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CONSTRUCTION OF 2-ADIC CHEBYSHEV POLYNOMIALS

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Abstract. Several 2-adic cosine and sine functions are constructed on the 2-adic field expressed by the $\tilde{\mathbb{S}}$-valued exponential functions and the characters v_n of the 2-adic additive group. Then follows the construction of some analogies of the Chebyshev polynomials on the 2-adic field $(\mathbb{I}, \overset{\bullet}{+}, \bullet)$ using these cosine and sine functions. Orthogonality of these Chebyshev polynomials is also investigated.

1. Introduction

Chebyshev polynomials are important for example in approximation theory (the resulting interpolation polynomial provides an approximation that is close to the polynomial of best approximation to a continuous function under the maximum norm), and other fields of applications. In classical analysis the Chebyshev polynomials of the first and second kind can be expressed through the identities

$$T_n(x) = \cos(n \arccos x), \quad U_n(x) = \frac{\sin\left[(n+1)\arccos x\right]}{\sin(\arccos x)} \quad (x \in [-1,1],\ n \geq 0),$$

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where the cosine and sine functions can be given by means of the exponential function: $\cos x = \frac{e^{ix}+e^{-ix}}{2}$ and $\sin x = \frac{e^{ix}-e^{-ix}}{2i}$. Each of the Chebyshev polynomials of the first and second kind form an orthogonal system with respect to the weight function $(1-x^2)^{-1/2}$ and $(1-x^2)^{1/2}$, respectively.

In this work we will construct some analogies of the Chebyshev polynomials on the 2-adic field $(\mathbb{I}, \overset{\bullet}{+}, \bullet)$ using several kinds of 2-adic cosine and sine functions. We present two opportunities to construct 2-adic trigonometric functions expressed by the additive characters $(v_n, n \in \mathbb{N})$ or by the $\tilde{\mathbb{S}}$-valued exponential functions, which is in connection with the multiplicative characters. In this way we will obtain first two dyadic martingale structure preserving transformations of $(v_n, n \in \mathbb{N})$, which will yield a UDMD-product system, thus complete and orthonormal. Then follows two further types of Chebyshev polynomials, which will also fulfil orthogonality.

The algebraic structure is presented in details in [5] and [4]. Denote by $\mathbb{A} := \{0,1\}$ the set of bits and by

$$\mathbb{B} := \{a = (a_j, j \in \mathbb{Z}) \mid a_j \in \mathbb{A} \text{ and } \lim_{j\to-\infty} a_j = 0\}$$

the set of bytes. Special bytes are $\theta := (0,0,\cdots)$, $e := (\delta_{n0}, n \in \mathbb{Z})$, and for $k \in \mathbb{Z}$ let $e_k := (\delta_{nk}, n \in \mathbb{Z})$ where δ_{nk} is the Kronecker-symbol. The order of a byte $x \in \mathbb{B} \setminus \{\theta\}$ is $\pi(x) := \min\{n \in \mathbb{N} \mid x_n = 1\}$, and set $\pi(\theta) := +\infty$. The norm of a byte x is defined by $\|x\| := 2^{-\pi(x)}$ for $x \in \mathbb{B} \setminus \{\theta\}$, and $\|\theta\| := 0$. By an interval in $\mathbb{B}$ of rank $n \in \mathbb{Z}$ and center $a \in \mathbb{B}$ we mean a set of the form $I_n(a) = \{x \in \mathbb{B} \mid x_j = a_j \text{ for } j < n\}$. Set $\mathbb{I}_n := I_n(\theta)$ $(n \in \mathbb{Z})$, $\mathbb{I} := \mathbb{I}_0$, and $\mathbb{S} := \{x \in \mathbb{I} \mid x_0 = 1\}$.

The 2-adic field $(\mathbb{B}, \overset{\bullet}{+}, \bullet)$ is given by the following operations. The *2-adic* (*or arithmetical*) *sum* $a \overset{\bullet}{+} b$ of elements $a = (a_n, n \in \mathbb{Z}), b = (b_n, n \in \mathbb{Z}) \in \mathbb{B}$ is defined by $a \overset{\bullet}{+} b := (s_n, n \in \mathbb{Z})$ where the bits $q_n, s_n \in \mathbb{A}$ $(n \in \mathbb{Z})$ are obtained recursively as follows:

$$\begin{aligned} &q_n = s_n = 0 \quad \text{ for } n < m := \min\{\pi(a), \pi(b)\}, \\ &\text{and } \ a_n + b_n + q_{n-1} = 2q_n + s_n \quad \text{ for } n \geq m. \end{aligned}$$

The *2-adic* (*or arithmetical*) *product* of $a, b \in \mathbb{B}$ is $a \bullet b := (p_n, n \in \mathbb{Z})$, where the sequences $q_n \in \mathbb{N}$ and $p_n \in \mathbb{A}$ $(n \in \mathbb{Z})$ are defined recursively by

$$\begin{aligned} &q_n = p_n = 0 \quad (n < m := \pi(a) + \pi(b)) \\ &\text{and } \sum_{j=-\infty}^{\infty} a_j b_{n-j} + q_{n-1} = 2q_n + p_n \quad (n \geq m). \end{aligned}$$

The *reflection* x^- of a byte $x = (x_j, j \in \mathbb{Z})$ is defined by:

$$(x^-)_j := \begin{cases} x_j, & \text{for } j \leqq \pi(x) \\ 1 - x_j, & \text{for } j > \pi(x). \end{cases}$$

We will use the following notation: $a \overset{\bullet}{-} b := a \overset{\bullet}{+} b^-$.

Definition 1. For $x \in \mathbb{I}$ and $n \in \mathbb{N}^*$ define $n \cdot x := \underbrace{x \overset{\bullet}{+} x \overset{\bullet}{+} \ldots \overset{\bullet}{+} x}_{n \text{ times}}$ and let $0 \cdot x := \theta$.

Note, that $2 \cdot x = x \overset{\bullet}{+} x = e_1 \bullet x$ $(x \in \mathbb{I})$ and $2^n \cdot x = e_n \bullet x$ $(x \in \mathbb{I}, n \in \mathbb{N})$. Recall, that multiplication by e_k shifts bytes: $(e_k \bullet x)_l = x_{l-k}$ $(k, l \in \mathbb{Z})$. For $n \in \mathbb{N}$ with dyadic expansion $n = \sum_{j=0}^{\infty} n_j 2^j$ the reversal of n is $\widehat{n} = \sum_{j=0}^{\infty} n_j 2^{-j-1}$. The reversal map is a bijection from $\mathbb{N}$ onto $\mathbb{Q} \cap [0, 1]$ with $\mathbb{Q} := \{p2^m \mid p, m \in \mathbb{Z}\}$. Consider the *Rademacher system* $(r_n, n \in \mathbb{N})$ with $r_n(x) := (-1)^{x_n}$ $(x \in \mathbb{I})$. Consider the Haar-measure μ on the field $(\mathbb{I}, \overset{\bullet}{+}, \bullet)$.

The concept of **UDMD systems** is due to F. Schipp. (See [4] and [5].) Denote with $\mathcal{A}$ the σ-algebra generated by the intervals $I_n(a)$ $(a \in \mathbb{I}, n \in \mathbb{N})$. $\mathbb{I}, \mathcal{A}$, and the restriction of μ on $\mathbb{I}$ gives a probability measure space $(\mathbb{I}, \mathcal{A}, \mu)$. Let $\mathcal{A}_n$ be the sub-σ-algebra of $\mathcal{A}$ generated by the intervals $I_n(a)$ $(a \in \mathbb{I})$. Let $L(\mathcal{A}_n)$ denote the set of $\mathcal{A}_n$-measurable functions on $\mathbb{I}$. The *conditional expectation* of an $f \in L^1(\mathbb{I})$ with respect to $\mathcal{A}_n$ is of the form

$$(\mathcal{E}_n f)(x) := \frac{1}{\mu(I_n(x))} \int_{I_n(x)} f d\mu \quad (x \in \mathbb{I}).$$

A sequence of functions $(f_n, n \in \mathbb{N})$ is called a *dyadic martingale* if each f_n is $\mathcal{A}_n$-measurable and $\mathcal{E}_n f_{n+1} = f_n$ $(n \in \mathbb{N})$. The *sequence of martingale differences* of $(f_n, n \in \mathbb{N})$ is the sequence $\phi_n := f_{n+1} - f_n$ $(n \in \mathbb{N})$. The martingale difference sequence $(\phi_n, n \in \mathbb{N})$ is called a *unitary dyadic martingale difference sequence* or a *UDMD sequence*, if $|\phi_n(x)| = 1$ $(n \in \mathbb{N})$. According to Schipp [4], $(\phi_n, n \in \mathbb{N})$ is a UDMD sequence if and only if

$$\phi_n = r_n g_n, \; g_n \in L(\mathcal{A}_n), \; |g_n| = 1 \quad (n \in \mathbb{N}). \tag{1}$$

A system $\psi = (\psi_m, m \in \mathbb{N})$ is called a UDMD product system if it is a product system generated by a UDMD system, i.e., there is a UDMD system $(\phi_n, n \in \mathbb{N})$ such that for each $m \in \mathbb{N}$, whose binary expansion is given by $m = \sum_{j=0}^{\infty} m_j 2^j$ $(m_j \in \mathbb{A})$, the function ψ_m satisfies $\psi_m = \prod_{j=0}^{\infty} \phi_j^{m_j}$ $(m \in \mathbb{N})$.

We consider $\varepsilon(t) = \exp(2\pi i t)$ $(t \in \mathbb{R})$. The character set of the group $(\mathbb{I}, \overset{\bullet}{+})$ is the product system $(v_m, m \in \mathbb{N})$ generated by the functions

$$v_{2^n}(x) = \varepsilon\left(\frac{x_n}{2} + \frac{x_{n-1}}{2^2} + \cdots + \frac{x_0}{2^{n+1}}\right) \quad (x \in \mathbb{I}, n \in \mathbb{N}),$$

that is, $v_m(x) = \prod_{j=0}^{\infty}(v_{2^j}(x))^{m_j}$ $(m \in \mathbb{N})$. It is well-known, that $(v_n, n \in \mathbb{N})$ is a UDMD-product system on $\mathbb{I}$.

The notion of **DMSP-function**s and some properties of compositions with them were presented by the author in I. Simon[8].

Definition 2. We call a function $B : \mathbb{I} \to \mathbb{I}$ a dyadic martingale structure preserving function or shortly a DMSP-function if it is generated by a system of bijections $(\vartheta_n, n \in \mathbb{N})$, $\vartheta_n : \mathbb{A} \to \mathbb{A}$, and an arbitrary system $(\eta_n, n \in \in \mathbb{N}^*)$, $\eta_n : \mathbb{A}^n \to \mathbb{A}$ in the following way:

$$\begin{aligned} &(B(x))_0 := \vartheta_0(x_0), \\ &(B(x))_n := \vartheta_n(x_n) + \eta_n(x_0, x_1, \ldots, x_{n-1}) \pmod 2 \quad (n \in \mathbb{N}^*). \end{aligned}$$

We will refer to some restrictions of DMSP-functions on dyadic intervals also as DMSP-functions, as they fulfil the same properties. We will use the following properties:

(i) For each bijection system $(\vartheta_n, n \in \mathbb{N})$ and arbitrary system $(\eta_n, n \in \in \mathbb{N}^*)$, the generated DMSP-function B is a bijection on $\mathbb{I}$ and its inverse function, B^{-1} is also a DMSP-function.

(ii) Let $B : \mathbb{I} \to \mathbb{I}$ be a DMSP-transformation. The function system $(f_n, n \in \in \mathbb{N})$ is a UDMD system on $\mathbb{I}$, if and only if $(f_n \circ B, n \in \mathbb{N})$ is a UDMD system on $\mathbb{I}$.

(iii) DMSP-transformations are measure-preserving.

(iv) Let $(B_n : \mathbb{I} \to \mathbb{I}, n \in \mathbb{N})$ be a system of DMSP-transformations. The function system $(f_n, n \in \mathbb{N})$ is a UDMD system on $\mathbb{I}$, if and only if $(f_n \circ B_n, n \in \mathbb{N})$ is a UDMD system on $\mathbb{I}$.

(v) The composition of DMSP-functions is also a DMSP-function.

The first three properties were proved in [8]. (iv) can be shown in the same way as (ii). (v) is trivial.

The $\tilde{\mathbb{S}}$-valued exponential function on $\mathbb{I}$: A 2-adic exponential function is presented in Schipp–Wade [5], pp. 59-60. We will use now a similar one determined by a slightly different base, starting from $b_1 = e \overset{\bullet}{+} e_2$ instead of $e \overset{\bullet}{+} e_1$. We consider first the following base:

Definition 3. Let $b_1 := e \overset{\bullet}{+} e_2, \qquad b_n := b_{n-1} \bullet b_{n-1} \qquad (n \geq 2).$

Definition 4. Let $\tilde{\mathbb{S}} := \{x \in \mathbb{S} \ : \ x_1 = 1\} = I_2(e \overset{\bullet}{+} e_1)$. Define the $\tilde{\mathbb{S}}$-valued exponential function on $\mathbb{I}$ by:

$$\zeta(x) := \prod_{j=1}^{\infty} b_j^{x_{j-1}} \qquad (x = (x_j, j \in \mathbb{N}) \in \mathbb{I}).$$

This function is similar to those defined in Schipp–Wade[5], pp. 59-60, thus with similar arguments we have the following three propositions:
a) ζ is a continuous function satisfying the functional-equation

$$\zeta(x \overset{\bullet}{+} y) = \zeta(x) \bullet \zeta(y) \qquad (x, y \in \mathbb{I}). \tag{2}$$

b) The base has the following structure:

$$b_n = e \overset{\bullet}{+} e_{n+1} \overset{\bullet}{+} d_{n+1} \ (n \geq 1) \text{ with } \pi(d_{n+1}) \geq n + 2. \tag{3}$$

c) With the notations of Definition 1, the function ζ has the following representation:

$$\zeta(x) = \prod_{j=1}^{\infty} (e \overset{\bullet}{+} e_{j+1} \overset{\bullet}{+} d_{j+1})^{x_{j-1}} = \prod_{j=1}^{\infty} \left[e \overset{\bullet}{+} x_{j-1}(e_{j+1} \overset{\bullet}{+} d_{j+1}) \right]. \tag{4}$$

Let us note, that as $b_n = b_1^{2^{n-1}}$ $(n \geq 1)$, we have

$$\zeta(x) = \prod_{j=1}^{\infty} b_1^{2^{j-1} x_{j-1}} = b_1^{\sum_{j=1}^{\infty} x_{j-1} 2^{j-1}} \qquad (x \in \mathbb{I}) \tag{5}$$

which yields $\zeta(x) = b_1^{\beta(x)}$ for $x \in \mathbb{I} \cap \mathbb{B}^+$ and also $\zeta(x) = b_1^{\alpha(\widehat{x})}$ for $x \in \mathbb{I}$, where $\mathbb{B}^+ := \{a \in \mathbb{B} \ | \ \lim_{j \to +\infty} a_j = 0\}$, $\beta(x) := \sum_{j=-\infty}^{\infty} x_j 2^j$ $(x \in \mathbb{B}^+)$ and $\alpha(x) := \sum_{j=-\infty}^{\infty} x_j 2^{-j-1}$ $(x \in \mathbb{B})$. Thus function ζ corresponds to function 5^x while we identify $\mathbb{B}^+ \cap \mathbb{I}$ with $\mathbb{N}$ by means of β or ζ corresponds to function $(\frac{5}{8})^x$ while we identify $\mathbb{I}$ with $[0, 1]$ by means of α.

The structure of this base will be essential, and we will need the first 6 digits of the first four exactly, which can be calculated simply:

$$\begin{aligned}
&b_2 = e \overset{\bullet}{+} e_3 \overset{\bullet}{+} e_4 = e \overset{\bullet}{+} e_3 \overset{\bullet}{+} d_3, \quad \pi(d_3) \geq 4, \\
&b_3 = e \overset{\bullet}{+} e_4 \overset{\bullet}{+} e_5 \overset{\bullet}{+} e_6 \overset{\bullet}{+} e_9 = e \overset{\bullet}{+} e_4 \overset{\bullet}{+} d_4, \quad \pi(d_4) \geq 5, \\
&b_4 = e \overset{\bullet}{+} e_5 \overset{\bullet}{+} e_6 \overset{\bullet}{+} e_7 \overset{\bullet}{+} e_8 \overset{\bullet}{+} \ldots = e \overset{\bullet}{+} e_5 \overset{\bullet}{+} d_5, \ \pi(d_5) \geq 6,
\end{aligned} \tag{6}$$

where $d_3 := e_4, \ d_4 := e_5 \overset{\bullet}{+} e_6 \overset{\bullet}{+} e_9 \ d_5 := e_6 \overset{\bullet}{+} \ldots$

The **reversal** $\hat{t}$ of a byte $t \in \mathbb{B}$ is defined by $\hat{t} := \beta(\alpha^{-1}(t))$ $(t \in \mathbb{Q}^+)$. That is, if the expansion of t is $t = \sum_{j=-\infty}^{\infty} t_j 2^{-j-1}$, then $\hat{t} = \sum_{j=-\infty}^{\infty} t_j 2^j$. This notion will be used in the proof of Theorem 2 and in perceiving the significance of the function system (COS_n) defined in Definition 6.

2. 2-adic sine and cosine functions

In this section we present two ways of constructions of 2-adic trigonometric functions. The first one is expressed by the $\tilde{\mathbb{S}}$-valued exponential functions, which is in connection with the 2-adic multiplicative characters. See [5], pp. 72–73. An other way of the construction is expressed by the additive characters and results a complex-valued function.

Definition 5. Define *the 2-adic cosine and sine function on* $\mathbb{I}$ as follows:

$$\cos x := (\zeta(x) \overset{\bullet}{+} \zeta(x^-)) \bullet e_{-1} \qquad (x \in \mathbb{I}),$$
$$\sin x := (\zeta(x) \overset{\bullet}{-} \zeta(x^-)) \bullet e_{-1} \qquad (x \in \mathbb{I}).$$

Definition 6. To any $n \in \mathbb{N}$ define *the 2-adic COS_n and SIN_n functions on* $\mathbb{I}$ as follows:

$$COS_n(x) := \frac{v_n(x) + v_n(x^-)}{2} \qquad (x \in \mathbb{I}, n \in \mathbb{N}),$$
$$SIN_n(x) := \frac{v_n(x) - v_n(x^-)}{2i} \qquad (x \in \mathbb{I}, n \geq 2).$$

As the reversal map establishes a contact between the discrete exponential system $(e^{inx}, n \in \mathbb{N})$ and the character system $(v_n(x), n \in \mathbb{N})$ $(v_{2^n}(x) = = e^{in\widehat{\beta(x)}})$, we have that the reversal map takes the classical real cosine system $(\cos(nx), n \in \mathbb{N})$ into $(COS_n(x), n \in \mathbb{N})$. Similar statement holds for $(\sin(nx), n \in \mathbb{N})$ and $(SIN_n(x), n \in \mathbb{N})$. Thus these systems can be perceived as the 2-adic discrete cosine and sine systems.

Addition formulas for 2-adic sine and cosine functions are a result of the functional equation (2) of the exponential function, and can be derived as in the real case but resulting slightly different coefficients. We state first that by $x^- = x \bullet e^-$ $(x \in \mathbb{B})$ and by the distributivity of the 2-adic operations we have $(x \overset{\bullet}{+} y)^- = x^- \overset{\bullet}{+} y^-$. Furthermore, $2a := a \overset{\bullet}{+} a = a \bullet e_1$, thus $a = (a \overset{\bullet}{+} a) \bullet e_{-1}$, and $e_{-1} \bullet e_{-1} = e_{-2}$. Now,

$$\begin{aligned}
\cos(x \overset{\bullet}{+} y) &= \Big(\zeta(x \overset{\bullet}{+} y) \overset{\bullet}{+} \zeta(x^- \overset{\bullet}{+} y^-)\Big) \bullet e_{-1} = \\
&= \Big(\zeta(x) \bullet \zeta(y) \overset{\bullet}{+} \zeta(x^-) \bullet \zeta(y^-)\Big) \bullet e_{-1} = \\
&= \Big([\zeta(x) \bullet \zeta(y) \overset{\bullet}{+} \zeta(x^-) \bullet \zeta(y)] \overset{\bullet}{+} [\zeta(x^-) \bullet \zeta(y^-) \overset{\bullet}{+} \zeta(x) \bullet \zeta(y^-)] \overset{\bullet}{+} \\
&\overset{\bullet}{+} [\zeta(x) \bullet \zeta(y) \overset{\bullet}{-} \zeta(x) \bullet \zeta(y^-)] \overset{\bullet}{+} [\zeta(x^-) \bullet \zeta(y^-) \overset{\bullet}{-} \zeta(x^-) \bullet \zeta(y)]\Big) \bullet e_{-2} = \\
&= \cos x \bullet \cos y \overset{\bullet}{+} \sin y \bullet \sin x.
\end{aligned}$$

Similarly, $\sin(x \overset{\bullet}{+} y) = \sin x \bullet \cos y \overset{\bullet}{+} \cos x \bullet \sin y$ $(x, y \in \mathbb{I})$. Clearly, cosine is even and sine is odd, that is, $\cos(x^-) = \cos(x)$, and $\sin(x^-) = (\sin(x))^-$ $(x \in \mathbb{I})$. Thus also holds $\cos(x \overset{\bullet}{-} y) = \cos x \bullet \cos y \overset{\bullet}{-} \sin x \bullet \sin y$, and so, by addition turns out, that

$$\cos(x \overset{\bullet}{+} y) \overset{\bullet}{+} \cos(x \overset{\bullet}{-} y) = \cos x \bullet \cos y \bullet e_1.$$

Thus the 2-adic cosine and sine functions satisfy the so-called d'Alembert equation and sine-cosine functional equation investigated also in Sahoo[2] and Staetker[3].

Evidently, we have

$$\begin{aligned} &\cos 2x = \cos^2 x \overset{\bullet}{+} \sin^2 x, \qquad \sin 2x = \sin x \bullet \cos x \bullet e_1, \\ &e = \cos(\theta) = \cos^2 x \overset{\bullet}{-} \sin^2 x, \\ &\cos u \overset{\bullet}{+} \cos v = \cos\Big([u \overset{\bullet}{+} v] \bullet e_{-1}\Big) \bullet \cos\Big([u \overset{\bullet}{-} v] \bullet e_{-1}\Big) \bullet e_1. \end{aligned}$$

Clearly, COS_n is even and SIN_n is odd, that is $COS_n(x^-) = COS_n(x)$, and $SIN_n(x^-) = -SIN_n(x)$ $(x \in \mathbb{I}, n \in \mathbb{N})$. Addition formulas are in this case also a result of the functional equation $v_n(x \overset{\bullet}{+} y) = v_n(x)v_n(y)$ of the characters:

$$\begin{aligned} &COS_n(x \overset{\bullet}{+} y) = COS_n(x)COS_n(y) - SIN_n(x)SIN_n(y), \\ &COS_n(x \overset{\bullet}{-} y) = COS_n(x)COS_n(y) + SIN_n(x)SIN_n(y), \quad \text{thus} \\ &COS_n(x \overset{\bullet}{+} y) + COS_n(x \overset{\bullet}{-} y) = COS_n(x)COS_n(y) \ (x, y \in \mathbb{I}, n \in \mathbb{N}). \end{aligned}$$

Thus COS_n and SIN_n satisfy the so-called d'Alembert equation and sine-cosine functional equation investigated for example in Sahoo [2] and in Staetker [3]. We have furthermore: $COS_n^2(x) + SIN_n^2(x) = 1$ $(x \in \mathbb{I},\ n \in \mathbb{N})$.

As the inverse function of cos is needed in the chosen construction of Chebyshev polynomials, we determine now a set $\tilde{\mathbb{S}}$, on which cos is bijective. It is not injective on the original domain $\mathbb{I}$, thus we consider its restriction on $\tilde{\mathbb{S}}$ and we determine the range also: $\mathbb{S}^\dagger$.

Notation 1. Recall that $\tilde{\mathbb{S}} := I_2(e \overset{\bullet}{+} e_1) = e \overset{\bullet}{+} e_1 \overset{\bullet}{+} \mathbb{I}_2 = \{x \in \mathbb{S} \ : \ x_1 = 1\}$. Consider the following sets of bytes

$$\begin{aligned} &\mathbb{S}^\natural := I_3(e) = e \overset{\bullet}{+} \mathbb{I}_3 = \{e \overset{\bullet}{+} t \ : \ t \in \mathbb{I}_3\} = \{x \in \mathbb{I} \ : \ x_0 = 1, x_1 = x_2 = 0\ \}, \\ &\mathbb{S}^\dagger := I_6(e \overset{\bullet}{+} e_3 \overset{\bullet}{+} e_5) = \{x \in \mathbb{I} \ : \ x_0 = x_3 = x_5 = 1, x_1 = x_2 = x_4 = 0\ \} \subset \mathbb{S}^\natural, \\ &\tilde{\mathbb{S}_l} := I_{l+2}(e_l \overset{\bullet}{+} e_{l+1}), \quad \mathbb{S}_l = e_l \overset{\bullet}{+} \mathbb{I}_{l+1} = I_{l+1}(e_1) \ (l \in \mathbb{N}). \end{aligned}$$

Theorem 1. a) *The function* cos *takes* $\mathbb{S}$ *to* $\mathbb{S}^{\dagger}$. *Specially,* $\cos : \tilde{\mathbb{S}} \subset \mathbb{S} \to \mathbb{S}^{\dagger}$ *is a bijection.*

b) *The function* cos *takes* $\mathbb{I}$ *to* $\mathbb{S}^{\natural}$.

Proof. a) If $x \in \mathbb{S}$, then $x_0 = (x^-)_0 = 1$ and $(x^-)_j = 1 - x_j$ $(j \geq 1)$. Thus with the notations of (3) and representation (4) we have:

$$\cos(x) = b_1^{x_0} \bullet \left(\prod_{j=2}^{\infty} b_j^{x_{j-1}} \dot{+} \prod_{j=2}^{\infty} b_j^{1-x_{j-1}} \right) \bullet e_{-1} =$$

$$b_1 \bullet e_{-1} \left(\prod_{j=2}^{\infty} \left[e \dot{+} x_{j-1}(e_{j+1} \dot{+} d_{j+1}) \right] \dot{+} \prod_{j=2}^{\infty} \left[e \dot{+} (1 - x_{j-1})(e_{j+1} \dot{+} d_{j+1}) \right] \right).$$

Now, set $z := (b_1)^{-1} \bullet e_1 \bullet \cos(x)$, which is the expression in the huge round brackets. Let us investigate the digits of z: each of the products belongs to $\mathbb{S}$, thus the first terms are $e \dot{+} e = e_1$, and the next possibly nonzero digit is z_3. So, we compute the digits from the 3rd to the 8th using the structure (6) of the base and establishing also the rests q_i determined by the 2-adic sum:

$$z_3 + 2q_3 = x_1 + (1 - x_1) = 1 \;\Rightarrow\; z_3 = 1,\; q_3 = 0$$

$$z_4 + 2q_4 = x_2 + (1 - x_2) + \underbrace{(d_3)_4}_{=1}(x_1 + (1 - x_1)) + q_3 = 2 \;\Rightarrow\; z_4 = 0,\; q_4 = 1$$

$$z_5 + 2q_5 = x_3 + (1 - x_3) + \underbrace{(d_3)_5}_{=0}(x_1 + (1 - x_1)) + \underbrace{(d_4)_5}_{=1}(x_2 + (1 - x_2)) + \underbrace{q_4}_{=1} = 3 \;\Rightarrow\; z_5 = q_5 = 1$$

(7)

$$z_6 + 2q_6 = x_4 + (1 - x_4) + \underbrace{(d_3)_6}_{=0} + \underbrace{(d_4)_6}_{=1} + \underbrace{(d_5)_6}_{=1} + \underbrace{q_5}_{=1} = 4 \;\Rightarrow\; z_6 = 0,\; q_6 = 2$$

$$z_7 + 2q_7 = \underbrace{x_5 + (1 - x_5)}_{\text{always}=1} + \underbrace{[x_1 x_2 + (1 - x_1)(1 - x_2)]}_{\text{depends on } x_1, x_2} \underbrace{(e_3 \bullet e_4)_7}_{=1} + (d_3)_7 + (d_4)_7 + (d_5)_7 + (d_6)_7 + q_6$$

$$z_8 = 1 + \underbrace{[x_1 x_3 + (1 - x_1)(1 - x_3)]}_{\text{depends on } x_1, x_3} + \varphi(x_1, x_2) \; (mod\ 2)$$

$$\vdots$$

$$z_k = 1 + \underbrace{[x_1 x_{k-5} + (1 - x_1)(1 - x_{k-5})]}_{\text{depends on } x_1, x_{k-5}} + \varphi(x_1, x_2, \ldots, x_{k-6}) \; (mod\ 2) \; (k \geq 7).$$

This computation resulted, that the 1st, 3rd and 5th digits of z were equal to 1, and the others were 0 until the 6th digit. Thus

$$\cos(x) = b_1 \bullet e_{-1} \bullet \left(e_1 \overset{\bullet}{+} e_3 \overset{\bullet}{+} e_5 \overset{\bullet}{+} \tilde{d}_6\right) = e \overset{\bullet}{+} e_3 \overset{\bullet}{+} e_5 \overset{\bullet}{+} d'_5$$

with some $\tilde{d}_6 \in \mathbb{I}_7,\ d'_5 \in \mathbb{I}_6$. Thus $y = \cos(x) \in \mathbb{S}^\dagger$ and $\cos : \mathbb{S} \to \mathbb{S}^\dagger$.

Computation (7) also implies, that z_7 can take either 0 or 1 depending on x_1 and x_2, and so do the following digits, too, but depending on further digits of x. Thus setting condition $x_1 = 1$, which is the case for $x \in \tilde{\mathbb{S}}$, the 7th digit of z determines x_2, the 8th one determines x_3, the k-th digit of z determines x_{k-5} $(k \geq 7)$, and by an inductive argument follows the existence of a unique $x \in \tilde{\mathbb{S}}$ with the required property. Thus to any given $y \in \mathbb{S}^\dagger$ there exists an $x \in \tilde{\mathbb{S}}$ uniquely such that $\cos x = y$.

b) When $x \in \mathbb{I} \setminus \mathbb{S}$, then only base elements b_i of higher indexes $(i \geq 2)$ will occur in $\cos(x)$, thus the nonzero coordinates except of the 0th are shifted to the right, so $\cos(x) \in \mathbb{S}$ and $(\cos(x))_1 = (\cos(x))_2 = 0$ holds in each case, thus the image of cos on $\mathbb{I}$ is a subset of $\mathbb{S}^\natural$. ■

Notation 2. Let us denote the inverse of $\cos : \tilde{\mathbb{S}} \to \mathbb{S}^\dagger$ by arccos, which has domain $\mathbb{S}^\dagger$.

We will use the following lemma in the next section.

Lemma 1. *$f(t) := \cos(e_{-4} \bullet t)$ is a DMSP-function on $\tilde{\mathbb{S}_4} = I_6(e_4 \overset{\bullet}{+} e_5)$, and also on $\mathbb{S}_4 \setminus \tilde{\mathbb{S}_4} = I_6(e_4)$.*

Proof. If $t \in \tilde{\mathbb{S}_4}$, than $x = e_{-4} \bullet t \in \tilde{\mathbb{S}}$. Computation (7) implies that if $x \in \tilde{\mathbb{S}}$ we have for $z = (b_1)^{-1} \bullet e_1 \bullet \cos(x)$ recursion form:

$$z_k = x_{k-5} + \varphi(x_2, x_3, \ldots, x_{k-6}) \qquad (\mod 2)\ (k \geq 6)$$

with some $\varphi : \mathbb{A}^{k-7} \to \mathbb{A}$. As $b_1 \in \mathbb{S}$, $b_1 \bullet z \in \mathbb{S}$ has the same type of recursion, furthermore follows for $y = \cos x = e_{-1} \bullet b_1 \bullet z$ the recursion form

$$y_k = x_{k-4} + \varphi(x_2, x_3, \ldots, x_{k-5}) \qquad (\mod 2)\ (k \geq 5) \tag{8}$$

with some $\varphi : \mathbb{A}^{k-6} \to \mathbb{A}$. As multiplying by e_{-4} shifts bytes, we have $x_{k-4} = (t \bullet e_{-4})_{k-4} = t_k$ $(k \in \mathbb{Z})$. Thus by (8) follows that $f(t) = \cos(e_{-4} \bullet t)$ is a DMSP-function on $\tilde{\mathbb{S}_4}$.

Computation (7) also implies that for $x \in \mathbb{S} \setminus \tilde{\mathbb{S}}$ we have $x_1 = 0$ and recursion

$$z_k = 1 - x_{k-5} + \varphi(x_2, x_3, \ldots, x_{k-6}) \qquad (\mod 2)\ (k \geq 6)$$

with some $\varphi : \mathbb{A}^{k-7} \to \mathbb{A}$. Thus similarly follows in this case also that $f(t) = \cos(e_{-4} \bullet t)$ is also a DMSP-function on $\mathbb{S}_4 \setminus \tilde{\mathbb{S}_4}$. ■

Remark. Similar investigation shows, that $\sin : \mathbb{S} \to I_3(e+e_2)$ is a bijection, and a simple recursion yields the digits of $y = e_{-2} \bullet \sin x$, thus $x \mapsto e_{-2} \bullet \sin(x)$ is a DMSP-function on $\mathbb{S}$.

Theorem 2. *The systems* $(COS_0, COS_1, \sqrt{2}COS_2, \sqrt{2}COS_3, \ldots)$ *and* $(\sqrt{2}SIN_n, n \geq 2)$ *are orthonormal systems.*

Proof. Let $n, m \in \mathbb{N}$.

$$\int_{\mathbb{I}} COS_n(x)\overline{COS_m(x)}d\mu(x) = \frac{1}{4}\int_{\mathbb{I}} v_n(x)\overline{v_m(x)}d\mu(x) +$$
$$+\frac{1}{4}\int_{\mathbb{I}} v_n(x)\overline{v_m(x^-)}d\mu(x) +$$
$$+\frac{1}{4}\int_{\mathbb{I}} v_n(x^-)\overline{v_m(x)}d\mu(x) + \frac{1}{4}\int_{\mathbb{I}} v_n(x^-)\overline{v_m(x^-)}d\mu(x) =:$$
$$=: \frac{1}{4}(I_1 + I_2 + I_3 + I_4).$$

Since $x \mapsto x^-$ is measure-preserving, $I_4 = I_1$. As the system $(v_n, n \in \mathbb{N})$ is orthonormal, we have $I_4 = I_1 = \delta_{n,m}$.

$v_n(x)\overline{v_m(x^-)} = v_n(x)v_m(x) = v_{n \overset{\bullet}{+} m}(x)$ where operation $\overset{\bullet}{+}$ on $\mathbb{N}$ is defined in the following way: $n \overset{\bullet}{+} m := \alpha(\alpha^{-1}(n) \overset{\bullet}{+} \alpha^{-1}(m)) = (\widehat{n} + \widehat{m} \ P \mod 2)\widehat{}$. Now, $n \overset{\bullet}{+} m = 0 \Leftrightarrow n = m = 0$ or $n = m = 1$. Thus, $I_2 = \int_{\mathbb{I}} v_n(x)\overline{v_m(x^-)}d\mu(x) = \int_{\mathbb{I}} v_{n \overset{\bullet}{+} m}(x)d\mu(x) = \delta_{mn}(\delta_{n0} + \delta_{n1})$.

In case of $(n,m) \in \{(0,0),(1,1)\}$ we make use of the definition $\mu(\mathbb{I}) = 1$, which implies $I_2 = I_3 = 1$, thus $\int_{\mathbb{I}} COS_n(x)\overline{COS_m(x)} = 1$. Otherwise $I_2 = I_3 = 0$ and so $\int_{\mathbb{I}} COS_n(x)\overline{COS_m(x)} = \frac{1}{2}\delta_{mn}$.

As $\int_{\mathbb{I}} SIN_n(x)\overline{SIN_m(x)}d\mu(x) = \frac{1}{4}(I_1 - I_2 - I_3 + I_4)$, the statement for $(SIN_n, n \in \mathbb{N} \setminus \{0,1\})$ follows similarly. ■

3. The 2-adic Chebyshev polynomials

It seems at first sight to have exaggerated in the next two definitions by using k twice in t_k, but the first one ensures that the system will be a UDMD-product system, and the second one belongs to the nature of Chebyshev polynomials.

Definition 7. Define *the 2-adic Chebyshev polynomials of the first kind* as the product system of $t_k(x) := v_{2^{k+6}}\left(\cos[(2k+1)\arccos(x)]\right)$ $(x \in \mathbb{S}^\dagger, k \in \mathbb{N})$, that is,

$$(9) \qquad T_n(x) := \prod_{k=0}^{\infty} \left[v_{2^{k+6}}\left(\cos[(2k+1)\arccos(x)]\right)\right]^{n_k} \qquad (x \in \mathbb{S}^\dagger, n \in \mathbb{N}).$$

Definition 8. Let us define *the 2-adic Chebyshev polynomials of the second kind* as the product system of $u_k(x) := v_{2^{k+3}}\left(\sin[(2k+1)\arccos x]\right)$ $(x \in \mathbb{S}^\dagger, k \in \in \mathbb{N})$, that is

$$(10) \qquad U_n(x) := \prod_{k=0}^{\infty} \left[v_{2^{k+3}}\left(\sin[(2k+1)\arccos(x)]\right)\right]^{n_k} \qquad (x \in \mathbb{S}^\dagger, n \in \mathbb{N}).$$

In order to see the orthogonality, we need first to examine the functions $x \mapsto \cos((2n+1)\arccos x)$ and $x \mapsto \sin((2n+1)\arccos x)$ $(x \in \mathbb{S}^\dagger)$.

Lemma 2. *The functions* $x \mapsto \cos((2n+1)\arccos x)$ $(x \in \mathbb{S}^\dagger)$ *and* $x \mapsto e_3 \bullet \bullet \sin((2n+1)\arccos x)$ $(x \in \mathbb{S}^\dagger, n \in \mathbb{N})$ *are DMSP-functions on* $\mathbb{S}^\dagger$*.*

Proof. The first function is obtained by a composition of functions

$$f_1(x) := e_4 \bullet \arccos(x), \qquad f_1 : \mathbb{S}^\dagger \to \tilde{\mathbb{S}_4}$$
$$f_2(x) := (2n+1) \cdot x = \underbrace{x \overset{\bullet}{+} x \overset{\bullet}{+} \ldots \overset{\bullet}{+} x}_{2n+1 \text{ times}}, \qquad f_2 : \tilde{\mathbb{S}_4} \to \mathbb{S}_4$$
$$f_3(x) := \cos(x \bullet e_{-4}), \qquad f_3 : \mathbb{S}_4 \to \mathbb{S}^\dagger.$$

The distributivity implies that $(2n+1) \cdot (e_4 \bullet y) = e_4 \bullet [(2n+1) \cdot y]$ $(y \in \mathbb{B})$, thus $(f_3 \circ f_2 \circ f_1)(x) = \cos((2n+1)\arccos x)$ $(x \in \mathbb{S}^\dagger)$.

We have already seen in Lemma 1, that f_3 is a DMSP-function on $\tilde{\mathbb{S}_4}$ and on $\mathbb{S}_4 \setminus \tilde{\mathbb{S}_4}$, too. Thus property i) of DMSP-functions results that f_1 is also a DMSP-function on $\mathbb{S}^\dagger$.

Let us examine f_2. With the dyadic expansion $n = \sum_{i=0}^{\infty} n_i 2^i$ we have $n \cdot x = \sum_{i=0}^{\infty} n_i(2^i \cdot x) = \sum_{i=0}^{\infty} n_i(e_i \bullet x)$, where the sum is taken in sense $\overset{\bullet}{+}$. Thus $(n \cdot x)_k = \sum_{i=0}^{k} n_i x_{k-i}$ $(k \in \mathbb{N},\ x \in \mathbb{I})$, which contains x_k if and only if $n_0 = 1$, that is, if n is odd. Thus $f_2(x) = (2n+1) \cdot x$ is a DMSP-function on $\tilde{\mathbb{S}_4}$. Given $n \in \mathbb{N}$ the range of f_2 is either $\tilde{\mathbb{S}_4}$ or $\mathbb{S}_4 \setminus \tilde{\mathbb{S}_4}$ depending on $n_1 \in \mathbb{A}$.

Property v) of DMSP-functions implies that $f_3 \circ f_2 \circ f_1$ is a DMSP-function on $\mathbb{S}^\dagger$. ■

Theorem 3. *The 2-adic Chebyshev polynomials of the first and second kind* $(T_n, n \in \mathbb{N})$ *and* $(U_n, n \in \mathbb{N})$ *form UDMD product systems, thus they are complete and orthonormal systems.*

Proof. As for each $c \in \mathbb{I}$ the system $(v_{2^{k+6}}, k \in \mathbb{N})$ is a UDMD-system on $\mathbb{I}_6(c)$, we have by property iii) of DMSP-transformations and Lemma 3 that $(t_n, n \in \mathbb{N})$ is a UDMD-system on $\mathbb{S}^\dagger$, which results that $(T_n, n \in \mathbb{N})$ is a UDMD-product system on $\mathbb{S}^\dagger$, thus complete and orthonormal. (See Schipp-Wade[5], pp. 92-94.) The proof is similar for the second kind Chebyshev polynomials. ■

Corollary 1. *Fourier series of any* $f \in L^p(\mathbb{I})$ $(p > 1)$ *with respect to systems* $(T_n, n \in \mathbb{N})$ *and* $(U_n, n \in \mathbb{N})$ *converges a.e. to* f*.*

This is a consequence of Theorem 4 in Schipp [6] stated in general for any UDMD-product systems.

Corollary 2. $(C, 1)$*-summability of any* $f \in L^1(\mathbb{I})$ *with respect to to systems* $(T_n, n \in \mathbb{N})$ *and* $(U_n, n \in \mathbb{N})$ *holds.*

This is a consequence of Theorem 15 in Gát[1] stated for Vilenkin-like systems, a generalization of UDMD-product systems.

Remarks: 1) Theorem 3 remains valid if we use any proper UDMD-systems instead of $v_{2^{k+6}}$ and $v_{2^{k+3}}$ $(k \in \mathbb{N})$.

2) The 2-adic Chebyshev polynomials of the first and second kind can be defined also on $\mathbb{I}$ by establishing a proper shift operation: $S : \mathbb{I} \to \mathbb{S}^\dagger = I_6(e \overset{\bullet}{+}$ $\overset{\bullet}{+} e_3 \overset{\bullet}{+} e_5)$, $S(x) := x \bullet e_6 \overset{\bullet}{+} e \overset{\bullet}{+} e_3 \overset{\bullet}{+} e_5$. Now,

$$\widetilde{T_n}(x) := \prod_{k=0}^{\infty} \left[v_{2^{k+6}} \left(\cos[(2k+1)\arccos(S(x))]\right)\right]^{n_k} \qquad (x \in \mathbb{I}, n \in \mathbb{N}),$$

$$\widetilde{U_n}(x) := \prod_{k=0}^{\infty} \left[v_{2^{k+3}} \left(\sin[(2k+1)\arccos(S(x))]\right)\right]^{n_k} \qquad (x \in \mathbb{I}, n \in \mathbb{N}).$$

Notation 3. Consider shift operations:

$$S : \mathbb{I} \to \mathbb{S}^\dagger, \ S(x) := x \bullet e_6 \overset{\bullet}{+} e \overset{\bullet}{+} e_3 \overset{\bullet}{+} e_5,$$

$$S' : \tilde{\mathbb{S}} \to \mathbb{I}, \ S'(x) := [x \overset{\bullet}{-} e \overset{\bullet}{-} e_1] \bullet e_{-2}.$$

Definition 9. Define *the 2-adic Chebyshev polynomials of the third and fourth kind* by

$$\begin{aligned} &\overline{T_n}(x) := COS_n[S'(\arccos(S(x)))] \qquad (x \in \mathbb{I}, n \in \mathbb{N}), \\ &\overline{U_n}(x) := SIN_n[S'(\arccos(S(x)))] \qquad (x \in \mathbb{I}, n \geq 2). \end{aligned} \tag{11}$$

Theorem 4. *The 2-adic Chebyshev polynomials of the third and fourth kind* $(\overline{T_n}, n \in \mathbb{N}), (\overline{U_n}, n \in \mathbb{N})$ *are orthogonal systems in* $L^2(\mathbb{I})$*.*

Proof. The variable transformation $B : x \mapsto S'(\arccos(S(x)))$ is a DMSP-transformation on $\mathbb{I}$, thus it is measure-preserving. Hence,

$$\int_{\mathbb{I}} f \circ B \, d\mu = \int_{\mathbb{I}} f d\mu \qquad (f \in L^1(\mathbb{I})). \tag{12}$$

Let $n, m \in \mathbb{N}^*$. By (12) and by the orthogonality of the systems $(COS_n, n \in \mathbb{N})$, $(SIN_n, n \in \mathbb{N})$ follows the statement:

$$\int_{\mathbb{I}} \overline{T_n}(x)\overline{T_m}(x) d\mu(x) = \int_{\mathbb{I}} COS_n(y) COS_m(y) d\mu(y) = 0 \ (n \neq m).$$

■

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