Annales Univ. Sci. Budapest., Sect. Comp. 43 (2014) 21–38

ARBITRATION SOLUTIONS TO BARGAINING AND GAME THEORY PROBLEMS

Peter Z. Revesz (Lincoln, Nebraska, USA)

Dedicated to András Benczúr on the occasion of his 70th birthday

Communicated by Attila Kiss

(Received May 31, 2014; accepted July 1, 2014)

Abstract. Decision theory includes arbitration, bargaining and game theory, whose relationships to each other are not well-understood. This paper introduces a particular weighted arbitration operator and applies it to King Solomon's dilemma and Nash's original bargaining problem. These applications allow a more meaningful comparison of the three theories and show that arbitration provides a more natural, simpler and flexible solution than the previously proposed bargaining and game theory solutions.

1. Introduction

Axiomatic arbitration theory originated when the late Prof. Alberto Mendelzon (1951–2005) introduced me to the axiomatic theory of belief revision [1] and database update [8, 5], while I was a postdoctoral student at the University of Toronto. In the otherwise beautiful descriptions he presented, it was disappointing to learn that the axioms of belief revision and database update operators both assume that the new information is more accurate than the old information. Realizing that information often needs to be combined from sources that have equivalent reliability, I developed an axiomatic description of

Key words and phrases: Arbitration, bargaining, game theory, Nash bargaining.
2010 Mathematics Subject Classification: 62Cxx,, 91Axx, 91Bxx, 91B10, 91B14, 91B26

operators, named arbitration operators, that treat the old and the new information in a more symmetric way [17, 18]. Some arbitration operators, often called consensus, combination, fusion, merging etc. operators, were already proposed by other researchers, but nobody gave before an axiomatic description of arbitration similar to the axiomatic descriptions of belief revision and database updates.

Prof. András Benczur, to whom this paper is dedicated on his seventieth birthday, and his students at ELTE University were early enthusiasts of arbitration theory. It is a pleasure to think back to my first visit to ELTE University in 1994 at the invitation of Prof. Benczur. A fruitful collaboration ensued that extended arbitration theory in new directions, including important cases of weighted arbitration [2, 3, 4]. This paper develops further the theme of arbitration by introducing a weight function for atomic propositions and giving based on that weight function a new arbitration operator in Section 3.3.

Arbitration theory belongs to the rich nexus of decision theory, which also includes game theory and bargaining theory. The exact relationship among these axiomatic theories is hard to understand not only because these theories have numerous variants but also because theories usually allow many particular operators that satisfy their axioms. However, the particular weighted arbitration operator introduced in this paper can be applied to some problems already considered in game theory and bargaining theory. The new arbitration application examples, including King Solomon's dilemma and Nash's original bargaining problem, allows a meaningful comparison of the three theories in practice. At least in the given examples, arbitration provides a more natural, simpler and flexible solution than either game theory or bargaining theory does.

This paper is organized as follows. Section 2 presents some basic definitions. Section 3 reviews propositional arbitration, which is also called model-fitting. Section 4 compares arbitration and game theory solutions to the problem of King Solomon's dilemma. Section 5 compares arbitration and bargaining solutions to a problem of bargaining presented in Nash [13]. Finally, Section 6 presents some conclusions and further work.

2. Basic definitions

Let $\mathcal{T}$ be a finite set of propositional *terms*. We build propositional *formulas* from terms using the unary connective $\neg$ denoting *boolean negation*, and the binary connectives $\wedge$ and $\vee$ denoting *boolean and* and *boolean or*. We call each $I \subseteq \mathcal{T}$ an interpretation. Let $\mathcal{M}$ be the set of interpretations $\{I : I \subseteq \mathcal{T}\}$. The set of *models* of a formula ϕ denoted by $Mod(\phi)$ is defined as follows:

$$\begin{array}{rcl} Mod(t) & = & \{I \in \mathcal{M} : t \in I\}, \\ Mod(\neg\phi_1) & = & \mathcal{M} \setminus Mod(\phi_1), \\ Mod(\phi_1 \vee \phi_2) & = & Mod(\phi_1) \cup Mod(\phi_2), \\ Mod(\phi_1 \wedge \phi_2) & = & Mod(\phi_1) \cap Mod(\phi_2). \end{array}$$

In this paper we will use the expression $form(I_1, \ldots, I_k)$ to denote the formula that has exactly the models $I_1, \ldots, I_k$.

A knowledge base K is a set of formulas $\{\psi_1, \ldots, \psi_n\}$. A theory is a deductively closed set of formulas. If we have a consequence relation *cn* and K is any knowledge base, then $cn(K)$ is a theory. Let $\perp$ denote falsity, that is the formula with no models. We say that a theory T is consistent if and only if $\perp \notin T$. A knowledge base K is consistent if and only if the theory $cn(K)$ is consistent.

We say that a knowledge base K is satisfiable (or consistent) if and only if the conjunction of all propositional formulas in K is satisfiable. The set $Mod(K)$ is the set of models of the conjunction of all the propositional formulas in K. We say that $K \wedge \mu$ is satisfiable if and only if there is an interpretation that satisfies all the formulas in K and also satisfies μ.

A *pre-order* $\leq$ over $\mathcal{M}$ is a reflexive and transitive relation on $\mathcal{M}$. A pre-order is *total* if for every pair $I, J \in \mathcal{M}$, either $I \leq J$ or $J \leq I$ holds. We define the relation $<$ as $I < J$ if and only if $I \leq J$ and $J \not\leq I$.

The set of *minimal models* of a subset $\mathcal{S}$ of $\mathcal{M}$ with respect to a pre-order $\leq$ is defined as

$$Min(\mathcal{S}, \leq) = \{I \in \mathcal{S} : \nexists I' \in \mathcal{S} \text{ where } I' < I\}.$$

2.1. Weighted knowledge bases

A *weighted knowledge base* is a function $\tilde{K}$ from model sets to nonnegative real numbers. We denote the weight of a model set M in weighted knowledge base $\tilde{K}$ as $\tilde{K}(M)$. The real numbers are intended to describe the relative degree of importance of the model sets within the weighted knowledge base.

If $\tilde{K_1}$ and $\tilde{K_2}$ are two weighted knowledge bases, we take their weighted union, denoted as $\tilde{K_1} \uplus \tilde{K_2}$, to be the weighted knowledge base $\tilde{K}$ such that for each model set M, $\tilde{K}(M) = \tilde{K_1}(M) + \tilde{K_2}(M)$.

Let the function *Form* map each model set S to a propositional formula that has S as its set of models. Further, if $S_1, \ldots S_n$ are model sets with nonzero weights in a weighted knowledge base $\tilde{K}$, then let $Form(\tilde{K})$ denote the conjunction of the set of formulas $Form(S_i)$ for each $1 \leq i \leq n$.

We say that a weighted knowledge base $\tilde{K}$ is satisfiable if and only if the intersection of all model sets with nonzero weights in $\tilde{K}$ is nonempty. A weighted knowledge base is unsatisfiable if and only if it is not satisfiable. We say that an interpretation I is a model of a weighted knowledge base $\tilde{K}$, written as $I \in Mod(\tilde{K})$, if and only if I is an element of each model set with nonzero weight in $\tilde{K}$.

3. Arbitration

Several sets of axioms for different cases of arbitration were presented in the original paper [17] and its journal version [18], which both considered regular and weighted knowledge bases. Below we present the axioms of arbitration in only one case, which is called *weighted model-fitting*.

Intuitively, a weighted model fitting operator $\triangleright$ takes as input a weighted knowledge base $\tilde{K}$ and an integrity constraint μ and returns those models of μ that are overall closest to $\tilde{K}$. If there is no integrity constraint, then μ is by default the set of all possible models. We make the notion of overall closest more precise after presenting the axioms.

We say that a knowledge base operator $\triangleright$ is a weighted model-fitting operator if and only if it satisfies the following axioms for all weighted propositional knowledge bases $\tilde{K}, \tilde{K_1}, \tilde{K_2}$ and propositional formulas μ and ϕ:

(W1) $\tilde{K} \triangleright \mu$ implies μ.

(W2) If $Form(\tilde{K}) \wedge \mu$ is satisfiable then $\tilde{K} \triangleright \mu \leftrightarrow Form(\tilde{K}) \wedge \mu$.

(W3) If μ is satisfiable then $\tilde{K} \triangleright \mu$ is also satisfiable.

(W4) If $\mu \leftrightarrow \phi$ then $\tilde{K} \triangleright \mu \leftrightarrow \tilde{K} \triangleright \phi$.

(W5) $(\tilde{K} \triangleright \mu) \wedge \phi$ implies $\tilde{K} \triangleright (\mu \wedge \phi)$.

(W6) If $(\tilde{K} \triangleright \mu) \wedge \phi$ is satisfiable then $\tilde{K} \triangleright (\mu \wedge \phi)$ implies $(\tilde{K} \triangleright \mu) \wedge \phi$.

(W7) $(\tilde{K_1} \triangleright \mu) \wedge (\tilde{K_2} \triangleright \mu)$ implies $(\tilde{K_1} \uplus \tilde{K_2}) \triangleright \mu$.

(W8) If $(\tilde{K_1} \triangleright \mu) \wedge (\tilde{K_2} \triangleright \mu)$ is satisfiable then $(\tilde{K_1} \uplus \tilde{K_2}) \triangleright \mu$ implies $(\tilde{K_1} \triangleright \mu) \wedge (\tilde{K_2} \triangleright \mu)$.

3.1. Restriction to multisets

A *multiset knowledge base* $K = \{\psi_1, \ldots, \psi_n\}$ is a multiset of propositional formulas. Note that the propositional formulas of K can be logically equivalent, that is, $Mod(\psi_i) = Mod(\psi_j)$ may be true for some $1 \leq i, j \leq n$. If K_1 and K_2 are multiset knowledge bases, then $K_1 \leftrightarrow K_2$ means that there is a bijection of propositional formulas in K_1 and K_2 that maps logically equivalent formulas to each other. Let $\sqcup$ be the multiset union operator.

A *natural weighted knowledge base* is a weighted knowledge base with only non-negative integer weights.

Proposition 3.1. *Every natural weighted knowledge base $\tilde{K}$ is equivalent to a multiset knowledge base K such that for each model set M the following holds*

$$\tilde{K}(M) \;=\; \sum_{\psi_i \in K} (Mod(\psi_i) \equiv M).$$

We define the function *Multiset* from natural weighted knowledge bases to multiset knowledge bases such that if $\mathit{Multiset}(\tilde{K}) = K$, then $\tilde{K}$ and K are equivalent as described in Proposition 3.1. Similarly, we define the function *Weigh* from multiset knowledge bases to natural weighted knowledge bases such that if $\mathit{Weigh}(K) = \tilde{K}$, then $\tilde{K}$ and K are equivalent as described in Proposition 3.1.

Proposition 3.2. *For each pair of natural weighted knowledge bases $\tilde{K_1}$ and $\tilde{K_2}$ and multiset knowledge bases $K_1 = \mathit{Multiset}(\tilde{K_1})$ and $K_2 = \mathit{Multiset}(\tilde{K_2})$ the following holds*

$$\tilde{K_1} \uplus \tilde{K_2} = K_1 \sqcup K_2.$$

Konieczny and Pino Pérez [9] say that for each propositional formula μ a knowledge base operator $\triangle_\mu$ is an *integrity constraint merging operator* if and only if it satisfies the following axioms for all multiset knowledge bases K, K_1, K_2 and propositional formulas ϕ, ϕ':

(IC0) $\triangle_\mu(K)$ implies μ.

(IC1) If μ is satisfiable then $\triangle_\mu(K)$ is also satisfiable.

(IC2) If $\bigwedge K \wedge \mu$ is satisfiable then $\triangle_\mu(K) \leftrightarrow \bigwedge K \wedge \mu$.

(IC3) If $K_1 \leftrightarrow K_2$ and $\mu \leftrightarrow \phi$ then $\triangle_\mu(K_1) \leftrightarrow \triangle_\phi(K_2)$.

(IC4) If ϕ implies μ and ϕ' implies μ, then
if $\triangle_\mu(\phi \sqcup \phi') \wedge \phi$ is satisfiable, then $\triangle_\mu(\phi \sqcup \phi') \wedge \phi'$ is satisfiable.

(IC5) $\triangle_\mu(K_1) \wedge \triangle_\mu(K_2)$ implies $\triangle_\mu(K_1 \sqcup K_2)$.

(IC6) If $\triangle_\mu(K_1) \wedge \triangle_\mu(K_2)$ is satisfiable then $\triangle_\mu(K_1 \sqcup K_2)$ implies $\triangle_\mu(K_1) \wedge \triangle_\mu(K_2)$.

(IC7) $\triangle_\mu(K) \wedge \phi$ implies $\triangle_{\mu \wedge \phi}(K)$.

(IC8) If $\triangle_\mu(K) \wedge \phi$ is satisfiable then $\triangle_{\mu \wedge \phi}(K)$ implies $\triangle_\mu(K) \wedge \phi$.

The next theorem shows that integrity constraint merging operators are a subclass of weighted model-fitting operators.

Theorem 3.3. *Let μ be any propositional formula. Then every integrity constraint merging operator $\triangle_\mu$ is equivalent to a weighted model-fitting operator $\triangleright$ such that the following holds for any multiset knowledge base K*

$$\triangle_\mu(K) \equiv \mathit{Weigh}(K) \triangleright \mu.$$

Proof. The weighted model-fitting axioms can be mapped into the integrity constraint merging axioms as follows: (W1) to (IC0), (W2) to (IC2), (W3) to (IC1), (W4) to (IC3), (W5) to (IC7), (W6) to (IC8), (W7) to (IC5), and (W8) to (IC6). ■

Remark 3.1. (IC4) is the only axiom that is not derived from (W1-W8). Hence any integrity constraint merging operator is a weighted model-fitting operator restricted to natural weighted knowledge bases and satisfying (IC4).

Lin and Mendelzon [11] say that a particular operator, denoted here as $\circ$, is a *majority merging operator* if for any multiset knowledge base $K = \{\psi_1, \ldots, \psi_n\}$ and literal, or atomic proposition or its negation, l, it satisfies the following axioms.

(MM1) $\circ(K)$ is satisfiable.

(MM2) If $\bigwedge K$ is satisfiable, then $\circ(K) \leftrightarrow \bigwedge K$.

(MM3) If $K_1 \leftrightarrow K_2$, then $\circ(K_1) \leftrightarrow \circ(K_2)$.

(MM4) $\forall l$ if $|\psi_i \models l| > |\psi_i \models \neg l| + |\psi_i \approx l|$, then $\circ(K) \models l$.

where $||$ means cardinality and $\psi_i \approx l$ means *partially satisfies*, that is, ψ is consistent and $\psi_1 \not\models l$ and $\psi_1 \not\models \neg l$.

Remark 3.2. Lin and Mendelzon gave only one example majority merging operator, denoted $\triangle_\mu^{\Sigma}$, which is the same as one of the example operators of Revesz [17]. However, there are other operators that satisfy axioms (MM1-MM4).

Example 3.4. If $\bigwedge K$ is consistent, then $\circ K = \bigwedge K$ as required by axiom (MM2). Identify all the literals l that satisfy the condition in axiom (MM4). These literals are restricted because $\circ(K) \models l$ must hold. Let us call *unrestricted* those atoms a which do not satisfy the condition of axiom (MM4). For unrestricted atoms, neither a nor $\neg a$ must occur in the models of $\circ(K)$. Now consider a majority merging operator that for all unrestricted atoms a makes $\circ(K) \models a$ if the number of propositional formulas is odd and $\circ(K) \models \neg a$ if the number of propositional formulas is even.

Let knowledgebases $K_1 = K_2$ each contain the following three propositional formulas: $\psi_1 = (a \wedge \neg b \wedge \neg c) \vee (\neg a \wedge b \wedge \neg c) \vee (\neg a \wedge \neg b \wedge c)$, $\psi_2 = a \wedge b \wedge c$ and $\psi = a \wedge \neg b \wedge c$. It can be calculated that in K_1 and in K_2 no literal satisfies the condition of axiom (MM4). Hence the $\circ(K_1) = \circ(K_2) = a \wedge b \wedge c$.

Let knowledgebase $K = K_1 \sqcup K_2$. Clearly, in K no literal satisfies the condition of axiom (MM4). Since K has an even number of propositional formulas, $\circ(K) = \neg a \wedge \neg b \wedge \neg c$. This shows that this new majority merging operator does not satify conditions (W7) and (W8).

Theorem 3.5. $\triangle_\mu^{Max}$ *does not satisfy (W8), (IC6).*

Proof. Let $\psi_1 = a \wedge \neg b \wedge \neg c$, $\psi_2 = \neg a \wedge b \wedge \neg c$, $K_1 = \{\psi_1\}$, $K_2 = \{\psi_2\}$ and $\mu = \psi_2 \vee (\neg a \wedge \neg b \wedge c)$. Then $Mod(\psi_1) = \{\{a\}\}$, $Mod(\psi_2) = \{\{b\}\}$ and $Mod(\mu) = \{\{b\}, \{c\}\}$. Hence $\triangle_\mu^{Max}(K_1) = \mu$ and $\triangle_\mu^{Max}(K_2) = \psi_2$ and $\triangle_\mu^{Max}(K_1 \sqcup K_2) = \mu$. Hence $\triangle_\mu^{Max}(K_1) \wedge \triangle_\mu^{Max}(K_2) = \psi_2$, which is satisfiable, but $\triangle_\mu^{Max}(K_1 \sqcup K_2) = \mu$ does not imply ψ. ■

3.2. Restriction to unweighted knowledge bases

An *unweighted knowledge base*, denoted K, is a weighted knowledge base $\tilde{K}$ where the weight of each model set is either zero or one. Since each model set M can be mapped into some propositional formula ψ such that $Mod(\psi) = M$, an unweighted knowledge base K is equivalent to a set of propositional formulas $\{\psi_1, \ldots, \psi_n\}$. Therefore, unweighted knowledge bases form a further restriction from multisets to sets of propositional formulas. Arbitration in the case of unweighted knowledge bases is also called *model fitting* and is defined as follows [17, 18].

Definition 3.1. *We say that a knowledge base change operator $\triangleright$ is a model fitting operator if and only if $\triangleright$ satisfies the following axioms for each (not necessarily consistent) propositional knowledge base K, and formulas μ and ϕ:*

(M1) *$K \triangleright \mu$ implies μ.*

(M2) *If $K \wedge \mu$ is satisfiable then $K \triangleright \mu \leftrightarrow K \wedge \mu$.*

(M3) *If μ is satisfiable then $K \triangleright \mu$ is also satisfiable.*

(M4) *If $K_1 \leftrightarrow K_2$ and $\mu \leftrightarrow \phi$ then $K_1 \triangleright \mu \leftrightarrow K_2 \triangleright \phi$.*

(M5) $(K \triangleright \mu) \wedge \phi$ *implies* $K \triangleright(\mu \wedge \phi)$.

(M6) *If $(K \triangleright \mu) \wedge \phi$ is satisfiable then $K \triangleright(\mu \wedge \phi)$ implies $(K \triangleright \mu) \wedge \phi$.*

(M7) $(K_1 \triangleright \mu) \wedge (K_2 \triangleright \mu)$ *implies* $(K_1 \cup K_2) \triangleright \mu$.

Remark 3.3. Note that there is no axiom (M8) similar to (W8). The reason is the incompatibility of the $\triangle_\mu^{Max}$ operator and (W8). The definition of model fitting operators allows $\triangle_\mu^{Max}$ on unweighted knowledge bases.

3.3. Example arbitration operator

We give below as an example an arbitration operator. Let $\mathcal{A}$ be the set of all possible atomic propositions and $\mathcal{R}^+$ be the set of positive real numbers. Let ω be a weight function from $\mathcal{A}$ to $\mathcal{R}^+$. For any set S, let $|S|$ denote its cardinality.

Definition 3.2. *The unweighted distance between interpretations I and J, denoted $dist(I, J)$, is*

$$dist(I, J) = |(I \setminus J) \cup (J \setminus I)| = \sum_{t \in (I \setminus J) \cup (J \setminus I)} 1.$$

The ω-weighted distance between interpretations I and J, denoted $dist_\omega(I, J)$, is

$$dist_\omega(I, J) = \sum_{t \in (I \setminus J) \cup (J \setminus I)} \omega(t).$$

Example 3.6. Assume that $I = \{A, B, C\}$ and $J = \{C, D, E\}$. Suppose that ω is a weight function that assigns $\omega(A) = 1$, $\omega(B) = 2$, $\omega(C) = 3$, $\omega(D) = 4$ and $\omega(E) = 5$. Then $dist(I, J) = 4$ and $dist_\omega(I, J)$, is $1+2+4+5 = 12$.

Definition 3.3. *Let $K = \{\psi_1, \ldots, \psi_n\}$ be an unweighted knowledge base, and let ω be a weight function from $\mathcal{A}$ to $\mathcal{R}^+$. The overall distance between a propositional formula ψ and an interpretation I, denoted $odist(\psi, I)$, is*

$$odist(\psi, I) = \min_{J \in Mod(\psi)} dist_\omega(I, J).$$

The overall distance between K and an interpretation I, denoted $odist(K,I)$, is

$$odist(K,I) = \max_{\psi \in K} odist(\psi, I).$$

Then we assign to each unweighted knowledge base K the total pre-order $\leq_K$ defined by $I \leq_K J$ if and only if $odist(K,I) \leq odist(K,J)$. We define the model fitting operator $\triangle^{Max}_{\omega,\mu}$ as the set of minimal models of μ with respect to the pre-order $\leq_K$. The $\triangle^{Max}_{\omega,\mu}$ operator extends the $\triangle^{Max}_{\mu}$ operator by using a weight function on atomic propositions.

A more complex extension of $\triangle^{Max}_{\mu}$ is when we use a different weight function ω_i for each propositional formula ψ_i in K. The more complex operator, denoted $\triangle^{Max}_{\omega_1,\ldots,\omega_n,\mu}$ is useful in modeling the preferences of several people.

Definition 3.4. *Let $K = \{\psi_1, \ldots, \psi_n\}$ be an unweighted knowledge base, and let ω_i be the weight function from $\mathcal{A}$ to $\mathcal{R}^+$ that is associated with ψ_i. The overall distance between a propositional formula ψ_i and an interpretation I, denoted $odist(\psi_i, I)$, is*

$$odist(\psi_i, I) = \min_{J \in Mod(\psi_i)} dist_{\omega_i}(I, J).$$

The overall distance between K and an interpretation I, denoted $odist(K,I)$, is

$$odist(K,I) = \max_{\psi_i \in K} odist(\psi_i, I).$$

As shown in the following sections, the above extension is useful in modeling game theory and bargaining problems where the preferences of several players need to be considered.

4. Game theory

Game theory, originated by von Neumann, is an extremely successful and broad subject. Nevertheless, it seems an overreach to try to apply game theory to arbitration problems. Some authors recently tried to solve "King Solomon's dilemma"–presumably named after the well-known "Prisoner's dilemma" problem–using game theory. At first, we present a simple solution to this problem using arbitration. Then we discuss game theoretic proposals for this problem and make some comparisons.

Example 4.1. According to a story in the Bible, King Solomon had to make a judgement between two women who both claimed to have given birth to the same baby. One women had a stillbirth and switched her stillborn baby with the live baby while its mother was asleep. When the two women appeared before him, King Solomon threatened to cut the baby in half if both women continue to claim the baby as theirs.

Let the propositions L, M and T represent, respectively, that the baby is kept alive, that the mother gets custody, and the thief gets custody. Let us assume that King Solomon considers four possible outcomes: (1) the baby is cut in half, that is, nobody gets the baby, (2) the mother gets the baby, (3) the thief gets the baby, and (4) neither woman gets the baby, but it is given to some other person. King Solomon's four options can be represented as a formula μ that has the following models:

$$Mod(\mu) = \{\{\}, \{L, M\}, \{L, T\}, \{L\}\}.$$

The mother wants the baby to be kept alive and gain custody. Hence

$$Mod(\psi_m) = \{\{L, M\}\}.$$

Assume that the mother's preferences are expressed by the weight function ω_m, which gives $\omega_m(L) = 8, \omega_m(M) = 6, \omega_m(T) = 1$. Hence, we calculate that

$$odist(\psi_m, \{\}) = 14,$$

$$odist(\psi_m, \{L, M\}) = 0,$$
$$odist(\psi_m, \{L, T\}) = 7,$$
$$odist(\psi_m, \{L\}) = 6.$$

The thief also wants the baby and keep its custody. Hence

$$Mod(\psi_t) = \{\{L, T\}\}.$$

Assume that the thief's preferences are expressed by the weight function ω_t, which gives $\omega_t(L) = 2, \omega_t(M) = 1, \omega_t(T) = 4$. Hence

$$odist(\psi_t, \{\}) = 6,$$

$$odist(\psi_t, \{L, M\}) = 5,$$
$$odist(\psi_t, \{L, T\}) = 0,$$
$$odist(\psi_t, \{L\}) = 4.$$

Our knowledge base $K = \{\psi_m, \psi_t\}$. Hence we find that

$$odist(K, \{\}) = \max(14, 6) = 14,$$

$$odist(K, \{L, M\}) = \max(0, 5) = 5,$$
$$odist(K, \{L, T\}) = \max(7, 0) = 7,$$
$$odist(K, \{L\}) = \max(6, 4) = 6.$$

Since the minimum value is five, we find that

$$\triangle^{Max}_{\omega_m,\omega_t,\mu}(K) = \{\{L, M\}\}.$$

Hence King Solomon's best choice is to give the live baby to its mother.

Game theory would represent the situation by a payoff matrix [19] as shown in Figure 1. In each entry of the payoff matrix, the top row shows a pair of numbers, where the first number is mother's payoff and the second number is the thief's payoff, and the bottom row shows one of the four outcomes. For arbitration, we calculated the distances from the desires of the mother and the thief to the outcomes. For game theory, we calculate the payoffs as the distance from the complement of the desires to the outcomes. For example, the mother's desire is $\{L, M\}$, its complement is $\{T\}$. This can be taken as the mother's worst fear. Any improvement from that is a payoff. For example, the weighted distance from $\{T\}$ to $\{L\}$ is $1 + 8 = 9$. The mother gets a payoff of eight because the baby's life is spared and a payoff of one because the baby is not given to the thief. Therefore, the mother gets a total of nine payoff.

For the mother, we calculate the payoffs

$$odist(\{T\}, \{\}) = 1,$$
$$odist(\{T\}, \{L, M\}) = 15,$$
$$odist(\{T\}, \{L, T\}) = 8,$$
$$odist(\{T\}, \{L\}) = 9.$$

For the thief, whose worst fear is $\{M\}$, the complement of $\{L, T\}$, we calculate the payoffs

$$odist(\{M\}, \{\}) = 1,$$
$$odist(\{M\}, \{L, M\}) = 2,$$
$$odist(\{M\}, \{L, T\}) = 7,$$
$$odist(\{M\}, \{L\}) = 3.$$

The above calculations are used to fill in the payoff matrix in Figure 1.

	thief: mine	**thief: not mine**
mother: mine	1, 1 $\{\}$	15, 2 $\{L, M\}$
mother: not mine	8, 7 $\{L, T\}$	9, 3 $\{L\}$

Figure 1. The payoff matrix for the baby custody dispute.

Game theory searches for the best strategies of the players. A Nash equilibrium point [14] is when none of the players has an incentive to change if the others also do not change. In this case, if we do not allow mixed (stochastic) strategies, then there are two Nash equilibria, namely, outcomes (2) and (3). In mixed strategies, the players choose between the two options that they have with a certain frequency. If the women do not coordinate their choices, then the baby may get killed.

To avoid the above outcome, Mihara [12], Olszewski [15], and Qin and Yang [16] proposed several more complicated solutions. These solutions have in common a two part process. At first, the two women are asked to bid in order for the game to take place. The second part, is the game itself with the condition that the looser has to pay the winner what she bade. Instead of having a game actually take place, the aim of these strategies is to make the bidding so high that the less eager participant, the thief, will back off from the whole process. The complexity of such a solution suggests that game theory is not the right approach to arbitration problems like King Solomon's dilemma.

Remark 4.1. In game theory, the players can eliminate some possible outcomes that are undesirable to them. In arbitration, the players cannot eliminate possible outcomes. The players may say that they prefer some outcomes, but an arbitrator is not bound to follow what they say. In the baby custody dispute example, King Solomon, the arbitrator, can see that the mother is only playing a strategy when she says that the baby is not hers. From the responses of the two women, King Solomon can guess the cost matrix of this game and then identify that the best solution is to give the baby back to its mother. In arbitration, the arbitrator can eliminate some possible outcomes.

5. The bargaining problem

The bargaining or negotiating problem has been also considered by many authors. In this section we focus on Nash bargaining [13] and show that it can be also represented by propositional arbitration.

Let $\mathcal{I} = \{I_1, \ldots, I_n\}$ be a set of $n \geq 2$ items and $\mathcal{P} = \{P_1, \ldots, P_m\}$ be a set of $m \geq 2$ persons. Let the proposition i_j represent that item I_i is given to person P_j. Assume also that each person P_j's weight (or utility) function, ω_j, assigns to each item I_i and corresponding proposition i_j a positive real number, which reflects P_j's desire of owning I_i.

Remark 5.1. The weight function ω_j also could assign some positive real number to the other propositions i_k where $k \neq j$. For example, if P_k is a friend of P_j, then $\omega_j(i_k)$ could be some positive real number because P_j could find some utility in P_k's ownership of I_i as P_j can now easier borrow item I_i from P_k than from the former owner. Arbitration can handle this friendship effect that most other models of bargaining ignore. To ignore friendship effects in arbitration modeling, let ω_j assign some small $\epsilon > 0$ to all i_k where $k \neq j$.

Arbitration assumes for each person P_j a propositional formula ψ_j that describes the exchanges desired by P_j. For example, if P_j is a completely selfish individual, then P_j would desire to get all the items owned by others. In this case, P_j's desire can be described by ψ_j with

$$Mod(\psi_j) = \{i_j : P_j \text{ does not own } I_i\}.$$

All of the desires $K = \{\psi_1, \ldots, \psi_m\}$ need arbitration by selecting the models of μ that are overall closest to K, where each model of μ describes a set of possible and allowed exchanges of items.

Remark 5.2. In a two-person bargaining the subscript of the i_j can be omitted because it is obvious who gets an item if the item is exchanged.

The next example gives an arbitration solution to Nash's original bargaining problem [13].

Example 5.1. Suppose that Bill and Jack own a set of items and weigh each item as shown in Figure 2, which lists the items in the same order as in [13]. In this case, Bill desires to exchange the following set of items

$$Mod(\psi_B) = \{\{pen, toy, knife, hat\}\},$$

while Jack desires to exchange the following set of items

$$Mod(\psi_J) = \{\{book, whip, ball, bat, box\}\}.$$

During the bargaining, any subset of the nine items could be exchanged between Bill and Jack. That is, there are 2^9 different possible exchanges. Each of these exchanges can be uniquely described as a subset of the nine items and

Bill and Jack's Weight (Utility) Functions

	book	whip	ball	bat	box	pen	toy	knife	hat
owner	Bill	Bill	Bill	Bill	Bill	Jack	Jack	Jack	Jack
ω_B	2	2	2	2	4	10	4	6	2
ω_J	4	2	1	2	1	1	1	2	2

Figure 2. Bill and Jack's ownnership and weight functions of a set of items.

a possible model of μ. Let us suppose that μ is such that it has exactly the following three models:

$$M_1 = \{book, whip, ball, bat, pen, knife\},$$
$$M_2 = \{book, whip, ball, bat, pen, toy, knife\},$$
$$M_3 = \{book, whip, bat, pen, toy, knife\}.$$

In the above M_1 describes that Bill gives Jack the book, the whip, the ball and the bat, and Jack gives Bill the pen and the knife. M_2 and M_3 can be interpreted similarly. For Bill the overall distance between ψ_B and M_1 is

$$odist(\psi_B, M_1) = dist_{\omega_B}(\{pen, toy, knife, hat\}, \{book, whip, ball, bat, pen, knife\})$$
$$= \omega_B(book) + \omega_B(whip) + \omega_B(ball) + \omega_B(bat) + \omega_B(toy) + \omega_B(hat) = 14.$$

Similarly, for Jack the overall distance between ψ_J and M_1 is

$$odist(\psi_J, M_1) = dist_{\omega_J}(\{book, whip, ball, bat, box\}, \{book, whip, ball, bat, pen, knife\})$$
$$= \omega_J(box) + \omega_J(pen) + \omega_J(knife) = 4.$$

For M_2 and M_3 we find that

$$odist(\psi_B, M_2) = 10,$$
$$odist(\psi_J, M_2) = 5,$$
$$odist(\psi_B, M_3) = 8,$$
$$odist(\psi_J, M_3) = 6.$$

Let the knowledge base be $K = \{\psi_B, \psi_J\}$. We calculate that:

$$odist(K, M_1) = \max(odist(\psi_B, M_1), odist(\psi_J, M_1)) = \max(14, 4) = 14,$$
$$odist(K, M_2) = \max(odist(\psi_B, M_2), odist(\psi_J, M_2)) = \max(10, 5) = 10,$$
$$odist(K, M_3) = \max(odist(\psi_B, M_3), odist(\psi_J, M_3)) = \max(8, 6) = 8.$$

The minimal model of μ with respect to the pre-order $\leq_K$ is M_3 because $odist(K, M_3) < odist(K, M_2) < odist(K, M_1)$. Therefore,

$$\triangle^{Max}_{\omega_B,\omega_J,\mu}(K) = M_3.$$

This shows that the best choice for arbitration is M_3. The choice of M_3 leaves each player the least disappointed. Compared to their original expectations, Bill's disappointment is 8 and Jack's disappointment is 6. Hence the largest disappointment of any player is only 8. All the other choices are more disappointing to some player.

Next we solve the problem of Example 5.1 using Nash bargaining.

Example 5.2. Nash bargaining starts by calculating for the possible bargining results the gains of both Bill and Jack as shown in Figure 3. Bill's gains are the number of total weights of Bill's items after the bargaining minus the total weights of Bill's items before the bargaining. Jack's gains are calculated similarly.

Possible Bargains between Bill and Jack

Case	Bill gives	Jack gives	Bill gains	Jack gains
1	book, whip, ball, bat	pen, knife	8	6
2	book, whip, ball, bat	pen, toy, knife	12	5
3	book, whip, bat	pen, toy, knife	14	4
⋮	⋮	⋮	⋮	⋮

Figure 3. Possible bargains between Bill and Jack and their utility gains.

Nash's bargaining solution optimizes the product of the two players' gains, which in this case happens to be largest in the second case, that is, when Bill gains 12 and Jack gains 5.

Remark 5.3. Since cases 1, 2 and 3 in Figure 3 correspond to models M_1, M_2 and M_3 in Example 5.1, the arbitration and the Nash bargaining solutions are different for this problem. Arbitration takes the approach that the two sides came together to bargain and neither side should be left greatly disappointed. The Nash bargaining solution leaves Bill quite disappointed. A major disappointment of either party endangers the entire deal. If Bill would be offered the Nash solution, then he may walk away because of feeling disappointment and resentment. Perhaps people are not always as rational as Nash assumed them to be. However, it should be also noted that they are not completely irrational either. In Example 5.1, we assumed that Bill and Jack have the maximalist desires of getting everything given to them free. Such desires are appropriate only for children, which Bill and Jack presumably are given the bargaining items in the problem. In practice, experienced negotiators always try to offer something that at least they think the other side may like.

Example 5.3. Suppose Bill anticipates giving Jack the book, given that Jack likes it better he does, and accepts the idea that it is unreasonable to ask Jack to give him the hat, which they both like equally. Moreover, Bill recognizes that exchanging one item for three items is unrealistic, hence he is prepared to offer Jack the whip too. In other words, Bill's desire regarding the outcome of the bargaining changes to

$$Mod(\psi_B) = \{\{book, whip, pen, toy, knife\}\}.$$

This reduces by six the overall distance from ψ_B to each M_i for $1 \le i \le 3$. Assuming that Jack's desire does not change, now we get

$$odist(K, M_1) = \max(odist(\psi_B, M_1), odist(\psi_J, M_1)) = \max(8, 4) = 8,$$

$$odist(K, M_2) = \max(odist(\psi_B, M_2), odist(\psi_J, M_2)) = \max(4, 5) = 5,$$

$$odist(K, M_3) = \max(odist(\psi_B, M_3), odist(\psi_J, M_3)) = \max(2, 6) = 6.$$

In this case, M_2 has the minimum distance and would be the arbitration solution in agreement with the Nash bargaining solution. Hence the arbitration solution is more general than the Nash bargaining solution. Arbitration matches the human experience that the participants' anticipations are crucial in determining which deal takes place.

Remark 5.4. Some other solutions to the bargaining problem also start by computing the gains of the participants but either choose the solution that maximizes the gain of the smallest gainer or the solution that gives the same proportion of gains to the players with respect to their maximum gains [6, 7].

6. Conclusions and further work

Axiomatic arbitration theory emerged as an alternative to belief revision and database updates, but, as shown in this paper, it has deep connections with bargaining and game theory, too. It would be interesting to explore further these connections considering also first-order arbitration as described in [18]. First-order formulas could provide more flexible and hence potentially more realistic descriptions of the participants' desires and the allowed arbitration outcomes. Another direction of research would be to apply arbitration to more complex conflict resolution problems, such as arms control negotiations or international trade disputes.

References

[1] **Alchourrón, C.E., P. Gärdenfors and D. Makinson,** On the logic of theory change: Partial meet contraction and revision functions, *Journal of Symbolic Logic*, **50** (1985), 510–530.

[2] **Benczur, A., A.B. Novak and P.Z. Revesz,** Classical and weighted knowledgebase transformations, *Computers and Mathematics with Applications*, **32** (1996), 85–98.

[3] **Benczur, A., A.B. Novak, and P.Z. Revesz,** On the semantics of revision and model-fitting in the weighted case, *Journal of Information and Organizational Sciences*, **20** (1996), 45–58.

[4] **Benczur, A., A.B. Novak, and P.Z. Revesz,** Weighted knowledgebase transformations, *Annales Univ. Sci. Bud. Sect. Comp.*, **17** (1998), 65–78.

[5] **Grahne, G., A.O. Mendelzon, and P.Z. Revesz,** Knowledgebase transformations, *Journal of Computer and System Sciences*, **54** (1997), 98–112.

[6] **Kalai, E.,** Propositional solutions to bargaining situations: Interpersonal utility comparisons, *Econometrica*, **45** (1977), 1623–1637.

[7] **Kalai, E. and M. Smorodinsky,** Other solutions to Nahs's bargaining problem and the Nash social welfare function, *Econometrica*, **43** (1975), 513–518.

[8] **Katsuno, H. and A.O. Mendelzon,** Propositional knowledge base revision and minimal change, *Artificial Intelligence*, **52** (1991), 263–294.

[9] **Konieczny, S. and R. Pino Pérez,** Merging information under constraints: A logical framework, *Journal of Logic and Computation*, **12** (2002), 773-808.

[10] **Liberatore, P. and M. Schaerf,** Arbitration (or how to merge knowledge bases), *IEEE Transactions on Knowledge and Data Engineering*, **10** (1998), 76–90.

[11] **Lin, J. and A.O. Mendelzon,** Knowledge base merging by majority, *Dynamic Worlds: From the Frame Problem to Knowledge Management*, ed. R. Pareschi, Kluwer Academic, 1999.

[12] **Mihara, H.R.,** The second-price auction solves King Solomon's dilemma, *Japanese Economic Review*, **63** (2012), 420–429.

[13] **Nash, J.F.,** The bargaining problem, *Econometrica*, **18** (1950), 155–162.

[14] **Nash, J.F.,** Equilibrium points in n-person games, *Proceedings of the National Academy of Sciences of the USA*, **36** (1950), 48–49.

[15] **Olzsewski, W.,** A simple and general solution to King Solomon's problem, *Games and Economic Behavior*, **42** (2003), 315–318.

[16] **Qin, C.-Z. and C.-L. Yang,** Make a guess: A robust mechanism for King Solomon's dilemma, *Economic Theory*, **39** (2009), 259–268.

[17] **Revesz, P.Z.,** On the semantics of theory change: Arbitration between old and new information, in: C. Beeri (Ed.) *Proc. of the Twelfth ACM SIGACT-SIGMOD-SIGART Symposium on Principles of Database Systems, Washington DC, 1993,* ACM Press, New York, 1993, 71–82.

[18] **Revesz, P.Z.,** On the semantics of arbitration, *International Journal of Algebra and Computation*, **7** (1997), 133–160.

[19] **von Neumann, J. and O. Morgenstern,** *Theory of Games and Economic Behavior*, Princeton University Press, Princeton, NJ, 1944.

Peter Z. Revesz
Department of Computer Science and Engineering
University of Nebraska-Lincoln
Lincoln
USA
revesz@cse.unl.edu