = \sum_{r=1}^{M} \theta(q_r) \sim \left(\sum_{r=1}^{M} q_r \right)P
\]

by the Prime Number Theorem, so

(4.8)

\[
L(\log n) \gtrsim \left(\sum_{r=1}^{M} \frac{q_r}{P}\right)^{1-\alpha}L(P).
\]

Finally (4.4), (4.7) and (4.8) give

\[
\limsup_{n \to \infty} \frac{\log f(n)}{L(\log n)} \leq \frac{\sum_{r=1}^{M} \Delta h(r)\left(\frac{q_r}{P}\right)^{1-\alpha}}{\left(\sum_{r=1}^{M} \frac{q_r}{P}\right)^{1-\alpha}} + \varepsilon \leq R_\alpha + \varepsilon.
\]
This holds for all \( \varepsilon > 0 \), so the above holds with \( \varepsilon = 0 \). Combining with Proposition 3.3 concludes the proof of Theorem 1.

**Proof of Theorem 2.** We already noted in the introduction that \( R_\alpha \leq S_\alpha \) where

\[
S_\alpha = \left( \sum_{n=1}^{\infty} \Delta h(n)^{1/\alpha} \right)^\alpha.
\]

But equality holds in (1.3) if \( a_n = c\Delta h(n)^{1/\alpha} \) for some constant \( c \). So we choose \( a_n \) as such (with \( c > 0 \)) which is valid as \( \Delta h(n)^{1/\alpha} \) is decreasing and summable. Thus \( R_\alpha = S_\alpha \) in this case.

**Proof of Theorem 3.** Consider \( \alpha = 0 \). For \( M \in \mathbb{N} \), let

\[
R_0(M) = \sup_{0 \leq a_M \leq \ldots \leq a_1} \frac{\sum_{n=1}^{M} \Delta h(n)a_n}{\sum_{n=1}^{M} a_n},
\]

the supremum being over all \( a_1, \ldots, a_M \) satisfying \( a_1 \geq \ldots \geq a_M \geq 0 \). It is clear that \( R_0(M) \to R_0 \) as \( M \to \infty \). We show that

\[
(4.9) \quad R_0(M) = \max_{n \leq M} h(n) \frac{n}{n}.
\]

Let \( a_1 \geq \ldots \geq a_M \geq 0 \) and put \( b_n = a_n - a_{n+1} \) (\( n = 1, \ldots, M \)) with \( a_{M+1} = 0 \). So \( a_n = \sum_{r=n}^{M} b_r \). Then

\[
\sum_{n=1}^{M} \Delta h(n)a_n = \sum_{n=1}^{M} \Delta h(n) \sum_{r=n}^{M} b_r = \sum_{r=1}^{M} b_r \sum_{n=1}^{r} \Delta h(n) = \sum_{r=1}^{M} b_r h(r),
\]

while \( \sum_{n=1}^{M} a_n = \sum_{r=1}^{M} r b_r \). Thus

\[
R_0(M) = \sup_{b_1, \ldots, b_M \geq 0} \frac{\sum_{n=1}^{M} h(n)b_n}{\sum_{n=1}^{M} nb_n} = \sup_{c_1, \ldots, c_M \geq 0} \frac{\sum_{n=1}^{M} h(n) c_n}{\sum_{n=1}^{M} c_n}
\]

on putting \( nb_n = c_n \). The expression on the right is \( \leq \max_{n \leq M} \frac{h(n)}{n} \) while, choosing \( c_k = 1 \) and \( c_n = 0 \) for \( n \neq k \) (\( k \) any fixed integer from 1, \ldots, \( M \)), we find \( R_0(M) \geq \frac{h(k)}{k} \). Thus (4.9), and hence, Theorem 3 follows. Note that the supremum is a maximum since \( h(n)/n \to 0 \).

**5. On the value of \( R_\alpha \)**

The evaluation of \( R_\alpha \) is an intriguing optimization problem in its own right. In the case \( \alpha = 0 \) and the case where \( \Delta h(n) \) is decreasing one obtains simple
explicit formulas for $R_\alpha$. In general, one can still evaluate $R_\alpha$ but there does not appear to be an elegant formula.

We can turn it into a finite-dimensional problem by defining, for $M \in \mathbb{N}$,

$$R_\alpha(M) = \sup_{a_1, \ldots, a_M \geq 0} \sum_{n=1}^{M} \Delta h(n) a_n^{1-\alpha}.$$  

We first prove that where $\Delta h(n)$ is increasing, we must take $a_n$ constant. In fact, we prove this for a slightly more general problem:

**Lemma 5.1.** Let $\alpha \in (0, 1)$, $\ell = (l_1, \ldots, l_M) \in \mathbb{N}^M$ and $\Lambda = (\lambda_1, \ldots, \lambda_M) \in \mathbb{R}^M$ with each $\lambda_i \geq 0$ and consider

$$R_\alpha(\Lambda, \ell; M) = \max_{a_1, \ldots, a_M \geq 0} \sum_{m=1}^{M} \lambda_m l_m a_m^{1-\alpha}.$$  

(i) Suppose that $\lambda_k < \lambda_{k+1}$ for some $k \in \{1, \ldots, M-1\}$. Then for the above maximum, we must take $a_k = a_{k+1}$.

(ii) If $\lambda_k \geq \lambda_{k+1}$ for every $k$, then $R_\alpha(\Lambda, \ell; M) = (\sum_{m=1}^{M} \lambda_m^{1/\alpha} l_m)^\alpha$.

**Proof.** (i) In any case $a_k \geq a_{k+1}$, so it suffices to show that if $a_k > a_{k+1}$ then there exists $a'_1 = (a'_1, \ldots, a'_M)$ with $a'_1 \geq \ldots \geq a'_M \geq 0$ and $\sum_{m=1}^{M} l_m a'_m = 1$ for which

$$\sum_{m=1}^{M} \lambda_m l_m a'_m^{1-\alpha} > \sum_{m=1}^{M} \lambda_m l_m a_m^{1-\alpha}. \quad (5.1)$$  

So, suppose $a_k > a_{k+1}$. Let $a'_n = a_n$ for $n \neq k, k+1$ and put

$$a'_k = a'_{k+1} = \frac{l_k a_k + l_{k+1} a_{k+1}}{l_k + l_{k+1}}.$$  

As such, $a'_1 \geq \ldots \geq a'_M \geq 0$ (since $a_{k+1} < a'_k < a_k$) and $\sum_{m=1}^{M} l_m a'_m = 1$ (since $l_k a'_k + l_{k+1} a'_{k+1} = l_k a_k + l_{k+1} a_{k+1}$) while

$$\sum_{m=1}^{M} \lambda_m l_m a'_m^{1-\alpha} - \sum_{m=1}^{M} \lambda_m l_m a_m^{1-\alpha} =$$

$$= (\lambda_k l_k + \lambda_{k+1} l_{k+1}) (a'_k)^{1-\alpha} - \lambda_k l_k a_k^{1-\alpha} + \lambda_{k+1} l_{k+1} a_{k+1}^{1-\alpha}) =$$

$$= \lambda_{k+1} a_k^{1-\alpha} \left\{ \frac{(l_k s + l_{k+1})}{(l_k + l_{k+1})^{1-\alpha}} - (l_k s + l_{k+1}) \right\}$$.
where $s = \frac{\lambda}{\lambda_{k+1}}$ and $t = \frac{a_{k+1}}{a_k}$. Note that $0 \leq s, t < 1$. Now put

$$F(x, y) = F_{m,n}(x, y) = \frac{(mx + n)(m + ny)^{1-\alpha}}{(m + n)^{1-\alpha}} - (mx + ny^{1-\alpha}) \quad (m, n \in \mathbb{N}).$$

So the RHS of (5.2) is $\lambda_{k+1}^{-\alpha} F_{k+1,k+2}(s, t)$. We claim that for any $x, y \in [0, 1]$, $F(x, y) \geq 0$ with equality if and only if $x = y = 1$. For

$$F(x, y) \geq 0 \quad \forall x, y \in [0, 1] \iff$$

$$\iff \frac{(mx + n)(m + ny)^{1-\alpha}}{(m + n)^{1-\alpha}} \geq mx + ny^{1-\alpha} \quad \forall x, y \in [0, 1] \iff$$

$$\iff mx \left\{1 - \left(\frac{m + ny}{m + n}\right)^{1-\alpha}\right\} \leq n \left\{(\frac{m + ny}{m + n})^{1-\alpha} - y^{1-\alpha}\right\} \quad \forall x, y \in [0, 1] \iff$$

$$\iff m \left\{1 - \left(\frac{m + ny}{m + n}\right)^{1-\alpha}\right\} \leq n \left\{(\frac{m + ny}{m + n})^{1-\alpha} - y^{1-\alpha}\right\} \quad \forall y \in [0, 1]$$

since the LHS is largest when $x = 1$. Rearranging, we see that this holds if and only if $G(y) \geq 0$ for $0 \leq y < 1$ where

$$G(y) = (m + n)^\alpha (m + ny)^{1-\alpha} - m - ny^{1-\alpha}.$$ 

But $G'(y) = (1-\alpha)ny^{-\alpha}((m+ny)^\alpha - 1) < 0$ for $0 < y < 1$. Thus $G$ is strictly decreasing in $[0, 1]$. Since $G(1) = 0$ the result follows.

For the second part, note that by Hölder’s inequality

$$\sum_{m=1}^M \lambda_m l_n a_m^{1-\alpha} = \sum_{m=1}^M \lambda_m a_m^{\alpha} (l_n a_m)^{1-\alpha} \leq \left(\sum_{m=1}^M \lambda_m a_m^{1-\alpha}\right)^\alpha.$$ 

Equality holds if $\lambda_m^{1/\alpha} = ca_m$ for some constant $c$, which is feasible if $\lambda_m$ is decreasing.

**Determining $R_\alpha$.** Thus, in the evaluation of $R_\alpha(\Lambda, \tau; M)$, for the optimal solution we need to take $a_n$ constant on intervals where $\lambda_n$ is strictly increasing. Partition $\{1, \ldots, M\}$ into consecutive intervals$^\dagger$ $\mathcal{L}_1, \ldots, \mathcal{L}_M$ and $\lambda_n$ is strictly increasing on each $\mathcal{L}_r$. Thus we can write $\mathcal{L}_r = \{L_{r-1} + 1, \ldots, L_r\}$ for $r = 1, \ldots, M'$ where $L_r$ is a strictly increasing sequence of integers with $L_0 = 0$ and $L_M = M$, and $\lambda_{n+1} > \lambda_n$ for $L_{r-1} < n < L_r$, while $\lambda_{n+1} \leq \lambda_n$ for $n = L_r$ $(1 \leq r < M')$. (If $\lambda_k$ is decreasing, we must take $\mathcal{L}_r = \{r\}$.) As such, we take $a_n$ constant on each $\mathcal{L}_r$. Writing

$^\dagger$That is; sets of the form $\{k, k+1, k+2, \ldots, l\}$ where $k, l \in \mathbb{N}$. 

\[ \ell'_r = \sum_{n \in \mathcal{L}_r} l_n \quad \text{and} \quad b_r = a_{L_r}, \]

gives \( \sum_{n=1}^{M} l_n a_n = \sum_{r=1}^{M'} (\sum_{n \in \mathcal{L}_r} l_n) a_{L_r} = \sum_{r=1}^{M'} \ell'_r b_r = 1, \)

while

\[ \sum_{n=1}^{M} \lambda_n l_n a_n = \sum_{r=1}^{M'} \left( \sum_{n \in \mathcal{L}_r} \lambda_n l_n \right) b_r = \sum_{r=1}^{M'} \lambda'_r l'_r b_r \]

where \( \lambda'_r = \frac{1}{l'_r} \sum_{n \in \mathcal{L}_r} \lambda_n l_n. \) Thus

\[ R_\alpha(\Lambda, \ell; M) = \max_{b_1 \geq \cdots \geq b_{M'} \geq 0} \sum_{m=1}^{M'} \lambda'_m l'_m b'_m = R_\alpha(\Lambda', \ell'; M') \]

where \( \Lambda' = (\lambda'_1, \ldots, \lambda'_{M'}) \) and \( \ell' = (l'_1, \ldots, l'_{M'}). \) Note that \( M' < M, \) unless \( \lambda_n \) is decreasing, in which case \( R_\alpha(M) \) can be evaluated. Now apply Lemma 5.1 to this optimization problem and continue the process repeatedly. Thus

\[ R_\alpha(\Lambda, \ell; M) = R_\alpha(\Lambda', \ell'; M') = \cdots = R_\alpha(\Lambda^*, \ell^*; M^*) \]

where the process stops when \( \Lambda^* \) is a decreasing set. This is guaranteed to happen when \( M^* = 1, \) but could happen earlier. Notice that at each stage, the forms for \( \Lambda \) and \( \ell \) are the same. Consider for example the second stage, where we have partitioned \( \{1, \ldots, M'\} \) into consecutive intervals \( \mathcal{L}_1', \ldots, \mathcal{L}_{M''} \) with corresponding \( \ell'' \) and \( \Lambda''. \) Then

\[ l''_k = \sum_{n \in \mathcal{L}_k''} l'_n = \sum_{n \in \mathcal{L}_k''} \sum_{m \in \mathcal{L}_k} l_m = \sum_{n \in \mathcal{L}} l_n \]

for some consecutive set \( \mathcal{L} \) (dependent on \( k \). Likewise

\[ \lambda''_k l''_k = \sum_{n \in \mathcal{L}_k''} \lambda'_n l'_n = \sum_{n \in \mathcal{L}_k''} \sum_{m \in \mathcal{L}_k} \lambda_m l_m = \sum_{n \in \mathcal{L}} \lambda_n l_n \]

In particular, this holds for \( \ell^* \) and \( \Lambda^*. \) Rewriting, the above shows that the optimal solution always has the form\(^5\)

\[ R_\alpha(\Lambda, \ell; M) = \left( \frac{\sum_{k=1}^{K} (q(m_k) - q(m_{k-1}))}{q(m_k) - q(m_{k-1})} \right)^{1/\alpha} \]

Another way to see this is to realise that at each stage more consecutive \( a_n \)s are equated until the corresponding \( \lambda'_n \)s (or \( \lambda''_n \)s etc.) are decreasing.
where \( q(r) = l_1 + \cdots + l_r \) and \( s(r) = \lambda_1 l_1 + \cdots + \lambda_r l_r \), for some sequence of integers \( m_k \) satisfying \( 0 = m_0 < m_1 < \cdots < m_K = M \). Being optimal, this requires that

\[
\frac{s(m_k) - s(m_{k-1})}{q(m_k) - q(m_{k-1})}
\]

is decreasing.

For the special case \( l_k \equiv 1 \) and \( \lambda_k = \Delta h(k) \), \( q(r) = r \) and \( s(r) = h(r) \). Thus

\[
R_\alpha(M) = \left( \sum_{k=1}^{K} (m_k - m_{k-1}) \left( \frac{h(m_k) - h(m_{k-1})}{m_k - m_{k-1}} \right)^{1/\alpha} \right)^{\alpha}
\]

for some such sequence \( m_k \) for which \( \frac{h(m_k) - h(m_{k-1})}{m_k - m_{k-1}} \) decreases.

6. Examples and final comments

Now we illustrate our results with a few examples.

**Example 1.** Let \( f \) be multiplicative with \( f(p^k) = \exp\{k^{\beta}p^{-\alpha}\} \) where \( 0 < \alpha < 1 \) and \( 0 < \beta < 1 - \alpha \) for prime powers \( p^k \). Thus \( h(k) = k^\beta \), which is increasing and \( \Delta h(k) \) is strictly decreasing as can be readily verified. In this case \( L(x) \sim \frac{x^{1-\alpha}}{(1-\alpha) \log x} \). Thus, by Theorem 2,

\[
\max_{n \leq N} \log f(n) \sim \left( \sum_{n=1}^{\infty} (n^\beta - (n-1)^\beta)^{1/\alpha} \right)^{\alpha} \frac{(\log N)^{1-\alpha}}{(1-\alpha) \log \log N}.
\]

(For \( \alpha = 0 \) the RHS is \( \frac{\log N}{\log \log N} \).) In some cases the constant can be evaluated in terms of \( \zeta \)-values. For example, taking \( \beta = \frac{1}{2} \) and \( \alpha = \frac{1}{3} \),

\[
\sum_{n=1}^{N} (\sqrt{n} - \sqrt{n-1})^3 = 4N^{3/2} + 3\sqrt{N} - 6 \sum_{n=1}^{\infty} \sqrt{n} \to -6 \left( -\frac{1}{2} \right),
\]

after suitable manipulations. By the functional equation for \( \zeta(s) \) this equals \( \frac{1}{2\pi} \zeta\left( \frac{3}{2} \right) \). That is, the maximal order of the multiplicative function with \( f(p^k) = \exp\{k^{\beta}/\sqrt{p}\} \) is

\[
\exp\left\{ \left( \frac{3}{2} \sqrt{\frac{3}{2\pi}} \zeta\left( \frac{3}{2} \right) + o(1) \right) \frac{(\log N)^{2/3}}{\log \log N} \right\}.
\]
Example 2. Theorem 2 can also be used in cases where \( \log f(p^k) \) is not the form \( h(k)l(p) \), but only asymptotically of this form. In [5], the maximal order of the function

\[
\eta_{\alpha,\gamma}(n) = \frac{1}{d(n)} \sum_{d|n} \sigma_{-\alpha}(d)^\gamma
\]

was required, where \( \sigma_{-\alpha}(n) = \sum_{d|n} d^{-\alpha} \) and \( d(n) = \sigma_0(n) \). It was shown that for \( \alpha \in (0, 1) \) and any \( \gamma > 0 \)

\[
\max_{n \leq N} \log \eta_{\alpha,\gamma}(n) \approx \frac{(\log N)^{1-\alpha}}{(1 - \alpha) \log \log N}
\]

but the true maximal order was left open. With Theorem 2, this can now be established.

Note that \( \eta_{\alpha,\gamma}(n) \) is multiplicative with

\[
\eta_{\alpha,\gamma}(p^k) = \frac{1}{k+1} \sum_{r=0}^{k} \sigma_{-\alpha}(p^r)^\gamma = \frac{1}{k+1} \left( 1 + \sum_{r=1}^{k} \left( 1 + \frac{1}{p^\alpha} + O\left( \frac{1}{p^{2\alpha}} \right) \right)^r \right) = 1 + \frac{\gamma k}{(k+1)p^\alpha} + O\left( \frac{1}{p^{2\alpha}} \right)
\]

the implied constants being independent of \( k \) (and \( p \)). Let \( s(n) \) denote the multiplicative function with \( s(p^k) = \exp \left( \frac{\gamma k}{k+1} \right) \). Then \( \eta_{\alpha,\gamma}(n) = s(n)t(n) \) and from the above, \( \sigma_{-2\alpha}(n)^{-\kappa} \leq t(n) \leq \sigma_{-2\alpha}(n)^{\kappa} \) for some \( \kappa > 0 \). It follows that \( \log t(n) \ll (\log n)^{1-2\alpha+\varepsilon} \) for every \( \varepsilon > 0 \). Thus the maximal order of \( \log \eta_{\alpha,\gamma}(n) \) is the same as for \( \log s(n) \), which can be found from Theorem 2. In this case \( h(k) = \frac{\gamma k}{k+1} \) which is increasing and \( \Delta h(k) = \frac{\gamma}{k(k+1)} \) which is decreasing, while \( l(p) = p^{-\alpha} \). Theorem 2 now gives

\[
\max_{n \leq N} \log \eta_{\alpha,\gamma}(n) \sim \max_{n \leq N} \log s(n) \sim \gamma \left( \sum_{n=1}^{\infty} \left( \frac{1}{n(n+1)} \right)^{1/\alpha} \right) \alpha \left( \frac{\log N}{1 - \alpha} \right)^{1-\alpha} \frac{\log \log N}{\log N}.
\]

For particular values of \( \alpha \) the constant may be evaluated. Take, say, \( \alpha = \frac{1}{2} \). Then the sum above becomes

\[
\sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+1} \right)^2 = \sum_{n=1}^{\infty} \left( \frac{1}{n^2} + \frac{1}{(n+1)^2} - \frac{2}{n(n+1)} \right) = 2\zeta(2) - 3.
\]

Hence, with say \( \gamma = 2 \),

\[
\max_{n \leq N} \log \eta_{\frac{1}{2},2}(n) \sim 4 \sqrt{\frac{\pi^2}{3}} \frac{\sqrt{\log N}}{\log \log N}.
\]
Example 3. Let \( f \) be multiplicative with \( \log f(p^k) = h(k)l(p) \) where \( h(k) = \lceil \sqrt{k} \rceil \). This time \( h(k) \) is increasing but \( \Delta h(k) \) is not, as \( \Delta h(k) = 1 \) for \( k \) a square and zero otherwise. Note that to apply Theorem 1, we require \( \alpha < \frac{1}{2} \). To calculate \( R_\alpha \) we use the method in Section 5. Thus

\[
R_\alpha = \sup_{a_n \in \mathbb{N}} \sum_{n=1}^{\infty} \Delta h(n)a_n^{1-\alpha} = \sup_{a_n \in \mathbb{N}} \sum_{n=1}^{\infty} a_n^{-\alpha}.
\]

Putting \( b_1 = a_1, b_2 = a_2 = a_3 = a_4, b_3 = a_5 = \cdots = a_9 \) etc. for the optimal solution gives

\[
R_\alpha = \sup_{\sum b_n > 0} \sum_{n=1}^{\infty} b_n^{1-\alpha} = \left( \sum_{n=1}^{\infty} \frac{1}{(2n-1)^{1/\alpha}} \right)^\alpha,
\]

by taking the optimal choice \( b_n = c(2n-1)^{-1/\alpha} \) for some \( c > 0 \). Thus, if \( l \) is decreasing and regularly varying of index \( -\alpha \) with \( 0 < \alpha < \frac{1}{2} \) then

\[
\max_{n \leq N} \log f(n) \sim (1 - 2^{1-\frac{1}{\alpha}})^\alpha \zeta\left(\frac{1}{\alpha} - 1\right) \sum_{p \leq \log N} l(p).
\]

Final comments

The constant appearing in the asymptotic formula in the theorems has the form of an \( l^p \)-norm. For \( a = (a_n) \) the \( l^p \)-norm is defined for \( 1 \leq p < \infty \) and \( p = \infty \) respectively by

\[
\|a\|_p = \left( \sum_{n=1}^{\infty} |a_n|^p \right)^{1/p}, \quad \|a\|_\infty = \sup_{n \in \mathbb{N}} |a_n|.
\]

Writing \( \alpha = 1/p \) \( (p > 1) \) we therefore see that, given the conditions of Theorem 2,

\[
\max_{n \leq N} \log f(n) \sim \|\Delta h\|_p L(\log N),
\]

while for Theorem 3, with \( \alpha = 0 \) corresponding to \( p = \infty \)

\[
\max_{n \leq N} \log f(n) \sim \|h_1\|_\infty L(\log N),
\]

where \( h_1(n) = h(n)/n \).

This type of formula is strangely similar to an asymptotic formula found for the following ‘quasi’-norm of an arithmetical operator (see [6]). Let

\[
M_f(T) = \sup_{g \in M^2} \frac{\|f * g\|_2}{\|g\|_2 T}
\]
where $\mathcal{M}^2$ is the set of square-summable multiplicative functions and $*$ is Dirichlet convolution. Taking $f \in \mathcal{M}^2$ to be completely multiplicative such that $f(p)$ is regularly varying with index $-\alpha$, it was proven in [6] that for $\frac{1}{2} < \alpha < 1$

$$\log M_f(T) \sim \left(\frac{1}{2} B\left(\frac{1}{\alpha}, 1 - \frac{1}{2\alpha}\right)\right)^{\alpha} F(\log T \log \log T)$$

where $F(x) = \sum_{p \leq x} f(p)$. Here $B(x, y)$ is the beta-function. Writing $p = 1/\alpha$, the constant can be rewritten as $\|h'\|_p$ where $h(x) = \sqrt{1 - e^{-2x}}$. With some heuristic reasoning, it was further suggested in the case where $f(n) = n^{-\alpha}$ that $M_f(T)$ represents the maximal order of $\zeta(\alpha + it)$ up to height $T$; i.e.

$$\max_{|t| \leq T} |\zeta(\alpha + it)| \sim \|h'\|_p \left(\frac{(\log T)^{1-\alpha}}{(1-\alpha)(\log \log T)^\alpha}\right)$$

where $\|h'\|_p = \left(\int_0^\infty |h'|^p \right)^{1/p}$ is now the $L_p$-norm. The similarity of form between these ‘discrete’ and ‘continuous’ cases is rather striking, and suggests that there might be a more general framework which combines these formulae.

7. Appendix

To put the results into a broader context, we consider a few classes of multiplicative functions of the form (1.1) where $h$ and $l$ satisfy slightly altered assumptions.

1. Case where $h$ is increasing, $l$ decreasing and such that $\sum_p l(p) < \infty$.

In this case we find the maximal order of $\log f(n)$ is of size $h\left(\frac{\log n}{\log 2}\right)$. More precisely, with $\lambda = \sum_p l(p)$

$$l(2)h\left(\frac{\log n}{\log 2}\right) \leq \log f(n) \leq \lambda h\left(\frac{\log n}{\log 2}\right),$$

where the RHS inequality holds for all $n$ and the LHS for infinitely many $n$, namely, $n = 2^k$.

**Proof.** Let $n = \prod_{p \leq P} p^{a_p}$ where $a_p$ can be taken to be decreasing after Lemma 4.1. Thus $\log n = \sum_{p \leq P} a_p \log p \geq a_2 \log 2$ and

$$\log f(n) = \sum_{p \leq P} h(a_p) l(p) \leq h(a_2) \sum_{p} l(p) = \lambda h(a_2) \leq \lambda h\left(\frac{\log n}{\log 2}\right).$$

On the other hand, with $n = 2^k$, $\log f(n) = l(2)h(k) = l(2)h\left(\frac{\log n}{\log 2}\right)$.
2. Case where \( h \) and \( \Delta h \) are increasing, and \( l \) decreasing. Now the maximum for \( f \) occurs when \( n = 2^k \) and

\[
\max_{n \leq N} f(n) = \exp\left\{ l(2)h\left(\frac{\log n}{\log 2}\right)\right\}.
\]

To see this, suppose \( p|n \) where \( p \) is an odd prime, so \( n = 2^k \ldots p^l \) for some \( k, l \in \mathbb{N} \). After Lemma 4.1 we can take \( k \geq l \). Then, with \( n' = \frac{2}{p} n \),

\[
\frac{f(n')}{f(n)} = \frac{f(2^{k+1})f(p^{l-1})}{f(2^k)f(p^l)} = \exp\{l(2)\Delta h(k + 1) - l(p)\Delta h(l)\} \geq 1.
\]

Thus, with \( K \) such that \( 2^K \leq N < 2^{K+1} \)

\[
\max_{n \leq N} f(n) = f(2^K) = e^{l(2)h(K)} = \exp\left\{ l(2)h\left(\frac{\log n}{\log 2}\right)\right\}.
\]

References


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