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UNIFORM DISTRIBUTION OF SOME ARITHMETICAL FUNCTIONS

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Dedicated to Professor András Benczúr on the occasion of his 70th birthday

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Abstract. We investigate the uniform distribution modulo 1 of sequence $\frac{a(n)}{b(n)}$ $(n \in \mathbb{N})$, where $a(n)$ and $b(n)$ are suitable additive or multiplicative functions. We prove that if $\kappa(n)$ is any of $\frac{An^r}{\omega(n)^h}, \frac{An^r}{a^{\omega(n)}}, \frac{An^r}{\tau(n^h)}$ $(r, h, A \in \mathbb{N})$ and $f(n)$ is an arbitrary additive function, then $\kappa(n) + f(n)$ is uniformly distributed mod 1.

1. Notation and introductory definitions

Let $\mathcal{A}$ be the set of additive, $\mathcal{M}$ be the set of multiplicative functions. A function $f : \mathbb{N} \to \mathbb{R}$ belongs to $\mathcal{A}$ if $f(nm) = f(n) + f(m)$ holds for every coprime pairs of integers n, m. A function $g : \mathbb{N} \to \mathbb{R}$ is multiplicative if $g(1) = 1$ and $g(nm) = g(n)g(m)$ holds for every coprime pairs of integers n, m.

Let $\mathcal{M}_1 := \{g \in \mathcal{M} \ : \ |g(n)| \leq 1\}$.

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Let $\mathcal{P}$ be the set of primes, $\omega(n)$ be the number of distinct prime divisors of n, $\tau(n)$ be the number of divisors of n.

Let $\{x\}$ = fractional part of x. We shall write $e(x)$ instead of $e^{2\pi ix}$.

Let $x_1, \cdots, x_N$ be real numbers. The discrepancy of $\{x_j\}$ $(j = 1, \cdots, N)$ is

$$\sup_{[a,b]\subset[0,1)} \left| \frac{1}{N} \sum_{\{x_j\}\in[a,b)} 1 - (b-a) \right| =: \mathcal{D}_N(\{x_j\}, j = 1, \cdots, N).$$

We say that an infinite sequence x_n $(n = 1, 2, \cdots)$ of real numbers is uniformly distributed modulo 1, if

$$\mathcal{D}_N(\{x_j\}, j = 1, \cdots, N) \to 0 \quad \text{as} \quad N \to \infty.$$

Let p runs over $\mathcal{P}$. Let $P(n)$ be the largest prime factor of n.

Let $\pi(x, k, \ell) = \natural\{p \le x \mid p \equiv \ell \pmod{k}, p \in \mathcal{P}\}$. We shall write UD mod 1 for the abbreviation of uniformly distributed mod 1.

2. Formulation of the theorems

Several papers have been published on the uniform distribution modulo 1 of the sequences $\alpha(n) = \frac{a(n)}{b(n)}$ $(n \in \mathbb{N})$, where $a(n)$ and $b(n)$ are either additive or multiplicative functions. See [5]–[9]. In [8] it is proved that $\frac{n}{\omega(n)}$, $\frac{n}{\tau(n)}$, $\frac{n}{a^{\omega(n)}}$ are UD mod 1. In [7] it is proved that $\nu(n) = \frac{\omega(n)}{g(n)}$, $\rho(n) = \frac{\omega(n+1)}{g(n)}$ are UD mod 1 if $g \in \mathcal{A}$, and $0 < g(p) < \frac{c_1}{p}$ and $0 < g(p^a) < c_2$ holds for all $p \in \mathcal{P}$ and $a \in \mathbb{N}$.

In this paper we shall investigate similar questions.

Theorem 1. *Let $K(n) : \mathbb{N} \to \mathbb{N}$ $(n \in \mathbb{N})$ be such a function for which*

$$K(n_1) = K(n_2) \quad \textit{if} \quad \frac{n_1}{P(n_1)} = \frac{n_2}{P(n_2)}. \tag{2.1}$$

Let

$$\kappa(n) = \frac{An^r}{K(n)}, \tag{2.2}$$

where $A \in \mathbb{N}$, $r \in \mathbb{N}$ are fixed integers.

Let

$$\Delta(n) := GCD(An^r, K(n)). \tag{2.3}$$

Assume that

$$\limsup_{x\to\infty} \frac{1}{x}\natural\{n \le x \mid \Delta(n) > \sqrt{K(n)}\} = 0, \tag{2.4}$$

$$\limsup_{x\to\infty} \frac{1}{x}\natural\{n \le x \mid \frac{K(n)}{\Delta(n)} < Y\} = 0 \text{ for every } Y > 0, \tag{2.5}$$

$$\limsup_{x\to\infty} \frac{1}{x}\natural\{n \le x \mid \Delta(n) > (\log n)^{\rho}\} \le c(\rho), \tag{2.6}$$

and $\delta(\rho) \to 0$ *as* $\rho \to \infty$.

Under the conditions (2.1)–(2.6) $\kappa(n)$ (mod 1) *is* UD mod 1.

Theorem 2. *Let*

$$\alpha(n) = \frac{An^r}{\omega(n)^h}, \quad \beta(n) = \frac{An^r}{a^{\omega(n)}}, \quad \gamma(n) = \frac{An^r}{\tau(n^h)} \quad (r, h, A \in \mathbb{N}).$$

Let $\kappa(n)$ *be any of* $\alpha(n)$, $\beta(n)$, $\gamma(n)$. *Let* $f(n)$ *be an arbitrary additive function. Then* $\kappa(n) + f(n)$ *is* UD mod 1.

3. Some lemmas

Lemma 1. (H. Weyl [10]) *A sequence* x_n $(n \in \mathbb{N})$ *is* UD mod 1 *if and only if*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} e(kx_n) = 0$$

for every $k \in \mathbb{N}$.

Lemma 2. (Siegel-Walfisz theorem [3]) *We have*

$$\pi(x, k, \ell) = \frac{li(x)}{\varphi(k)}\Big(1 + O\Big(\frac{1}{(\log x)^5}\Big)\Big)$$

uniformly as $x \ge 2, (k, \ell) = 1, k \le (\log x)^c$, *where* c *is an arbitrary positive constant. The constant implied by the error term may depend on* c.

Lemma 3. (Erdős–Turán inequality [1]) *We have*

$$\mathcal{D}_N(x_1 \cdots, x_N) \le c_1\Big(\sum_{k=1}^{M} \frac{1}{k}\Big|\frac{1}{N}\sum_{n=1}^{N} e(kx_n)\Big|\Big) + \frac{c_1}{M},$$

where c_1 *is an absolute constant,* M *is an arbitrary positive integer.*

Lemma 4. ([4]) *Let $t : \mathbb{N} \to \mathbb{R}$. Assume that for every $K > 0$ there exists a finite set $\mathcal{P}_K$ of primes $p_1 < p_2 < \cdots < p_k$ such that*

$$\sum_{j=1}^{R} \frac{1}{p_j} > K,$$

and that for the sequences

$$\eta_{i,j}(m) := t(p_i m) - t(p_j m),$$

the relation

$$\lim_{x\to\infty} \frac{1}{x} \sum_{m=1}^{[x]} e(\eta_{i,j}(m)) = 0$$

holds whenever $i \neq j, i, j \in \{1, 2, \cdots, R\}$. Then there exists a function ρ_x which tends to zero as $x \to \infty$ and such that

$$\sup_{f\in\mathcal{M}_1} \frac{1}{x} \Big| \sum_{n\le x} f(n)e(t(n)) \Big| \le \rho_x.$$

Lemma 5. (Hua [2]) *Let*

$$f(x) = a_r x^r + \cdots + a_1 x \in \mathbb{Z}[x], \quad (a_r, \cdots, a_1, q) = 1.$$

Then

$$\Big| \sum_{x=1}^{q} e\Big(\frac{f(x)}{q}\Big) \Big| \le c_1(r, \epsilon) q^{1-\frac{1}{r}+\epsilon},$$

where ϵ is an arbitrary positive number.

Lemma 6. *Let $r, A \in \mathbb{N}$ be fixed. Let $\ell_1, \dots, \ell_{\varphi(q)}$ be the set of reduced residues modulo q,*

$$y_\nu = \frac{A\ell_\nu^r}{q} \quad (\nu = 1, \dots, \varphi(q)).$$

Then, for every $k \in \mathbb{N}$,

$$\Big| \frac{1}{\varphi(q)} \sum_{\nu=1}^{\varphi(q)} e\big(ky_\nu\big) \Big| \to 0 \quad as \quad q \to \infty, k \in \mathbb{N},$$

and so

$$\mathcal{D}_{\varphi(q)}(y_1 \dots, y_{\varphi(q)}) \to 0 \quad as \quad q \to \infty.$$

Proof. We have

$$\sum_{\nu=1}^{\varphi(q)} e\big(ky_\nu\big) = \sum_{\ell=1}^{q} e\Big(\frac{kA\ell^r}{q}\Big) \sum_{\delta|(\ell,q)} \mu(\delta) = \sum_{\delta|q} \mu(\delta) \sum_{m=1}^{q/\delta} e\Big(\frac{kA\delta^r m^r}{q}\Big) = \\ = \sum_{\delta|q} \mu(\delta) \sum\nolimits_\delta,$$

where

$$\sum\nolimits_\delta := \sum_{m=1}^{q/\delta} e\Big(\frac{kA\delta^r m^r}{q}\Big).$$

We shall estimate the sums for $\delta \le z$.

Let

$$\frac{kA\delta^r}{q} = \frac{U}{V}, \quad (U,V)=1.$$

Since $(Ak\delta^r, q) \le Akz^r$, therefore $V \ge \frac{q}{Akz^r}$.

Let us apply Lemma 5. We have

$$\sum\nolimits_\delta = \sum_{m=1}^{q/\delta} e\Big(\frac{kA\delta^r m^r}{q}\Big) = \frac{q}{\delta V} \sum_{m=1}^{V} e\Big(\frac{Um^r}{V}\Big).$$

Thus

$$\big|\sum\nolimits_\delta\big| \le \frac{(Ak\delta^r, q)}{\delta} c_1(r,\epsilon) V^{1-\frac{1}{r}+\epsilon} \le c_1(r,\epsilon) q^{1-\frac{1}{r}+\epsilon} \frac{(Ak\delta^r,q)^{\frac{1}{r}-\epsilon}}{\delta} \le \\ \le c_2 c_1(r,\epsilon) \frac{1}{\delta^{\epsilon r}},$$

where $c_2 = Ak$. Since

$$\big|\sum\nolimits_\delta\big| \le q/\delta,$$

therefore

$$\Big|\sum_{\nu=1}^{\varphi(q)} e\big(ky_\nu\big)\Big| \le c_2 c_1(r,\epsilon) q^{1-\frac{1}{r}+\epsilon} \sum_{\substack{\delta \le z \\ \delta|q}} \frac{|\mu(\delta)|}{\delta^{\epsilon r}} + q \sum_{\substack{\delta > z \\ \delta|q}} \frac{|\mu(\delta)|}{\delta}.$$

Let $z = \sqrt{q}$. Observe that

$$\sum_{\substack{\delta \le z \\ \delta|q}} \frac{|\mu(\delta)|}{\delta^{\epsilon r}} \le 2^{\omega(q)} \le c_3 q^{c/\log\log q},$$

and that

$$q\sum_{\substack{\delta>\sqrt{q}\\ \delta|q}}\frac{|\mu(\delta)|}{\delta}\le q^{\frac12}2^{\omega(q)}\le Cq^{\frac12+\epsilon}.$$

Hence we obtain that

$$\Big|\frac{1}{\varphi(q)}\sum_{\nu=1}^{\varphi(q)}e\big(ky_\nu\big)\Big|\le\frac{cc_2c_1(r,\epsilon)q^{1-\frac1r+\epsilon}q^{c_3/\log\log q}}{\varphi(q)}+c\frac{q^{1/2+\epsilon}}{\varphi(q)}.$$

The right hand side tends to zero as $q\to\infty$, for every $k\in\mathbb{N}$. Lemma 6 follows from Lemma 4.

4. Proof of Theorem 1

Assume that the function $K(n)$ and positive integers A,r satisfy (2.1)–(2.6), where $\kappa(n)=\frac{An^r}{K(n)}$. From Lemma 1 it is enough to prove that

$$\frac1x\sum_{n\le x}e(k\kappa(n))\to 0\quad(x\to\infty)$$

for every $k\in\mathbb{N}$.

Let $J_x=\{n|x\le n<2x\}$. Let ϵ,Y,ρ be positive integers. Let $\mathcal{R}_x(\epsilon,Y,\rho)$ be the set of those $n\in J_x$ for which at least one of the next conditions hold:

$$\frac{K(n)}{\Delta(n)}<Y, \tag{4.1}$$

$$\Delta(n)>\sqrt{K(n)}, \tag{4.2}$$

$$\Delta(n)>(\log n)^\rho, \tag{4.3}$$

$$P(n)<x^\epsilon, \tag{4.4}$$

(4.5) n has two prime divisors p,q such that $Y<p<q<2p$.

Taking into account the relations (2.4), (2.5), (2.6) the number of $n\in J_x$ for which one of (2.4), (2.5), (2.6) holds is less than $O_Y(1)x+c\epsilon x+c\delta(\rho)x$. The number of $n\in J_x$ satisfying (4.4) is less than

$$cx\Pi_{x^\epsilon<p<x}(1-\frac1p)\le c\epsilon x.$$

The number of $n \in J_x$ for which (4.5) holds is less than

$$2x \sum_{p>Y} \frac{1}{p} \sum_{p<q<2p} \frac{1}{q} \le cx \sum_{p>Y} \frac{1}{p \log p} \le \frac{c_1 x}{\log Y}.$$

Thus

$$\frac{1}{x} \natural \mathcal{R}_x(\epsilon, Y, \rho) \le o_Y(1) + c\epsilon + c\delta(\rho).$$

Let $\mathcal{J}_x = J_x \setminus \mathcal{R}_x(\epsilon, Y, \rho)$.

Let us classify the elements of $\mathcal{J}_x$.

Let $n \in \mathcal{J}_x, p = P(n)$. Then $n = pm$. Let us consider the primes q in $[\frac{x}{m}, \frac{2x}{m}]$. We have $P(m) \le \frac{x}{m}$, since in the opposite case $mP(m) \ge x, mp < 2x$, and so $P(m) < p \le 2P(m)$, thus n is in $\mathcal{R}_x(\epsilon, Y, \rho)$.

Let

$$\mathcal{T}_x = \Big\{ mp \in \mathcal{J}_x \mid p \in \Big[\frac{x}{m}, \frac{2x}{m}\Big]\Big\}.$$

Let

$$T_k(m) = \sum_{n \in \mathcal{T}_k(m)} e(k\kappa(n)).$$

If $n = pm \in \mathcal{T}_k(m), p = P(m)$, then $\kappa(n) = \frac{Am^r p^r}{K(mp)}$ and $K(mp)$ depends only on m. Let

$$\frac{Am^r}{\Delta(n)} : \frac{K(mp)}{\Delta(n)} = \frac{B_m}{D_m},\ kB_m \pmod{D_m} \equiv H_m \pmod{m},$$

$$\frac{H_m}{D_m} \equiv \frac{U_m}{V_m} \pmod 1,\ (U_m, V_m) = 1,\ 1 \le U_m < V_m.$$

Since $V_m \le K(mp) \le (\log n)^\rho$, therefore we can apply the Siegel–Walfisz theorem. We have

$$T_k(m) = \sum_{\substack{\ell \pmod{V_m} \\ (\ell, V_m) = 1}} e\Big(\frac{U_m \ell^n}{V_m}\Big) \cdot \sum\nolimits_\ell,$$

$$\sum\nolimits_\ell = \sum_{\substack{p \equiv \ell \pmod{V_m} \\ p \in [\frac{x}{m}, \frac{2x}{m}]}} 1 = \pi\Big(\frac{2x}{m}, V_m, \ell\Big) - \pi\Big(\frac{x}{m}, V_m, \ell\Big) =$$

$$= \frac{1}{\varphi(V_m)}\Big(\pi\Big(\frac{2x}{m}\Big) - \pi\Big(\frac{x}{m}\Big)\Big) + O\Big(\frac{x}{\varphi(V_m) m (\log x)^5 \epsilon^5}\Big).$$

We have $\natural T_k(m) = \Big(\pi\Big(\frac{2x}{m}\Big) - \pi\Big(\frac{x}{m}\Big)$. So

$$\frac{T_k(m)}{\natural T_k(m)} = \frac{1}{\varphi(V_m)} \sum_{\substack{\ell \pmod{V_m} \\ (\ell, V_m) = 1}} e\Big(\frac{U_m \ell^n}{V_m}\Big) + O\Big(\frac{1}{(\log x)^5 \epsilon^5}\Big).$$

Since $V_m > Y$, from Lemma 6 we obtain that the right hand side is less than $O_Y(1) + O\Big(\frac{1}{(\log x)^4\epsilon^5}\Big)$.

Since this is true uniformly for every m, we obtain that

$$\Big|\sum_{n\in J_x} e(k\kappa(n))\Big| \le [o_y(1) + c\epsilon + c(\rho)]x.$$

Since

$$[1,2x] = J_x \cup J_{x/2} \cup \cdots \cup J_{x/2^L}, \quad \frac{1}{2} < x/2^L \le 1,$$

we obtain immediately that

$$\limsup_{x\to\infty} \frac{1}{x}\Big|\sum_{n\le x} e(k\kappa(n))\Big| \le o_Y(1) + c\epsilon + c(\rho).$$

Since the right hand side holds for every ϵ, Y and ρ, therefore the right hand side is zero.

Theorem 1 is true. ■

5. Proof of Theorem 2

We shall prove it only for $\gamma(n)$. The proof for $\alpha(n), \beta(n)$ is similar, therefore they are omitted.

Let T be a large constant. We modify the function τ. Let $\tau_T(n^h)$ be a multiplicative function, such that

$$\tau_T(p^{\alpha h}) = \begin{cases} \alpha h + 1 & \text{if } \ p \le T \\ (h+1)^\alpha & \text{if } \ p > T \end{cases}.$$

Let

$$\gamma_T(n) = \frac{An^r}{\tau_T(n^h)}.$$

Let $t_T(n) = k\gamma_T(n)$. Let K be a large number, $\mathcal{P}_K$ be a set of primes

$$(T <)p_1 < \cdots < p_R$$

such that

$$\sum_{j=1}^{R} \frac{1}{p_j} > K.$$

Let

$$\eta_{i,j}(m) = t_T(p_i m) - t_T(p_j m).$$

Then

$$\eta_{i,j}(m) = \frac{A(p_i^r - p_j^r)k}{h+1} \frac{m^r}{\tau_T(m^h)}.$$

Repeating the argument used in the proof of Theorem 1 with small changing we obtain that

$$\lim_{x\to\infty} \frac{1}{x} \sum_{m\le x} e(\eta_{i,j}(m)) = 0.$$

From Lemma 4, we obtain that

$$\sup_{g\in\mathcal{M}_1} \Big| \frac{1}{x} \sum_{n\le x} g(n) e(k\gamma_T(n)) \Big| \le \rho_x, \quad \rho_x \to 0 \ \text{ as } \ x\to\infty. \tag{5.1}$$

Let $f \in \mathcal{A}$. Choose $g(n) = e(kf(n)$. Observe that

$$\Big| \sum_{n\le x} e(kf(n)) e(k\gamma(n)) - \sum_{n\le x} e(kf(n)) e(k\gamma_T(n)) \Big| \le$$

$$\le 2 \sum_{\substack{n\le x \\ \gamma(n)\ne\gamma_T(n)}} 1 \le 2x \sum_{p>T} \frac{1}{p^2} \le \frac{cx}{T\log T}.$$

Hence, and from (5.1), we have that

$$\Big| \frac{1}{x} \sum_{n\le x} e(kf(n)) e(k\gamma(n)) \Big| \le \rho_x + \frac{c}{T\log T}.$$

Thus

$$\limsup_{x\to\infty} \frac{1}{x} \Big| \sum_{n\le x} e\big(k[f(n)+\gamma(n)]\big) \Big| \le \frac{c}{T\log T}. \tag{5.2}$$

Since this holds for every T, therefore the right hand side of (5.2) is zero. The proof is complete. ■

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