Annales Univ. Sci. Budapest., Sect. Comp. 43 (2014) 193–215

DISCRETE UNIFORMLY CONVERGENT PROCESSES ON THE ROOTS OF FOUR KINDS OF CHEBYSHEV POLYNOMIALS

Zsolt Németh (Budapest, Hungary)

Communicated by Ferenc Schipp

(Received February 18, 2014; accepted April 7, 2014)

Abstract. Starting from discrete Fourier series we construct approximation processes on the roots of four kinds of Chebyshev polynomials generated by suitable summation functions φ. We prove a general result stating that if the Fourier transform of φ is integrable then these processes are uniformly convergent on the whole interval $[-1, 1]$ in some weighted spaces of continuous functions. We also examine necessary and sufficient conditions for the interpolation. As applications, we obtain various new results for the arithmetic means of the Lagrange interpolation, the Grünwald, the de la Vallée Poussin and the Hermite–Fejér interpolation.

1. Introduction

Let $C(I)$ represent the linear space of continuous functions defined on an interval $I \subset \mathbb{R}$,

$$w_{\gamma,\delta}(x) := (1-x)^{\gamma}(1+x)^{\delta} \qquad \big(x \in [-1,1],\ \gamma,\delta \geq 0\big)$$

Key words and phrases: Lagrange interpolation, Hermite–Fejér interpolation, weighted interpolation, Chebyshev polynomials, summation processes, Grünwald processes, de la Vallée Poussin means.
2010 Mathematics Subject Classification: 41A05, 41A10, 40A05.

be a weight function and define the weighted function space

$$C_{w_{\gamma,\delta}} := \Big\{ f \in C(-1,1) \mid \lim_{\pm 1}(f w_{\gamma,\delta}) = 0 \Big\},$$

if $\gamma, \delta > 0$. Otherwise, if $\gamma = 0$ (respectively $\delta = 0$) let $C_{w_{\gamma,\delta}}$ consists of all continuous functions on $(-1,1]$ (respectively on $[-1,1)$) and

$$\lim_{-1}(f w_{\gamma,\delta}) = 0 \qquad \big(\text{resp.} \ \lim_{1}(f w_{\gamma,\delta}) = 0\big).$$

Finally, if $\gamma = \delta = 0$ (i.e. $w_{\gamma,\delta} \equiv 1$) then let $C_{w_{\gamma,\delta}} = C[-1,1]$.

Then

$$\|f\|_{w_{\gamma,\delta}} := \|f w_{\gamma,\delta}\|_\infty := \max_{x \in [-1,1]} |(f w_{\gamma,\delta})(x)| \qquad (f \in C_{w_{\gamma,\delta}})$$

is a norm on $C_{w_{\gamma,\delta}}$ and $(C_{w_{\gamma,\delta}}, \|\cdot\|_{w_{\gamma,\delta}})$ is a Banach space.

If $X_M := \{x_{k,M}\} \subset (-1,1)$ $(M \in \mathbb{N}^+ := \{1,2,\dots\})$ is an interpolatory matrix, that is

$$-1 < x_{M,M} < x_{M-1,M} < \cdots < x_{2,M} < x_{1,M} < 1$$

and $f : [-1,1] \to \mathbb{R}$ is a given function then we denote the Lagrange interpolation polynomial of f on X_M by $L_M(f, X_M, \cdot)$.

Using [25, Theorem 2.2] we have a *Faber type result* for the weighted approximation of the Lagrange interpolation, namely if $\gamma, \delta \geq 0$ then for the matrix of nodes X_M there exists a function $f \in C_{w_{\gamma,\delta}}$ for which the relation

$$\|f - L_M(f, X_M, \cdot)\|_{w_{\gamma,\delta}} \to 0 \quad \text{as} \quad n \to +\infty \tag{1.1}$$

does not hold.

Therefore we can ask how to construct such discrete processes which are uniformly convergent in suitable spaces of continuous functions.

One possibility to achieve this aim is to loosen the strict condition on the degree of interpolating polynomials (see [15, Chapter II], [18], [24], [7]). The success of a construction like this strongly depends on the matrix of nodes.

Another way to obtain uniformly convergent processes is to consider suitable means of the Lagrange interpolation polynomials (see [20], [21]).

In this paper we use a mixture of the above techniques to obtain wide classes of uniformly convergent weighted processes on the roots of the four kinds of Chebyshev polynomials using a summation function φ.

Let $w_{\alpha,\beta}(x) := (1-x)^\alpha (1+x)^\beta$ be a Jacobi weight $(\alpha, \beta > -1)$ and consider the sequence of *orthonormal* polynomials $p_j^{(\alpha,\beta)}(x)$ having positive main coefficients $(j \in \mathbb{N} := \{0,1,\dots\})$ with respect to the weight $w_{\alpha,\beta}$:

$$\int_{-1}^{1} p_i^{(\alpha,\beta)}(x) p_j^{(\alpha,\beta)}(x) w_{\alpha,\beta}(x) \, dx = \delta_{i,j} \qquad (i, j \in \mathbb{N}). \tag{1.2}$$

Let us denote by

$$X_M^{(\alpha,\beta)} := \{x_{k,M} := x_{k,M}^{(\alpha,\beta)} : k = 1, 2, \dots, M\} \qquad (M \in \mathbb{N}^+)$$

the M different roots of $p_M^{(\alpha,\beta)}$, indexed in decreasing order.

The Lagrange interpolation polynomial of a function f on $X_M^{(\alpha,\beta)}$ $(M \in \mathbb{N}^+)$ will be denoted by $L_M(f, X_M^{(\alpha,\beta)}, \cdot)$ and can be expressed (see [16, Theorem 3.2.2 and 3.4.6]) as

$$L_M(f, X_M^{(\alpha,\beta)}, x) = \sum_{j=0}^{M-1} c_{j,M}(f) p_j^{(\alpha,\beta)}(x) \qquad (x \in [-1, 1]), \tag{1.3}$$

where

$$c_{j,M}(f) := c_{j,M}^{(\alpha,\beta)}(f) = \sum_{k=1}^{M} f(x_{k,M}) p_j^{(\alpha,\beta)}(x_{k,M}) \lambda_{k,M}^{(\alpha,\beta)} \tag{1.4}$$

for $j = 0, 1, \dots, M-1$, and $\lambda_{k,M}^{(\alpha,\beta)}$ denote the Christoffel numbers with respect to the weight $w_{\alpha,\beta}$.

The definition of the coefficients $c_{j,M}^{(\alpha,\beta)}(f)$ may be extended for all $j \in \mathbb{N}$ by the above formula (1.4), and the series

$$\sum_{j \in \mathbb{N}} c_{j,M}(f) p_j^{(\alpha,\beta)} \tag{1.5}$$

can be considered as a *discrete Fourier series* of f.

Section 2. contains the construction of φ-summations for the parameters $|\alpha| = |\beta| = \frac{1}{2}$. We discuss the convergence and the interpolation property of these processes in general.

In Section 3. we consider some well-known methods, i.e. the arithmetic means of Lagrange interpolation; the Grünwald, de la Vallée Poussin and Hermite–Fejér interpolations.

The proofs of our statements can be found in Section 4.

2. General results

We shall consider only the special cases

$$|\alpha| = |\beta| = \frac{1}{2}, \tag{2.1}$$

i.e. the node systems $X_M^{(\alpha,\beta)}$ contains the roots of one of the four kinds of Chebyshev polynomials. With the notations $x = \cos\vartheta$, $x \in [-1,1]$, $\vartheta \in \in [0,\pi]$ we recall the orthonormal *first, second, third and fourth kind Chebyshev polynomials*, respectively:

$$p_n^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(x) = \sqrt{\tfrac{2}{\pi}}T_n(x) = \sqrt{\tfrac{2}{\pi}}\cos n\vartheta \tag{2.2}$$

if $n \in \mathbb{N}^+$ and

$$p_0^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(x) = \sqrt{\tfrac{1}{\pi}}T_0(x) = \sqrt{\tfrac{1}{\pi}}.$$

$$p_n^{\left(\frac{1}{2},\frac{1}{2}\right)}(x) = \sqrt{\tfrac{2}{\pi}}U_n(x) = \sqrt{\frac{2}{\pi}}\,\frac{\sin(n+1)\vartheta}{\sin\vartheta}, \tag{2.3}$$

$$p_n^{\left(-\frac{1}{2},\frac{1}{2}\right)}(x) = \sqrt{\tfrac{1}{\pi}}V_n(x) = \sqrt{\frac{1}{\pi}}\,\frac{\cos(2n+1)\frac{\vartheta}{2}}{\cos\frac{\vartheta}{2}}, \tag{2.4}$$

$$p_n^{\left(\frac{1}{2},-\frac{1}{2}\right)}(x) = \sqrt{\tfrac{1}{\pi}}W_n(x) = \sqrt{\frac{1}{\pi}}\,\frac{\sin(2n+1)\frac{\vartheta}{2}}{\sin\frac{\vartheta}{2}}. \tag{2.5}$$

For these α and β, let us define the values

$$\gamma := \frac{\alpha}{2} + \frac{1}{4} \quad \text{and} \quad \delta := \frac{\beta}{2} + \frac{1}{4}. \tag{2.6}$$

Fix a *summation function* $\varphi : [0,+\infty) \to \mathbb{R}$ with $\operatorname{supp}\varphi \subset [0,1]$. For a function $f \in C_{w_{\gamma,\delta}}$ we define the *φ-means of the discrete Fourier series* (1.5) on the node system $X_M^{(\alpha,\beta)}$ as

$$S_{n,M}^{\varphi}\left(f, X_M^{(\alpha,\beta)}, x\right) := \sum_{j=0}^{n} \varphi\left(\frac{j+\gamma+\delta}{n+2\gamma+2\delta}\right) c_{j,M}(f) p_j^{(\alpha,\beta)}(x) \qquad \left(x \in [-1,1];\ M \in \mathbb{N}^+;\ n \in \mathbb{N}\right), \tag{2.7}$$

where the coefficients $c_{j,M}(f)$ are given by (1.4). The degree of this polynomial is $\le n$. Note that the above polynomials have simple explicit, easily computable forms (the exact roots are known).

Remark. We remark that the "usual" way to define φ summation polynomials would be by the formula (cf. e.g. [19], [17])

$$\sum_{j=0}^{n} \varphi\left(\frac{j}{n}\right) c_{j,M}(f) p_j^{(\alpha,\beta)}(x) \qquad \left(x \in [-1,1],\ f \in C_{w_{\gamma,\delta}},\ n \in \mathbb{N}\right).$$

From the two-parameter operator family $\left(S^{\varphi}_{n,M}, n, M \in \mathbb{N}\right)$ we can choose a one-parameter family using two arbitrary index sequences $(n_m,\ m \in \mathbb{N})$ for the degree, and $(M_m,\ m \in \mathbb{N})$ for the number of nodes. Thus we obtain a sequence of bounded linear operators

$$S^{\varphi}_{n_m,M_m} : C_{w_{\gamma,\delta}} \to \mathcal{P}_{n_m} \qquad (m \in \mathbb{N}), \tag{2.8}$$

where $\mathcal{P}_m$ denotes the linear space of algebraic polynomials of degree $\leq m$.

Denote by $L^1(\mathbb{R}^+)$ $(\mathbb{R}^+ := [0, +\infty))$ the linear space of measurable functions $g : \mathbb{R}^+ \to \mathbb{R}$ for which the Lebesgue integral $\int_{\mathbb{R}^+} |g|$ is finite. The function

$$\|g\|_{L^1(\mathbb{R}^+)} := \int_{\mathbb{R}^+} |g| \qquad \left(g \in L^1(\mathbb{R}^+)\right)$$

is a norm on $L^1(\mathbb{R}^+)$ and $\left(L^1(\mathbb{R}^+), \|\cdot\|_{L^1(\mathbb{R}^+)}\right)$ is a Banach space.

The (cosine) Fourier transform of $g \in L^1(\mathbb{R}^+)$ is defined by

$$\hat{g}(x) := \frac{1}{\pi} \int_0^{+\infty} g(t) \cos(tx)\, dt \qquad \left(x \in \mathbb{R}^+\right).$$

The following theorem shows that if the Fourier transform of a suitable summation function φ is Lebesgue integrable on $\mathbb{R}^+$ then a sequence of polynomials (2.7) tends to f uniformly for any f from the weighted space $C_{w_{\gamma,\delta}}$.

Theorem 2.1. *Let $|\alpha| = |\beta| = \frac{1}{2}$ and (γ, δ) is given by* (2.6)*. Suppose that $\varphi : [0, +\infty) \to \mathbb{R}$ is a continuous function with supp $\varphi \subset [0, 1]$ and $\varphi(0) = 1$, moreover*

$$n_m \to +\infty \ (m \to +\infty) \quad \textit{and} \quad n_m \leq 2M_m.$$

If $\hat{\varphi} \in L^1(\mathbb{R}^+)$ then for any $f \in C_{w_{\gamma,\delta}}$ we have

$$\|f - S^{\varphi}_{n_m,M_m}(f, X^{(\alpha,\beta)}_{M_m}, \cdot)\|_{w_{\gamma,\delta}} \to 0 \qquad (m \to +\infty), \tag{2.9}$$

where the polynomials $S^{\varphi}_{n_m,M_m}$ are defined by (2.7)*.*

The direct verification of $\hat{\varphi} \in L^1(\mathbb{R}^+)$ is generally not easy, but the following sufficient condition is known.

Theorem A. ([12, p. 176]) *If $g : \mathbb{R}^+ \to \mathbb{R}$ is a continuous function supported in $[0, 1]$, and $g \in \operatorname{Lip} \eta$ $(\eta > 1/2)$ on $[0, 1]$ then $\hat{g} \in L^1(\mathbb{R}^+)$.*

Using these results one can easily choose the summation function φ such that the conditions of Theorem A hold, and construct many uniformly convergent discrete processes with simple computable explicit forms.

Remarks. 1. G.M. Natanson and V.V. Zuk [12, p. 168] proved that φ-summation of trigonometric Fourier series is uniformly convergent if and only if the Fourier transform of φ is Lebesgue integrable. The order of convergence was investigated by L. Szili and P. Vértesi [19].

2. The first results with respect to the general φ-summability of *discrete* trigonometric Fourier series are due to F. Schipp and J. Bokor [13]. Their results are sharpened by L. Szili and P. Vértesi [19]. F. Schipp and F. Weisz [14]. investigated the multi-dimensional case. In [3] J. Bokor and F. Schipp studied another type discrete series.

3. In algebraic interpolation, a similar result was presented in [17] for the unweighted case.

We also investigate the interpolatory properties of the polynomials (2.7). The following theorem states that these polynomials interpolate the function $f \in C_{w_{\gamma,\delta}}$ at the points $X_M^{(\alpha,\beta)}$, i.e.

$$f(x_{k,M}) = S^{\varphi}_{n_k,M_k}\big(f, X_M^{(\alpha,\beta)}, x_{k,M}\big) \qquad \big(x_{k,M} \in X_M^{(\alpha,\beta)}\big)$$

if and only if some values of the summation function φ are symmetrical to the center $(x_0, 1/2)$, where

$$x_0 = \frac{M+\gamma+\delta}{n+2\gamma+2\delta}.$$

This property is displayed on Figure 1.

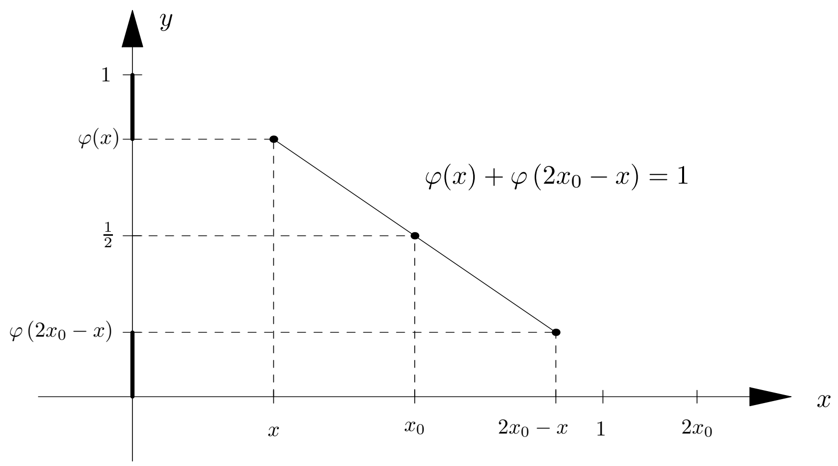

Figure 1.

Theorem 2.2. *Let $|\alpha| = |\beta| = \frac{1}{2}$ and $\gamma, \delta \geq 0$ arbitrary real numbers, moreover suppose that $M \geq 2$, $M \leq n \leq 2M$ $(n, M \in \mathbb{N}^+)$ and $\varphi : [0, +\infty) \to \mathbb{R}$. The polynomial $S^{\varphi}_{n,M}(f, X_M^{(\alpha,\beta)}, x)$ interpolates the function $f \in C_{w_{\gamma,\delta}}$ at the points $X_M^{(\alpha,\beta)}$ if and only if*

$$\varphi\left(\frac{j + \delta + \gamma}{n + 2\delta + 2\gamma}\right) + \varphi\left(\frac{2M - j + \delta + \gamma}{n + 2\delta + 2\gamma}\right) = 1$$

for every $j = 0, 1, \ldots, n,\ j \neq M$.

Remark. Similar necessary and sufficient condition is known in the cases of trigonometric interpolation [19] and unweighted (algebraic) interpolation [17].

3. Results in special cases

For a function $f \in C_{w_{\gamma,\delta}}$ and $M \in \mathbb{N}^+$ the *Lagrange interpolation* polynomials $L_M(f, X_M^{(\alpha,\beta)}, \cdot)$ can be obtained as special cases of (2.7). Indeed, let $n := M$ and

$$\varphi_L(t) := \begin{cases} 1, & \text{if } t \in [0, 1] \\ 0, & \text{if } t \in (1, +\infty). \end{cases}$$

Using Theorem 2.2 it is clear that $S^{\varphi_L}_{M,M}(f, X_M^{(\alpha,\beta)}, \cdot)$ interpolates f at the points $X_M^{(\alpha,\beta)}$, and the degree of the summation polynomial cannot exceed $M-1$ (since $c_{M,M}(f) = 0$, see Lemma 4.1), so it must be the Lagrange interpolation polynomial of f.

As we have already mentioned (see (1.1)), the sequence of these polynomials generally does not tend uniformly to f in $(C_{w_{\gamma,\delta}}, \|\cdot\|_{w_{\gamma,\delta}})$.

3.1. Arithmetic means of Lagrange interpolation

Let $M \in \mathbb{N}^+$ and for $m = 0, 1, \ldots, M-1$ define the polynomials

$$L_{m,M}(f, X_M^{(\alpha,\beta)}, x) := \sum_{j=0}^{m} c_{j,M}(f) p_j^{(\alpha,\beta)}(x), \qquad (f \in C_{w_{\gamma,\delta}}, x \in [-1, 1]).$$

Note that $L_{M-1,M}(f, X_M, \cdot)$ is the Lagrange interpolation polynomial.

We shall consider the following arithmetic means of Lagrange interpolation:

$$\sigma_M(f, X_M^{(\alpha,\beta)}, \cdot) := \frac{1}{M} \sum_{m=0}^{M-1} L_{m,M}(f, X_M^{(\alpha,\beta)}, \cdot).$$

Theorem 3.1. *Let* $|\alpha| = |\beta| = \frac{1}{2}$ *and* (γ, δ) *is given by* (2.6). *Then for any* $f \in C_{w_{\gamma,\delta}}$ *we have*

$$\lim_{M\to+\infty} \|f - \sigma_M(f, X_M^{(\alpha,\beta)}, \cdot)\|_{w_{\gamma,\delta}} = 0.$$

Remarks. 1. Theorem 3.1 is a discrete version of Fejér's theorem about the $(C, 1)$ summability of Fourier series. Analogue results in interpolation theory are due to S. N. Bernstein [1] and J. Marcinkiewicz [8] in the unweighted case.

2. The same result was already obtained in [21] (for more general parameters $\alpha, \beta, \gamma, \delta$), but our proof differs from the one presented there. A similar result was proved for the four kinds of Chebyshev nodes in [17], where the author supplemented the node systems with additional points instead of using weights.

3. We also note that by Theorem 2.2, $\sigma_M(f, X_M^{(\alpha,\beta)}, \cdot)$ does not interpolate f at the points of $X_M^{(\alpha,\beta)}$.

3.2. Grünwald–Rogosinski type processes

Let us consider the summation function

$$\varphi_G(t) := \begin{cases} \cos t\pi, & \text{if } t \in [0, \frac{1}{2}] \\ 0, & \text{if } t \in (\frac{1}{2}, +\infty). \end{cases}$$

Theorem 3.2. *Let* $|\alpha| = |\beta| = \frac{1}{2}$ *and* (γ, δ) *is given by* (2.6) *and suppose that* $f \in C_{w_{\gamma,\delta}}$.

(i) *For* φ_G *we have the Rogosinski type average of Lagrange interpolation, i.e. for* $f \in C_{w_{\gamma,\delta}}$ *the relation*

$$w_{\gamma,\delta} S_{2M,M}^{\varphi_G}(f, X_M^{(\alpha,\beta)}, x) =$$
$$= \frac{1}{2}\Big\{\mathcal{L}_{M,w_{\gamma,\delta}}(f, X_M^{(\alpha,\beta)}, x_+) + \mathcal{L}_{M,w_{\gamma,\delta}}(f, X_M^{(\alpha,\beta)}, x_-)\Big\}$$

holds, where

$$\mathcal{L}_{M,w_{\gamma,\delta}}(f, X_M^{(\alpha,\beta)}, \cdot) := w_{\gamma,\delta} \cdot L_M(f, X_M^{(\alpha,\beta)}, \cdot)$$

and

$$x_{\pm} := x\cos t_M \pm \sqrt{1-x^2}\sin t_M, \qquad t_M := \frac{\pi}{2(M+\gamma+\delta)}.$$

(ii) *For these polynomials we have*

$$\lim_{M\to+\infty} \|f - S_{2M,M}^{\varphi_G}(f, X_M^{(\alpha,\beta)}, \cdot)\|_{w_{\gamma,\delta}} = 0.$$

Remarks. 1. If $\alpha = \beta = -\frac{1}{2}$ and $\gamma = \delta = 0$, then we obtain Grünwald's classical result [6] for first kind Chebyshev roots in the unweighted case.

2. M.S. Webster [28] proved that for $\alpha = \beta = \frac{1}{2}$ the uniform convergence (without weight) is true only for closed subintervals of $(-1,1)$. In [27] P. Vértesi generalized Webster's result for arbitrary $\alpha, \beta > -1$. Theorem 3.2 shows that for $|\alpha| = |\beta| = \frac{1}{2}$ the uniform convergence holds on the whole interval $[-1,1]$, if we use suitable weight function. If $\alpha = \beta = \frac{1}{2}$ then Á. Chripkó [4] proved that the convergence is true if $\frac{1}{2} \le \gamma \le 2$.

3. We also note that by Theorem 2.2, $S_{2M,M}^{\varphi_G}(f, X_M^{(\alpha,\beta)}, \cdot)$ does not interpolate f at the points of $X_M^{(\alpha,\beta)}$.

3.3. De la Vallée Poussin type interpolation

Fix a number $\kappa \in (0,1)$ and let

$$\varphi_\kappa := \begin{cases} 1, & \text{if } t \in [0, \frac{1-\kappa}{2}) \\ -\frac{1}{\kappa}\left(t - \frac{1+\kappa}{2}\right), & \text{if } t \in [\frac{1-\kappa}{2}, \frac{1+\kappa}{2}] \\ 0, & \text{if } t \in (\frac{1+\kappa}{2}, +\infty). \end{cases}$$

Theorem 3.3. *Let $|\alpha| = |\beta| = \frac{1}{2}$ and (γ,δ) is given by* (2.6) *and suppose that $f \in C_{w_{\gamma,\delta}}$.*

(i) *For any fixed $\kappa \in (0,1)$ and $M \in \mathbb{N}^+$, the degree of the polynomial*

$$S_{2M,M}^{\varphi_\kappa}(f, X_M^{(\alpha,\beta)}, \cdot)$$

is $\le M(1+\kappa)$ and it interpolates f at the points of $X_M^{(\alpha,\beta)}$.

(ii) *For any $f \in C_{w_{\gamma,\delta}}$ we have*

$$\lim_{M\to+\infty} \|f - S_{2M,M}^{\varphi_\kappa}(f, X_M^{(\alpha,\beta)}, \cdot)\|_{w_{\gamma,\delta}} = 0.$$

Remarks. 1. For the values $\kappa = 1$ and $\kappa = 0$, we would obtain the Lagrange interpolation and the weighted Hermite–Fejér type interpolation (see in the next subsection), respectively. In trigonometric interpolation, S. N. Bernstein has analogue results [2] for a class of interpolatory polynomials.

2. We remark that Theorem 3.3 can also be considered as a discrete version of the de la Vallée Poussin summation of Fourier series.

3. This result also shows similarity to a result of P. Erdős [5, Theorem 1] in the classical (unweighted) case, where he proved that if the interpolatory point system $(X_M,\ M \in \mathbb{N}^+)$ is such that the fundamental polynomials of Lagrange interpolation are uniformly bounded, then for any $f \in C[-1,1]$ there exists a sequence of polynomials Q_M $(M \in \mathbb{N}^+)$ of degree $\le M(1+\kappa)$ tending uniformly to f, and Q_M interpolates f at the points of X_M for every $M \in \mathbb{N}^+$. For our four point systems, we now have a weighted analogue of this result.

3.4. Weighted Hermite–Fejér type interpolation

Let us define the summation function

$$\varphi_H(t) := \begin{cases} 1-t, & \text{if } t \in [0,1] \\ 0, & \text{if } t \in (1,+\infty). \end{cases}$$

The next theorem states that the weighted Hermite–Fejér type interpolatory polynomials can be obtained by using suitable summation function.

Theorem 3.4. *Let* $|\alpha| = |\beta| = \frac{1}{2}$ *and* (γ,δ) *is given by* (2.6) *and suppose that* $f \in C_{w_{\gamma,\delta}}$.

(i) *For any* $M = 1, 2, \ldots$ *the polynomials*

$$S^{\varphi_H}_{2M,M}\big(f, X^{(\alpha,\beta)}_M, x\big) = \sum_{j=0}^{2M} \left(1 - \frac{j+\gamma+\delta}{2M+2\gamma+2\delta}\right) c_{j,M}(f)\, p^{(\alpha,\beta)}_j(x)$$

(*see* (1.4) *and* (2.7)) *satisfy the following Hermite–Fejér type interpolatory properties*

$$S^{\varphi_H}_{2M,M}(f, X^{(\alpha,\beta)}_M, x_{k,M}) = f(x_{k,M}), \tag{3.1}$$

$$\big(w_{\gamma,\delta} S^{\varphi_H}_{2M,M}(f, X^{(\alpha,\beta)}_M, \cdot)\big)'(x_{k,M}) = 0, \tag{3.2}$$

for all $x_{k,M} \in X^{(\alpha,\beta)}_M$.

(ii) *For any* $f \in C_{w_{\gamma,\delta}}$ *we have*

$$\lim_{M\to+\infty} \|f - S^{\varphi_H}_{2M,M}(f, X^{(\alpha,\beta)}_M, \cdot)\|_{w_{\gamma,\delta}} = 0.$$

Remarks. 1. If $\alpha = \beta = -\frac{1}{2}$ and $\gamma = \delta = 0$, then we obtain Fejér's classical result for first kind Chebyshev roots in the unweighted case. (See e.g. [15, p. 165], [26].)

2. In [7] Ágota P. Horváth proved a general convergence theorem for the above type weighted Hermite–Fejér interpolation process on $\varrho(w)$-normal point systems (especially on Jacobi roots, see [7, Example (2)]); but Theorem 2 of her paper does not contain our Theorem 3.4.

3. G. Mastroianni and J. Szabados [11] investigated an other type weighted Hermite–Fejér interpolation process based on Jacobi nodes.

4. Proofs

4.1. On the coefficients $c_{j,M}(f)$

Now we prove some results which will be useful later on. First we take a closer look at the coefficients $c_{j,M}^{(\alpha,\beta)}(f)$ (see (1.4)), if $|\alpha| = |\beta| = \frac{1}{2}$.

Lemma 4.1. *Let us fix the positive integer M. For any $x_{k,M} \in X_M^{(\alpha,\beta)}$ ($k = 1, 2, \dots, M$) and $j = 0, 1, \dots, M-1$ we have*

$$p_j^{(\alpha,\beta)}(x_{k,M}) = -p_{2M-j}^{(\alpha,\beta)}(x_{k,M}), \tag{4.1}$$

and

$$p_M^{(\alpha,\beta)}(x_{k,M}) = 0. \tag{4.2}$$

For a function $f \in C_{w_{\gamma,\delta}}$ the coefficients $c_{j,M}^{(\alpha,\beta)}(f)$ have the properties

$$c_{j,M}(f) = -c_{2M-j,M}(f) \qquad (j = 0, 1, \dots, M-1), \tag{4.3}$$

and

$$c_{M,M}(f) = 0. \tag{4.4}$$

Proof. (4.2) obviously holds since the elements of $X_M^{(\alpha,\beta)}$ are the roots of $p_M^{(\alpha,\beta)}$. The equality (4.1) follows from certain trigonometric identities. The proofs are similar in each four cases for α, β. We shall discuss only the case $\alpha = \beta = \frac{1}{2}$, when for $k = 1, 2, \dots, M$ we have (see (2.3))

$$X_M^{(\frac{1}{2},\frac{1}{2})} \ni x_{k,M} = \cos\vartheta_{k,M} = \cos\frac{k}{M+1}\,\pi.$$

Since for any $j = 0, 1, \ldots, M-1$ we have

$$\sin(j+1)\vartheta_{k,M} = \sin\Big[\big(2M+2-(2M-j+1)\big)\frac{k\pi}{M+1}\Big] = \\ = -\sin(2M-j+1)\vartheta_{k,M},$$

consequently by (2.3)

$$p_j^{(\frac{1}{2},\frac{1}{2})}(x_{k,M}) = p_j^{(\frac{1}{2},\frac{1}{2})}(\cos\vartheta_{k,M}) = \sqrt{\frac{2}{\pi}}\,\frac{\sin(j+1)\vartheta_{k,M}}{\sin\vartheta_{k,M}} = \\ = -\sqrt{\frac{2}{\pi}}\,\frac{\sin(2M-j+1)\vartheta_{k,M}}{\sin\vartheta_{k,M}} = -p_{2M-j}^{(\frac{1}{2},\frac{1}{2})}(\cos\vartheta_{k,M}) = -p_{2M-j}^{(\frac{1}{2},\frac{1}{2})}(x_{k,M}),$$

which proofs (4.1).

Now from the definition of the coefficients (1.4) immediately follow (4.3) and (4.4). ■

4.2. Discrete orthogonality

It is possible to convert the (continuous) orthogonality relationship (1.2) with respect to the system $\big(p_n^{(\alpha,\beta)},\ n\in\mathbb{N}\big)$, into a discrete orthogonality relationship simply by replacing the integral with a certain summ. For the four kinds of orthonormal Chebyshev polynomials the following *discrete orthogonality* properties hold:

Lemma 4.2. *For a fixed $M \in \mathbb{N}^+$ and $i, j = 0, 1, \ldots, M-1$ we have*

$$\sum_{k=1}^{M} p_i^{(\alpha,\beta)}(x_{k,M})\, p_j^{(\alpha,\beta)}(x_{k,M})\, w_{\gamma,\delta}^2(x_{k,M})\, C_M(w_{\alpha,\beta}) = \begin{cases} 1, & \text{if} \quad i=j \\ 0, & \text{if} \quad i\neq j, \end{cases}$$

where $x_{k,M} \in X_M^{(\alpha,\beta)}$ and

$$C_M(w_{\alpha,\beta}) = \begin{cases} \dfrac{\pi}{M}, & \text{if} \quad \alpha=\beta=-\frac{1}{2} \\ \dfrac{\pi}{M+1}, & \text{if} \quad \alpha=\beta=\frac{1}{2} \\ \dfrac{2\pi}{2M+1}, & \text{otherwise.} \end{cases}$$

Proof. From the Gauss–Jacobi quadrature formula (see [16, Theorem 3.4.1]) we have the following discrete orthogonality relation for $i+j \le 2M-1$

$$\int_{-1}^{1} p_i^{(\alpha,\beta)}(x) p_j^{(\alpha,\beta)}(x) w_{\alpha,\beta}(x)\, dx = \sum_{k=1}^{M} p_i^{(\alpha,\beta)}(x_{k,M})\, p_j^{(\alpha,\beta)}(x_{k,M})\, \lambda_{k,M} = \delta_{i,j}$$

where $\lambda_{k,M}$'s are the Christoffel numbers, for which in the cases $|\alpha| = |\beta| = \frac{1}{2}$ by [16, pp. 352–353] we have

$$\lambda_{k,M}^{(\alpha,\beta)} = C_M(w_{\alpha,\beta}) \cdot w_{\gamma,\delta}^2(x_{k,M}) \qquad (k = 1, \dots, M), \tag{4.5}$$

which proves the statement. ∎

4.3. An other form of $S_{n,M}^{\varphi}$

For the proof of (2.9) we have to estimate the expression

$$\left| f(x) - S_{n,M}^{\varphi}(f, X_M, x) \right| w_{\gamma,\delta}(x) = \left| f(x)\, w_{\gamma,\delta}(x) - S_{n,M}^{\varphi}(f, X_M, x) w_{\gamma,\delta}(x) \right|,$$

i.e. we approximate the function $f w_{\gamma,\delta}$ with the weighted polynomial $S_{n,M}^{\varphi}\, w_{\gamma,\delta}$.

Now we give another form of $S_{n,M}^{\varphi}\, w_{\gamma,\delta}$. From (2.7) and (1.4) we have

$$\begin{aligned} &S_{n,M}^{\varphi}(f, X_M, x) w_{\gamma,\delta}(x) = \\ &= \sum_{k=1}^{M} \left(w_{\gamma,\delta} f \right)(x_{k,M}) \cdot K_{n,M}^{\varphi}(w_{\alpha,\beta}, w_{\gamma,\delta}, x_{k,M}, x) \cdot \frac{\lambda_{k,M}}{w_{\gamma,\delta}^2(x_{k,M})}\,, \end{aligned} \tag{4.6}$$

where the kernel function $K_{n,M}^{\varphi}$ is defined as

$$\begin{aligned} &K_{n,M}^{\varphi}(w_{\alpha,\beta}, w_{\gamma,\delta}, x_{k,M}, x) := K_{n,M}^{\varphi}(x_{k,M}, x) := \\ &:= \sum_{j=0}^{n} \varphi\left(\frac{j + \gamma + \delta}{n + 2\gamma + 2\delta} \right) \cdot \left(w_{\gamma,\delta} p_j^{(\alpha,\beta)} \right)(x_{k,M}) \cdot \left(w_{\gamma,\delta} p_j^{(\alpha,\beta)} \right)(x) \end{aligned} \tag{4.7}$$

for an $x_{k,M} \in X_M$.

Lemma 4.3. *Let us fix $n \in \mathbb{N}$, the positive integer M and the node $x_{k,M} = \cos\vartheta_{k,M} \in X_M^{(\alpha,\beta)}$. For $\vartheta \in [0, \pi]$ we have*

$$\begin{aligned} &K_{n,M}^{\varphi}(\cos\vartheta_{k,M}, \cos\vartheta) = \\ &= \frac{1}{\pi} \cdot \begin{cases} D_n^{\varphi}(\vartheta + \vartheta_{k,M}) + D_n^{\varphi}(\vartheta - \vartheta_{k,M}), & \text{if} \quad \alpha = \beta = -\frac{1}{2} \\ D_{n+2}^{\varphi}(\vartheta - \vartheta_{k,M}) - D_{n+2}^{\varphi}(\vartheta + \vartheta_{k,M}), & \text{if} \quad \alpha = \beta = \frac{1}{2} \\ D_{n+1}^{\varphi}(\vartheta + \vartheta_{k,M}) + D_{n+1}^{\varphi}(\vartheta - \vartheta_{k,M}), & \text{if} \quad \alpha = -\frac{1}{2},\ \beta = \frac{1}{2} \\ D_{n+1}^{\varphi}(\vartheta - \vartheta_{k,M}) - D_{n+1}^{\varphi}(\vartheta + \vartheta_{k,M}), & \text{if} \quad \alpha = \frac{1}{2},\ \beta = -\frac{1}{2}, \end{cases} \end{aligned}$$

where

$$D_n^{\varphi}(\vartheta) := \frac{1}{2} + \sum_{j=1}^{n} \varphi\left(\frac{j}{n} \right) \cos j\vartheta.$$

Proof. The proof is based on the trigonometric form of the polynomials $p_j^{(\alpha,\beta)}$. It is similar in each four cases for α,β, so we give the proof only for $\alpha=\beta=\frac{1}{2}$. In this case $\gamma=\delta=\frac{1}{2}$, so for $j=0,1,\dots,n$ we have

$$\varphi\left(\frac{j+\gamma+\delta}{n+2\gamma+2\delta}\right)=\varphi\left(\frac{j+1}{n+2}\right).$$

From (2.3) and (4.7) with $x=:\cos\vartheta$, ($\vartheta\in[0,\pi]$) we obtain

$$K_{n,M}^{\varphi}(\cos\vartheta_{k,M},\cos\vartheta)=\frac{2}{\pi}\sum_{j=0}^{n}\varphi\left(\frac{j+1}{n+2}\right)\sin(j+1)\vartheta_{k,M}\sin(j+1)\vartheta,$$

since

$$w_{\frac{1}{2},\frac{1}{2}}(\cos\vartheta)\cdot p_j^{(\frac{1}{2},\frac{1}{2})}(\cos\vartheta)=\sin\vartheta\cdot\sqrt{\frac{2}{\pi}}\frac{\sin(j+1)\vartheta}{\sin\vartheta}=\sqrt{\frac{2}{\pi}}\sin(j+1)\vartheta,$$

where $j=0,1,\dots,n$. Thus we get

$$K_{n,M}^{\varphi}(\cos\vartheta_{k,M},\cos\vartheta)=$$
$$=\frac{1}{\pi}\Big[\sum_{j=1}^{n+1}\varphi\left(\frac{j}{n+2}\right)\cos j(\vartheta-\vartheta_{k,M})-\sum_{j=1}^{n+1}\varphi\left(\frac{j}{n+2}\right)\cos j(\vartheta+\vartheta_{k,M})\Big],$$

and using the fact that $\varphi(1)=0$, the expression for $K_{n,M}^{\varphi}(\cos\vartheta_{k,M},\cos\vartheta)$ also equals to

$$\frac{1}{\pi}\Big[\frac{1}{2}+\sum_{j=1}^{n+2}\varphi\left(\frac{j}{n+2}\right)\cos j(\vartheta-\vartheta_{k,M})-\frac{1}{2}-\sum_{j=1}^{n+2}\varphi\left(\frac{j}{n+2}\right)\cos j(\vartheta+\vartheta_{k,M})\Big],$$

consequently

$$K_{n,M}^{\varphi}(\cos\vartheta_{k,M},\cos\vartheta)=\frac{1}{\pi}\left[D_{n+2}^{\varphi}(\vartheta-\vartheta_{k,M})-D_{n+2}^{\varphi}(\vartheta+\vartheta_{k,M})\right]. \qquad \blacksquare$$

4.4. Proof of Theorem 2.1

Let $|\alpha|=|\beta|=\frac{1}{2}$. We shall use the Banach–Steinhaus theorem. The polynomials $\{p_i^{(\alpha,\beta)}\ :\ i\in\mathbb{N}\}$ form a closed system in the space $(C_{w_{\gamma,\delta}},\|\cdot\|_{w_{\gamma,\delta}})$ (see e.g. [20, Section 3]), therefore we have to show that

$$\left\|S_{n_m,M_m}^{\varphi}(p_i,X_{M_m},\cdot)-p_i\right\|_{w_{\gamma,\delta}}\to 0 \qquad (m\to+\infty) \tag{4.8}$$

for every fixed $i \in \mathbb{N}$, moreover the norms of the operators $S^{\varphi}_{n_m,M_m}$ is uniformly bounded, i.e. there exists $c > 0$ independent of m such that

$$\|S^{\varphi}_{n_m,M_m}\|_{w_{\gamma,\delta}} \leq c \qquad (m \in \mathbb{N}), \tag{4.9}$$

where

$$\|S^{\varphi}_{n_m,M_m}\|_{w_{\gamma,\delta}} := \sup_{\|f\|_{w_{\gamma,\delta}}=1} \left\{ \|S^{\varphi}_{n_m,M_m}(f, X_{M_m}, \cdot)\|_{w_{\gamma,\delta}} \;:\; f \in C_{w_{\gamma,\delta}} \right\}.$$

In order to prove (4.8), let us fix $i \in \mathbb{N}$ and assume that m is large enough, i.e. $\min\{M_m, n_m\} > i$. Now by Lemma 4.2, for $j = 0, 1, \dots, M_m - 1$ we have

$$c_{j,M_m}(p_i) = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j, \end{cases}$$

so considering $n_m \leq 2M_m$ and (4.3), the equality

$$S^{\varphi}_{n_m,M_m}(p_i, X_{M_m}, \cdot) = \\ = \varphi\left(\frac{i+\gamma+\delta}{n_m+2\gamma+2\delta}\right) p_i - \varphi\left(\frac{2M_m - i+\gamma+\delta}{n_m+2\gamma+2\delta}\right) p_{2M_m-i}$$

holds. It is clear that

$$\lim_{m\to+\infty} \varphi\left(\frac{i+\gamma+\delta}{n_m+2\gamma+2\delta}\right) = \varphi(0) = 1.$$

Since $n_m \leq 2M_m$ $(m \in \mathbb{N})$ and

$$\liminf_{m\to+\infty} \frac{2M_m - i+\gamma+\delta}{n_m+2\gamma+2\delta} = \liminf_{m\to+\infty} \left(\frac{2M_m+2\gamma+2\delta}{n_m+2\gamma+2\delta} - \frac{i+\gamma+\delta}{n_m+2\gamma+2\delta}\right) \geq 1,$$

moreover $\varphi(x) = 0$ if $x \geq 1$, thus we have

$$\lim_{m\to+\infty} \varphi\left(\frac{2M_m - i+\gamma+\delta}{n_m+2\gamma+2\delta}\right) = 0,$$

therefore we proved (4.8).

Next we show (4.9). Using (4.5), (4.6) and $C_{M_m} = C_{M_m}(w_{\alpha,\beta})$, the norm can be expressed as

$$\|S^{\varphi}_{n_m,M_m}(f, X_{M_m}, \cdot)\|_{w_{\gamma,\delta}} = \\ = \left\| \sum_{k=1}^{M_m} w_{\gamma,\delta}(x_{k,M_m}) f(x_{k,M_m}) \cdot C_{M_m} \cdot K^{\varphi}_{n_m,M_m}(x_{k,M_m}, \cdot) \right\|_{\infty},$$

so if $\|f\|_{w_{\gamma,\delta}} = \sup_{x\in[-1,1]} |(w_{\gamma,\delta}f)(x)| = 1$, then we obtain

$$\|S^{\varphi}_{n_m,M_m}\|_{w_{\gamma,\delta}} = \sup_{x\in[-1,1]} C_{M_m} \sum_{k=0}^{M_m} \left|K^{\varphi}_{n_m,M_m}(x_{k,M_m}, x)\right|.$$

By Lemma 4.3 the kernel can be uniformly expressed as

$$K^{\varphi}_{n_m,M_m}(\vartheta_{k,M_k}, \vartheta) = \frac{1}{\pi}\left[D^{\varphi}_{n_m+2\gamma+2\delta}(\vartheta - \vartheta_{k,M_k}) \pm D^{\varphi}_{n_m+2\gamma+2\delta}(\vartheta + \vartheta_{k,M_k})\right],$$

so

$$\|S^{\varphi}_{n_m,M_m}\|_{w_{\gamma,\delta}} \le$$
$$\le \frac{C_{M_m}}{\pi} \max_{\vartheta\in[0,\pi]} \sum_{k=1}^{M_m} \left\{\left|D^{\varphi}_{n_m+2\gamma+2\delta}(\vartheta + \vartheta_{k,M_k})\right| + \left|D^{\varphi}_{n_m+2\gamma+2\delta}(\vartheta - \vartheta_{k,M_k})\right|\right\}.$$

Let

$$\|D^{\varphi}_n\|_{M,1} := \frac{1}{2M} \sup_{\vartheta\in[0,\pi]} \sum_{k=1}^{M} \left\{|D^{\varphi}_n(\vartheta + \vartheta_{k,M})| + |D^{\varphi}_n(\vartheta - \vartheta_{k,M})|\right\}.$$

Then

$$\|D^{\varphi}_n\|_{M,1} \le \left(1 + \frac{2n\pi}{M}\right)\|D^{\varphi}_n\|_1 := \left(1 + \frac{2n\pi}{M}\right) \cdot \frac{1}{2\pi}\int_{-\pi}^{\pi} |D^{\varphi}_n(t)|\, dt$$

(see [22, pp. 242]) and

$$2\sup_{n\in\mathbb{N}} \|D^{\varphi}_n\|_1 = \|\hat{\varphi}\|_{L^1(\mathbb{R}^+)}$$

(see Theorem 2 in §24 of [12]). Consequently if $\hat{\varphi} \in L^1(\mathbb{R}^+)$ and $n_m \le 2M_m$, then there exists $c > 0$ such that

$$\|S^{\varphi}_{n_m,M_m}\|_{w_{\gamma,\delta}} \le C_{M_m}\frac{M_m}{\pi}\left(1 + \frac{n_m + 2\gamma + 2\delta}{M_m}\pi\right)\|\hat{\varphi}\|_{L^1(\mathbb{R}^+)} < c,$$

since $C_{M_m}(w_{\alpha,\beta}) \le \frac{\pi}{M_m}$ for any $|\alpha| = |\beta| = \frac{1}{2}$. This completes the proof of (4.9) and consequently of Theorem 2.1. ■

4.5. Proof of Theorem 2.2

Let $f \in C_{w_{\gamma,\delta}}$ and $M \ge 2, M \in \mathbb{N}$. Using (4.3) and the fact that $M \le n \le$ $\le 2M$ we can write the polynomial $S^{\varphi}_{n,M}(f, X^{(\alpha,\beta)}_M, x)$ (see (2.7)) in the form

$$S^{\varphi}_{n,M}(f, X^{(\alpha,\beta)}_M, x) =$$
$$= \sum_{j=0}^{M-1} c_{j,M}(f)\left[\varphi\left(\frac{j+\gamma+\delta}{n+2\gamma+2\delta}\right)p_j(x) - \varphi\left(\frac{2M-j+\gamma+\delta}{n+2\gamma+2\delta}\right)p_{2M-j}(x)\right],$$

since for $0 \le j < 2M - n$ we have $n < 2M - j \le 2M$, and

$$\varphi\left(\frac{2M - j + \gamma + \delta}{n + 2\gamma + 2\delta}\right) = 0.$$

Now for an arbitrary $x_{i,M} \in X_M^{(\alpha,\beta)}$, $(i = 1, 2, \ldots, M)$ by (4.1) we get

$$S_{n,M}^{\varphi}(f, X_M^{(\alpha,\beta)}, x_{i,M}) =$$
$$= \sum_{j=0}^{M-1} c_{j,M}(f) \cdot \left[\varphi\left(\frac{j + \gamma + \delta}{n + 2\gamma + 2\delta}\right) + \varphi\left(\frac{2M - j + \gamma + \delta}{n + 2\gamma + 2\delta}\right)\right] \cdot p_j(x_{i,M}),$$

and considering (1.3), $S_{n,M}^{\varphi}(f, X_M^{(\alpha,\beta)}, x_{i,M})$ also equals to

$$L_M(f, X_M^{(\alpha,\beta)}, x_{i,M}) +$$
$$+ \sum_{j=0}^{M-1} c_{j,M}(f) \cdot \left[\varphi\left(\frac{j + \gamma + \delta}{n + 2\gamma + 2\delta}\right) + \varphi\left(\frac{2M - j + \gamma + \delta}{n + 2\gamma + 2\delta}\right) - 1\right] \cdot p_j(x_{i,M}).$$

Since

$$L_M(f, X_M^{(\alpha,\beta)}, x_{i,M}) = f(x_{i,M}) \qquad (\forall\ x_{i,M} \in X_M^{(\alpha,\beta)}),$$

the equation

$$S_{n,M}^{\varphi}(f, X_M^{(\alpha,\beta)}, x_{i,M}) = f(x_{i,M})$$

holds for every $x_{i,M} \in X_M^{(\alpha,\beta)}$ if and only if the polynomial

$$\sum_{j=0}^{M-1} c_{j,M}(f) \cdot \left[\varphi\left(\frac{j + \gamma + \delta}{n + 2\gamma + 2\delta}\right) + \varphi\left(\frac{2M - j + \gamma + \delta}{n + 2\gamma + 2\delta}\right) - 1\right] \cdot p_j(x)$$

equals to zero at every point $x_{i,M} \in X_M^{(\alpha,\beta)}$, so it has M distinct roots and its degree $\le M - 1$, consequently it is the zero polynomial.

So $S_{n,M}^{\varphi}(f, X_M^{(\alpha,\beta)}, \cdot)$ interpolates f if and only if

$$\varphi\left(\frac{j + \gamma + \delta}{n + 2\gamma + 2\delta}\right) + \varphi\left(\frac{2M - j + \gamma + \delta}{n + 2\gamma + 2\delta}\right) - 1 = 0$$

for $j = 0, 1, \ldots, M - 1$. This completes the proof. ■

4.6. Proof of Theorem 3.1

A simple calculation shows that

$$\sigma_M(f, X_M^{(\alpha,\beta)}, \cdot) = S_{2M,M}^{\varphi_F}(f, X_M^{(\alpha,\beta)}, \cdot),$$

where

$$\varphi_F(t) := \begin{cases} 1-2t, & \text{if } t \in [0, \frac{1}{2}] \\ 0, & \text{if } t \in (\frac{1}{2}, +\infty). \end{cases}$$

For the Fourier transform of φ_F we have

$$\hat{\varphi}_F(x) = \frac{1}{4\pi}\left(\frac{\sin(x/4)}{x/4}\right)^2,$$

consequently $\hat{\varphi}_F(x) \in L^1(\mathbb{R}^+)$, and by Theorem 2.1, our proof is complete. ■

4.7. Proof of Theorem 3.2

The verification of **(i)** is based on the trigonometric form of the polynomials $p_j^{(\alpha,\beta)}$. It is similar in each four cases for α, β, so we give the proof only for $\alpha = \beta = \frac{1}{2}$. In this case $\gamma = \delta = \frac{1}{2}$.

Using the notation $x =: \cos\vartheta$, $(\vartheta \in [0, \pi])$ a simple calculation shows that

$$x_\pm = \cos(\vartheta \mp t_M).$$

Now by (2.3) for $j = 0, 1, \ldots, M-1$ we have

$$\begin{gathered}\big(w_{\frac{1}{2},\frac{1}{2}} p_j^{(\frac{1}{2},\frac{1}{2})}\big)(x_+) + \big(w_{\frac{1}{2},\frac{1}{2}} p_j^{(\frac{1}{2},\frac{1}{2})}\big)(x_-) = \\ = \sin(\vartheta - t_M)\cdot\frac{\sin\big[(j+1)(\vartheta - t_M)\big]}{\sin(\vartheta - t_M)} + \sin(\vartheta + t_M)\cdot\frac{\sin\big[(j+1)(\vartheta + t_M)\big]}{\sin(\vartheta + t_M)} = \\ = 2\cos(j+1)t_M \cdot \sin(j+1)\vartheta,\end{gathered}$$

and thus (by (1.3))

$$\begin{aligned}&\frac{1}{2}\Big\{\mathcal{L}_{M,w_{\gamma,\delta}}(f, X_M^{(\alpha,\beta)}, x_+) + \mathcal{L}_{M,w_{\gamma,\delta}}(f, X_M^{(\alpha,\beta)}, x_-)\Big\} = \\ &\quad = \frac{1}{2}\sum_{j=0}^{M-1} 2\cos(j+1)t_M \cdot c_{j,M}(f)\sin(j+1)\vartheta = \\ &\quad = \sin\vartheta \sum_{j=0}^{2M} \varphi\left(\frac{j+1}{2M+2}\right)\cdot c_{j,M}(f)\frac{\sin(j+1)\vartheta}{\sin\vartheta},\end{aligned}$$

where

$$\varphi\left(\frac{j+1}{M+2}\right) = \cos(j+1)t_M = \cos\frac{(j+1)\pi}{2M+2}$$

for $j = 0, 1, \ldots, M-1$, and $\varphi\left(\frac{j+1}{M+2}\right) = 0$, otherwise.

Consequently

$$\frac{1}{2}\Big\{\mathcal{L}_{M,w_{\gamma,\delta}}(f, X_M^{(\alpha,\beta)}, x_+) + \mathcal{L}_{M,w_{\gamma,\delta}}(f, X_M^{(\alpha,\beta)}, x_-)\Big\} =$$
$$= w_{\frac{1}{2},\frac{1}{2}} \cdot S_{2M,M}^{\varphi_G}(f, X_M^{(\frac{1}{2},\frac{1}{2})}, x).$$

(ii) For the Fourier transform of φ_G we have

$$\hat{\varphi}_G(x) = \frac{\sin(x-\pi)/2}{x^2-\pi^2} \qquad (x \in \mathbb{R}^+),$$

so $\hat{\varphi}_G \in L^1(\mathbb{R}^+)$. By Theorem 2.1, we obtain our statement. ■

4.8. Proof of Theorem 3.3

An easy calculation shows that

$$\hat{\varphi}_\kappa(x) = \frac{1}{2(1-\kappa)\pi}\frac{\sin^2(x/2) - \sin^2(1+\kappa)x}{(x/2)^2} \qquad (x \in \mathbb{R}^+),$$

so $\hat{\varphi}_\kappa \in L^1(\mathbb{R}^+)$, and also

$$\varphi_\kappa(t) + \varphi_\kappa(1-t) = 1 \qquad (t \in [0,1]),$$

thus from Theorem 2.1 and 2.2 we obtain the statement. ■

4.9. Proof of Theorem 3.4

(i) The summation function φ_H obviously satisfies the symmetry property of Theorem 2.2, which proves the interpolatory properties (3.1).

For the proof of (3.2) we shall use the following result regarding some values of the derivatives of the functions $w_{\gamma,\delta}p_j^{(\alpha,\beta)}$.

Lemma 4.4. *Let $|\alpha| = |\beta| = \frac{1}{2}$ and (γ,δ) is given by (2.6). Fix the positive integer M. Then for any node $x_{k,M} \in X_M^{(\alpha,\beta)}$ we have*

$$(2M - j + \gamma + \delta)\Big(w_{\gamma,\delta}p_j^{(\alpha,\beta)}\Big)'(x_{k,M}) = (j + \gamma + \delta)\Big(w_{\gamma,\delta}p_{2M-j}^{(\alpha,\beta)}\Big)'(x_{k,M}),$$

where $j = 0, 1, \ldots, M-1$.

Proof. The proof is based on the trigonometric form of the polynomials $p_j^{(\alpha,\beta)} =: p_j$, and is similar in each four cases for α, β, so we give the proof only for $\alpha = \beta = \frac{1}{2}$. In this case $\gamma = \delta = \frac{1}{2}$,

$$X_M^{(\frac{1}{2},\frac{1}{2})} \ni x_{k,M} = \cos\frac{k\pi}{M+1} =: \cos\vartheta_{k,M},$$

and by (2.3)

$$\big(w_{\frac{1}{2},\frac{1}{2}}\, p_j\big)(x) = \sin\big((j+1)\arccos x\big), \qquad (j = 0, 1, \dots, M-1).$$

For an arbitrary $j = 0, 1, \dots, M-1$ and $\vartheta \in [0, \pi]$ we have

$$\Big(w_{\frac{1}{2},\frac{1}{2}}\, p_j\Big)'(\cos\vartheta) = \sqrt{\frac{2}{\pi}}\frac{(j+1)\cdot\cos(j+1)\vartheta}{\sin\vartheta},$$

and since

$$\cos\frac{(j+1)k\pi}{M+1} = \cos\frac{(2M+2-(2M-j+1))k\pi}{M+1} = \cos\frac{(2M-j+1)k\pi}{M+1},$$

thus

$$\Big(w_{\frac{1}{2},\frac{1}{2}}\, p_j\Big)'(x_{k,M}) = \sqrt{\frac{2}{\pi}}\frac{(j+1)\cdot\cos(2M-j+1)\vartheta_{k,M}}{\sin\vartheta_{k,M}}, \qquad (x_{k,M} \in X_M).$$

Observe that the expression on the right side equals to

$$\frac{j+1}{2M-j+1}\cdot\sqrt{\frac{2}{\pi}}\frac{(2M-j+1)\cdot\cos(2M-j+1)\vartheta_{k,M}}{\sin\vartheta_{k,M}} = \\ = \frac{j+1}{2M-j+1}\Big(w_{\frac{1}{2},\frac{1}{2}}\, p_{2M-j}\Big)'(x_{k,M}),$$

proving our statement. ■

Let $\varphi := \varphi_H$. Then we have

$$\Big(w_{\gamma,\delta}S^{\varphi}_{2M,M}(f, X_M^{(\alpha,\beta)}, \cdot)\Big)' = \sum_{j=0}^{2M}\varphi\left(\frac{j+\gamma+\delta}{2M+2\gamma+2\delta}\right)c_{j,M}(f)\cdot(w_{\gamma,\delta}p_j)'.$$

By (4.3), this equals to

$$\sum_{j=0}^{M-1}\Big[\varphi\big(\tfrac{j+\gamma+\delta}{2M+2\gamma+2\delta}\big)(w_{\gamma,\delta}p_j)' - \varphi\big(\tfrac{2M-j+\gamma+\delta}{2M+2\gamma+2\delta}\big)(w_{\gamma,\delta}p_{2M-j})'\Big]\cdot c_{j,M}(f).$$

Using

$$\varphi\left(\frac{j+\gamma+\delta}{2M+2\gamma+2\delta}\right)+\varphi\left(1-\frac{j+\gamma+\delta}{2M+2\gamma+2\delta}\right)=1.$$

and Lemma 4.4 together leads us to

$$\left(w_{\gamma,\delta}S^{\varphi}_{2M,M}(f,X_M,\cdot)\right)'(x_{k,M})=$$
$$=\sum_{j=0}^{M-1}\left[\varphi\left(\frac{j+\gamma+\delta}{2M+2\gamma+2\delta}\right)-\frac{2M-j+\gamma+\delta}{j+\gamma+\delta}\left(1-\varphi\left(\frac{j+\gamma+\delta}{2M+2\gamma+2\delta}\right)\right)\right]\cdot$$
$$\cdot c_{j,M}(f)\big(w_{\gamma,\delta}p_j\big)'(x_{k,M}),$$

which equals to 0 for every $x_{k,M}\in X_M$ if

$$\frac{2M+2\gamma+2\delta}{j+\gamma+\delta}\cdot\varphi\left(\frac{j+\gamma+\delta}{2M+2\gamma+2\delta}\right)-\frac{2M-j+\gamma+\delta}{j+\gamma+\delta}=0$$

for $j=0,1,\dots,M-1$, or in another form,

$$\varphi\left(\frac{j+\gamma+\delta}{2M+2\gamma+2\delta}\right)=1-\frac{j+\gamma+\delta}{2M+2\gamma+2\delta},\qquad (j=0,1,\dots,M-1).$$

Since φ_H satisfies this condition and the interpolatory condition as well, so the proof of (3.2) is complete.

(ii) Since

$$\hat{\varphi}_H(x)=\frac{1}{2\pi}\left(\frac{\sin(x/2)}{x/2}\right)^2\qquad (x\in\mathbb{R}^+)$$

belongs to $L^1(\mathbb{R}^+)$, therefore by Theorem 2.1 we obtain the statement. ∎

References

[1] **Bernstein, S.N.,** On trigonometric interpolation by the method of least squares (in Russian), *Doklady Akademii Nauk SSSR,* **4** (1934), 1–8.

[2] **Bernstein, S.N.,** Sur une classe de formules d'interpolation, *Collected Works*, Vol. II, Moscow, 1954, 146–154. (Russian)

[3] **Bokor, J. and F. Schipp,** Approximate identification in Laguerre and Kautz bases, *Automatica,* **34 (4)** (1998), 463–468.

[4] **Chripkó, Á.,** On the weighted Grünwald–Rogosinski process, *Annales Univ. Sci. Budapest., Sect. Math.*, submitted

[5] **Erdős, P.,** On some convergence properties of the interpolation polynomials, *Ann. of Math.,* **44** (1943), 330–337.

[6] **Grünwald, G.,** On a convergence theorem for the Lagrange interpolation polynomials, *Bull. Amer. Math. Soc.,* **47** (1941), 271–275.

[7] **Horváth, Á.P.,** $\varrho(w)$-normal point systems, *Acta Math. Hungar.,* **85** (1999), 9–27.

[8] **Marcinkiewicz, J.,** Sur l'interpolation I, II, *Studia Math.,* **6** (1936), 1–17 and 67–81.

[9] **Mason, J.C. and D.C. Handscomb,** *Chebyshev Polynomials,* CRC Press LLC, Boca Raton, Florida, 2003.

[10] **Mastroianni, G. and G.V. Milovanović,** *Interpolation Processes: Basic Theory and Applications*, Springer-Verlag, Berlin-Heidelberg, 2008.

[11] **Mastroianni, G. and J. Szabados,** Hermite–Fejér interpolation with Jacobi weights, *Studia Sci. Math. Hungar.,* **48** (2011), 408–420.

[12] **Natanson, G.M. and V.V. Zuk,** *Trigonometric Fourier Series and Approximation Theory* (Russian), Izdat. Leningrad Unta, Leningrad, 1983.

[13] **Schipp, F. and J. Bokor,** L^∞ System Approximation Algorithms Generated by φ Summations, *Automatica*, **33 (11)** (1997), 2019–2024.

[14] **Schipp, F. and F. Weisz,** Multi-dimensional discrete summability, *Acta Sci. Math. (Szeged)*, **75** (2009), 219–231.

[15] **Szabados, J. and P. Vértesi,** *Interpolation of Functions*, World Sci. Publ., Singapore-New Jersey-London-Hong Kong, 1990.

[16] **Szegő, G.,** *Orthogonal Polynomials*, AMS Coll. Publ., Vol. 23, Providence, 1978.

[17] **Szili, L.,** Uniform convergent discrete processes on the roots of four kinds of Chebyshev polynomials, *Annales Univ. Sci. Budapest., Sect. Math.*, **44** (2002), 35–62.

[18] **Szili, L. and P. Vértesi,** Some Erdős-type convergence processes in weighted interpolation, in: *Advanced Problems in Contructive Approximation* (eds. M. D. Buchmann and D. H. Mache), International Series of Numerical Mathematics Vol. 142., Birkhaüser Verlag (Basel, 2002), pp. 237–253.

[19] **Szili, L. and P. Vértesi,** On uniform convergence of sequences of certain linear operators, *Acta Math. Hungar.,* **91 (1–2)** (2001), 159–186.

[20] **Szili, L. and P. Vértesi,** On summability of weighted Lagrange interpolation. I (General weights), *Acta Math. Hungar.,* **101 (4)** (2003), 323–344.

[21] **Szili, L. and P. Vértesi,** On summability of weighted Lagrange interpolation. III (Jacobi weights), *Acta Math. Hungar.,* **104 (1–2)** (2004), 39–62.

[22] **Timan, A.F.,** *Theory of Approximation of Functions of a Real Variable*, Pergamon Press, Oxford-London-New York-Paris, 1963.

[23] **Varma, A.K. and T.M. Mills,** On the summability of Lagrange interpolation, *J. Approx. Theory,* **9** (1973), 349–356.

[24] **Vértesi, P.,** An Erdős-type convergence process in weighted interpolation. I (Freud-type weights), *Acta Math. Hungar.,* **91 (3)** (2001), 195–215.

[25] **Vértesi, P.,** On the Lebesgue function of weighted Lagrange interpolation II, *J. Austral. Math. Soc. (Series A),* **65** (1998), 145–162.

[26] **Vértesi, P.,** Classical (unweighted) and weighted interpolation, in: *A Panorama of Hungarian Mathemetics in the Twentieth Century. I*, Bolyai Society Mathematical Studies **14**, 2005, 71–117.

[27] **Vértesi, P.,** Notes on a paper of G. Grünwald, *Acta Math. Hungar.,* **26** (1975), 191–197.

[28] **Webster, M.S.,** A convergence theorem for certain Lagrange interpolation polynomials, *Bull. Amer. Math. Soc.,* **49** (1943), 114–119.

Zs. Németh
Department of Numerical Analysis
Loránd Eötvös University
Pázmány P. sétány I/C
H–1117 Budapest, Hungary
birka0@gmail.com