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ON UNINORMS WITH FIXED VALUES ALONG THEIR BORDER

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Dedicated to András Benczúr on the occasion of his 70th birthday

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Abstract. Since their introduction, uninorms have been studied deeply by numerous authors from theoretical and also from application points of view. Recently, a characterization of the class of uninorms with strict underlying t-norm and t-conorm was presented. It is also known that uninorms with nilpotent underlying t-norm and t-conorm belong to U_{min} or U_{max}. In this paper, some further construction methods of uninorms with fixed values along the borders are discussed and sufficient and necessary conditions are presented.

1. Introduction

A triangular norm (*t-norm* for short) T is a binary operation on the closed unit interval $[0,1]$ such that $([0,1],T)$ is an abelian semigroup with neutral element 1 which is totally ordered, i.e., for all x_1, x_2, y_1, $y_2 \in [0,1]$ with

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$x_1 \le x_2$ and $y_1 \le y_2$ we have $T(x_1, y_1) \le T(x_2, y_2)$, where $\le$ is the natural order on $[0,1]$ [8, 10].

Standard examples of t-norms are the minimum $T_{\mathbf{M}}$, the product $T_{\mathbf{P}}$, the Łukasiewicz t-norm $T_{\mathbf{L}}$ given by $T_{\mathbf{L}}(x,y) = \max(x+y-1, 0)$, and the drastic product $T_{\mathbf{D}}$ with $T_{\mathbf{D}}(1,x) = T_{\mathbf{D}}(x,1) = x$, and $T_{\mathbf{D}}(x,y) = 0$ otherwise. Clearly, $T_{\mathbf{M}}$ and $T_{\mathbf{D}}$ are the greatest and smallest t-norm, respectively, i.e., for each t-norm T we have $T_{\mathbf{D}} \le T \le T_{\mathbf{M}}$.

A triangular conorm (*t-conorm* for short) S is a binary operation on the closed unit interval $[0,1]$ such that $([0,1], S)$ is an abelian semigroup with neutral element 0 which is totally ordered. Standard examples of t-conorms are the maximum $S_{\mathbf{M}}$, the probabilistic sum $S_{\mathbf{P}}$, the Łukasiewicz t-conorm $S_{\mathbf{L}}$ given by $S_{\mathbf{L}}(x,y) = \min(x+y, 1)$, and the drastic sum $S_{\mathbf{D}}$ with $S_{\mathbf{D}}(0,x) = S_{\mathbf{D}}(x,0) = x$, and $S_{\mathbf{D}}(x,y) = 1$ otherwise. Clearly, $S_{\mathbf{M}}$ and $S_{\mathbf{D}}$ are the smallest and greatest t-conorms, respectively, i.e., for each t-conorm S we have $S_{\mathbf{M}} \le S \le S_{\mathbf{D}}$. A continuous t-norm T is said to be *Archimedean* if $T(x,x) < x$ holds for all $x \in (0,1)$. A continuous Archimedean T is called *strict* if T is strictly monotone; i.e. $T(x,y) < T(x,z)$ whenever $x \in (0,1]$ and $y < z$, and *nilpotent* if there exist $x, y \in (0,1)$ such that $T(x,y) = 0$.

From the duality between t-norms and t-conorms, we can easily derive the following properties. A continuous t-conorm S is said to be *Archimedean* if $S(x,x) > x$ holds for every $x, y \in (0,1)$. A continuous Archimedean S is called *strict* if S is strictly monotone; i.e. $S(x,y) < S(x,z)$ whenever $x \in [0,1)$ and $y < z$, and *nilpotent* if there exist $x, y \in (0,1)$ such that $S(x,y) = 1$.

A t-norm is said to be *positive*, if $x, y > 0$ implies $T(x,y) > 0$.

The concept of uninorms was introduced in [13] as a generalization of both t-norms and t-conorms (see also [5]).

Definition 1.1. *A mapping $U : [0,1] \times [0,1] \to [0,1]$ is a uninorm, if it is commutative, associative, nondecreasing and there exists $e \in [0,1]$ such that $U(e,x) = x$ for all $x \in [0,1]$.*

The structure of uninorms was first examined in [7] (see also [6]). First we recall two classes of uninorms from [7] that play a key role in this paper.

Proposition 1.1. *Suppose that U is a uninorm with neutral element $e \in]0,1[$ and both functions $x \to U(x,1)$ and $x \to U(x,0)$ $(x \in [0,1])$ are continuous except perhaps at the point $x = e$. Then U is given by one of the following forms.*

1. If $U(0,1)=0$ then

$$U(x,y)=\begin{cases} eT\left(\frac{x}{e},\frac{y}{e}\right), & (x,y)\in[0,e]^2;\\ e+(1-e)S\left(\frac{x-e}{1-e},\frac{y-e}{1-e}\right), & (x,y)\in[e,1]^2;\\ min(x,y), & otherwise.\end{cases} \tag{1.1}$$

2. If $U(0,1)=1$ then

$$U(x,y)=\begin{cases} eT\left(\frac{x}{e},\frac{y}{e}\right), & (x,y)\in[0,e]^2;\\ e+(1-e)S\left(\frac{x-e}{1-e},\frac{y-e}{1-e}\right), & (x,y)\in[e,1]^2;\\ max(x,y), & otherwise.\end{cases} \tag{1.2}$$

The class of uninorms having form (1.1) is denoted by U_{min}, while the class with form (1.2) is denoted by U_{max}.

Uninorms turned out to be useful in many fields like expert systems [2], aggregation [12] and fuzzy integral [9, 3]. Idempotent uninorms were characterized in [1]. Recently, a characterization of the class of uninorms with strict underlying t-norm and t-conorm was presented in [4]. In [11] the authors show that uninorms with nilpotent underlying t-norm and t-conorm belong to U_{min} or U_{max}.

2. Results

In this section some further construction methods of uninorms from given t-norms and t-conorms are discussed and sufficient and necessary conditions are presented.

Proposition 2.1. *(See also [11].) Let T be a strict t-norm, S be a strict t-conorm and $e\in]0,1[$. The function*

$$U_1(x,y)=\begin{cases} eT\left(\frac{x}{e},\frac{y}{e}\right), & (x,y)\in[0,e]^2;\\ e+(1-e)S\left(\frac{x-e}{1-e},\frac{y-e}{1-e}\right), & (x,y)\in[e,1]^2;\\ 1, & x=1 \text{ or } y=1;\\ min(x,y), & otherwise\end{cases} \tag{2.1}$$

is a uninorm with neutral element e (see Figure 1a).

Proof. To prove that U_1 is a uninorm, we have to show that it is associative, commutative and that it has a neutral element $e \in (0,1)$. Commutativity is obvious. From the properties of t-norms and t-conorms it follows immediately that $e \in (0,1)$ is a neutral element.

Note that U_1 differs from U_{min} only at points where either $x = 1$ or $y = 1$. Since the associativity of U_{min} is already known from [7], we only need to concentrate on the border lines where at least one of the variables of U_1 is 1.

To examine the associative equation

$$U_1(x, U_1(y,z)) = U_1(U_1(x,y), z), \tag{2.2}$$

we need to take the following possibilities into consideration:

1. If $x = 1$ or $z = 1$, then $U_1(x, U_1(y,z)) = 1 = U_1(U_1(x,y), z)$.
2. If $U_1(y,z) = 1$ or $U_1(x,y) = 1$, then by using the strict monotonicity of $S(x,y)$, we get $x = 1$ or $y = 1$. Thus $U_1(x, U_1(y,z)) = 1 = U_1(U_1(x,y), z)$.

Remark 2.1. *Note that the strict property of S cannot be omitted in Proposition 2.1 (i.e. the statement does not hold for arbitrary t-conorms). For a counterexample let us choose $T_{\mathbf{P}}$, $S_{\mathbf{L}}$, $e = 0.3$, $x = 0.7$, $y = 0.8$, and $z = 0$. In this case $U_1(0.7, U_1(0.8, 0)) = 0$, while $U_1(U_1(0.7, 0.8), 0) = 1$.*

Proposition 2.2. *(See also Theorem 4 in [11].) U_1 in (2.1) is a uninorm if and only if S is dual to a positive t-norm.*

Proof. The condition is sufficient, since in this case the proof is similar to that of Proposition 2.1.

Now we show that it is also necessary.

Let us assume indirectly that there exist $x_0, y_0 \neq 1$, for which $U_1(x_0, y_0) = 1$. Obviously, $x_0, y_0 > e$. Let $z_0 < e$, $z_0 \neq 1$ so that $U(y_0, z_0) \neq 1$. In this case the right hand side of the associativity equation in (2.2) is trivially 1, while the left hand side is z_0, which is a contradiction.

Proposition 2.3. *Let T be a strict t-norm, S be a strict t-conorm and $e \in]0,1[$. The function*

$$U_2(x,y) = \begin{cases} eT\left(\frac{x}{e}, \frac{y}{e}\right), & (x,y) \in [0,e]^2; \\ e + (1-e)S\left(\frac{x-e}{1-e}, \frac{y-e}{1-e}\right), & (x,y) \in [e,1]^2; \\ 1, & x = 1, y \neq 0 \text{ or } x \neq 0, y = 1; \\ min(x,y), & \text{otherwise} \end{cases} \tag{2.3}$$

is a uninorm with neutral element e (see Figure 1b).

Proof. To prove that U_2 is a uninorm, we have to show that it is associative, commutative and that it has a neutral element $e \in (0,1)$. Commutativity is obvious. From the properties of t-norms and t-conorms it follows immediately that $e \in (0,1)$ is a neutral element.

Note that U_2 differs from U_1 only at points $(1,0)$ and $(0,1)$. Since the associativity of U_1 is already known (see Proposition 2.1), we only need to concentrate on the vertices of the unit square.

Since we examine the equation

$$U_2(x, U_2(y,z)) = U_2(U_2(x,y),z), \tag{2.4}$$

we need to take the following possibilities into consideration:

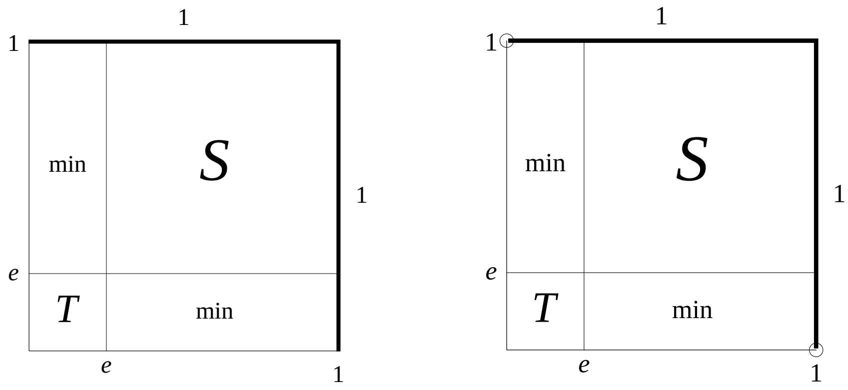
1. For $x = 0$ or $z = 0$ the two sides of the associativity equation in (2.4) are trivially 0.

2. For $x = 1$ and $U_2(y,z) = 0$ by using the strict monotonicity of T we obtain that either $y = 0$ or $z = 0$. This obviously means that the two sides of the associativity equation in (2.4) are equally 0. The proof is similar for $z = 1$ and $U_2(x,y) = 0$.

Remark 2.2. *Note that the strict property cannot be omitted in the Proposition 2.3 (i.e. the statement does not hold for arbitrary t-norms and t-conorms). For a counterexample let us choose* $T_{\mathbf{L}}$, $S_{\mathbf{P}}$, $e = 0.3$, $x = 1$, $y = 0.1$, *and* $z = 0.1$. *In this case* $U_2(1, U_2(0.1, 0.1)) = 0$, *while* $U_2(U_2(1, 0.1), 0.1) = 1$.

Proposition 2.4. *(See also Theorem 5 in [11].)* U_2 *in (2.3) is a uninorm if and only if* $T(x,y)$ *is a positive t-norm, and* $S(x,y)$ *is dual to a positive t-norm.*

Proof. This condition is sufficient, since in this case the proof is similar to that of Proposition 2.3.



(a) U_1 (b) U_2

Figure 1

Now we show that it is also necessary. From the proof of Proposition 2.2 the necessity of the second condition is trivial. We only need to show that if $U_2(x, y) = 0$ does not imply $x = 0$ or $y = 0$, then the associativity does not hold. Let us assume indirectly that there exist $y_0, z_0 \neq 0$, for which $U_2(y_0, z_0) = 0$. Obviously, $y_0, z_0 < e$. For $x = 1$ the left hand side of the associativity equation in (2.4) is trivially 0, while the left hand side is 1, which is a contradiction.

Proposition 2.5. *(See also [11].) Let T be a strict t-norm, S be a strict t-conorm and $e \in]0, 1[$. The function*

$$U_3(x,y) = \begin{cases} eT\left(\frac{x}{e}, \frac{y}{e}\right), & (x,y) \in [0,e]^2; \\ e + (1-e)S\left(\frac{x-e}{1-e}, \frac{y-e}{1-e}\right), & (x,y) \in [e,1]^2; \\ 0, & x = 0 \ or \ y = 0; \\ max(x,y), & otherwise \end{cases} \tag{2.5}$$

is a uninorm with neutral element e (see Figure 2a).

Proof. To prove that U_3 is a uninorm, we have to show that it is associative, commutative and that it has a neutral element $e \in (0, 1)$. Commutativity is obvious. From the properties of t-norms and t-conorms it follows immediately that $e \in (0, 1)$ is a neutral element.

Note that U_3 differs from U_{max} only at points where either $x = 0$ or $y = 0$. Since the associativity of $U_{max}(x, y)$ is already known from [7], we only need to concentrate on the border lines where at least one of the variables of U_3 is 0.

To examine the associative equation

$$U_3(x, U_3(y,z)) = U_3(U_3(x,y), z), \tag{2.6}$$

we need to take the following possibilities into consideration:

1. If $x = 0$ or $z = 0$, then $U_3(x, U_3(y,z)) = 0 = U_3(U_3(x,y), z)$.
2. If $U_3(y,z) = 0$ or $U_3(x,y) = 1$, then by using the strict monotonicity of $S(x,y)$, we get $x = 0$ or $y = 0$. Thus $U_3(x, U_3(y,z)) = 0 = U_3(U_3(x,y), z)$.

Remark 2.3. *Note that the strict property of T cannot be omitted in Proposition 2.5 (i.e. the statement does not hold for arbitrary t-norms). For a counterexample let us choose $T_{\mathbf{L}}$, $S_{\mathbf{P}}$, $e = 0.3$, $x = 0.1$, $y = 0.1$, and $z = 0.8$. In this case $U_3(0.1, U_3(0.1, 0.8)) = 0.8$, while $U_3(U_3(0.1, 0.1), 0.8) = 0$.*

Proposition 2.6. *(See also Theorem 4 in [11].) U_3 in (2.5) is a uninorm if and only if T is a positive t-norm.*

Proof. The condition is sufficient, since in this case the proof is similar to that of Proposition 2.5.

Now we show that it is also necessary.

Let us assume indirectly that there exist $x_0, y_0 \neq 0$, for which $U_3(x_0, y_0) = 0$. Obviously, $x_0, y_0 < e$. Let $z_0 \neq 0$ so that $U_3(y_0, z_0) \neq 0$. (It is easy to see that such z_0 always exists, since we can always choose $z_0 > e$.) In this case the right hand side of the associativity equation in (2.6) is trivially 0, while the left hand side is z_0, which is a contradiction.

Remark 2.4. *Note that Proposition 2.6 is dual to Proposition 2.2.*

Proposition 2.7. *(See also [11].) Let T be a strict t-norm, S be a strict t-conorm and $e \in]0,1[$. The function*

$$U_4(x,y) = \begin{cases} eT\left(\frac{x}{e}, \frac{y}{e}\right), & (x,y) \in [0,e]^2; \\ e + (1-e)S\left(\frac{x-e}{1-e}, \frac{y-e}{1-e}\right), & (x,y) \in [e,1]^2; \\ 0, & x = 0, y \neq 1 \text{ or } x \neq 1, y = 0; \\ max(x,y), & otherwise \end{cases} \tag{2.7}$$

is a uninorm with neutral element e (see figure 2b).

Proof. To prove that U_4 is a uninorm, we have to show that it is associative, commutative and that it has a neutral element $e \in (0,1)$. Commutativity is obvious. From the properties of t-norms and t-conorms it follows immediately that $e \in (0,1)$ is a neutral element.

Note that U_4 differs from U_3 only at points $(1, 0)$ and $(0, 1)$. Since the associativity of U_3 is already known (see Proposition 2.5), we only need to concentrate on the vertices of the unit square.

Since we examine the equation

$$U_4(x, U_4(y, z)) = U_4(U_4(x, y), z), \tag{2.8}$$

we need to take the following possibilities into consideration:

1. For $x = 1$ or $z = 1$ the two sides of the associativity equation in (2.4) are trivially 1.

2. For $x = 0$ and $U_4(y, z) = 1$ by using the strict monotonicity of $S(x, y)$ we obtain that either $y = 1$ or $z = 1$. This obviously means that the two sides of the associativity equation in (2.4) are equally 1. The proof is similar for $z = 0$ and $U_4(x, y) = 1$.

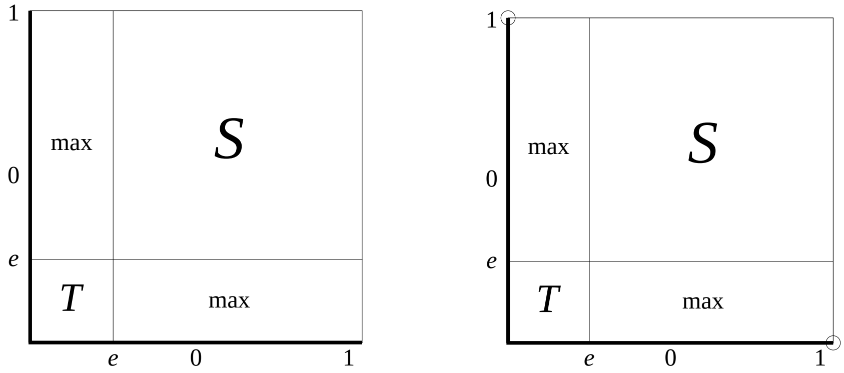
Remark 2.5. *Note that the strict property cannot be omitted in the Proposition 2.7 (i.e. the statement does not hold for arbitrary t-norms and t-conorms). For a counterexample let us choose* $T_{\mathbf{P}}$, $S_{\mathbf{L}}$, $e = 0.3$, $x = 0$, $y = 0.8$, *and* $z = 0.9$ *In this case* $U_4(0, U_4(0.8, 0.9)) = 0$, *while* $U_4(U_4(0, 0.8), 0.9) = 1$.

Proposition 2.8. *(See also Theorem 5 in [11].)* U_4 *in (2.7) is a uninorm if and only if* T *is a positive t-norm, and* S *is dual to a positive t-norm.*

Proof. The condition is sufficient, since in this case the proof is similar to that of Proposition 2.7.

Now we show that it is also necessary. From the proof of Proposition 2.6 the necessity of the first condition is trivial. We only need to show that if $U_4(x, y) = 1$ does not imply $x = 1$ or $y = 1$, then the associativity does not hold. Let us assume indirectly that there exists $y_0, z_0 \neq 1$, for which $U_4(y_0, z_0) = 1$. Obviously, $y_0, z_0 > e$. For $x = 0$ the left hand side of the associativity equation in (2.8) is trivially 1, while the left hand side is 0, which is a contradiction.

Remark 2.6. *Note that Proposition 2.8 is dual to Proposition 2.4.*



(a) U_3 (b) U_4

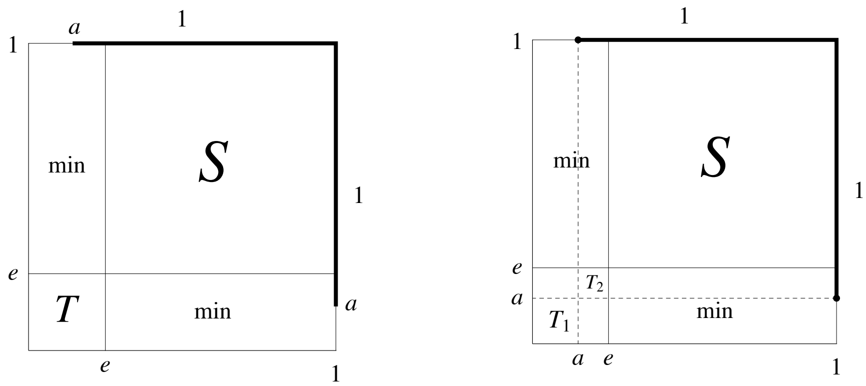
Figure 2

Let us now consider a function U_5 such that

(2.9)

$$U_5(x,y) = \begin{cases} eT\left(\frac{x}{e}, \frac{y}{e}\right), & (x,y) \in [0,e]^2; \\ e + (1-e)S\left(\frac{x-e}{1-e}, \frac{y-e}{1-e}\right), & (x,y) \in [e,1]^2; \\ 1, & x = 1 \textit{ and } y \geq a \textit{ or } y = 1 \textit{ and } x \geq a; \\ \min(x,y), & \textit{otherwise}, \end{cases}$$

where T is a t-norm, S is a t-conorm, $e \in]0,1[$, $a \in]0,e[$ (see Figure 3a).



(a) U_5 (b) U_5

Figure 3

We consider the conditions under which U_5 can be a uninorm. Suppose that U_5 is a uninorm with neutral element e.

Proposition 2.9. *If U_5 is a uninorm with neutral element e, then $U_5(a,a) = a$.*

Proof. From the conjunctive property of t-norms it follows immediately that $U_5(a,a) \leq a$. Suppose $U_5(a,a) < a$. Then by the definition of U_5, $U_5(1, U_5(a,a)) < 1$. On the other hand, by assiociativity, $U_5(U_5(1,a),a) = U_5(1,a) = 1$, a contradiction.

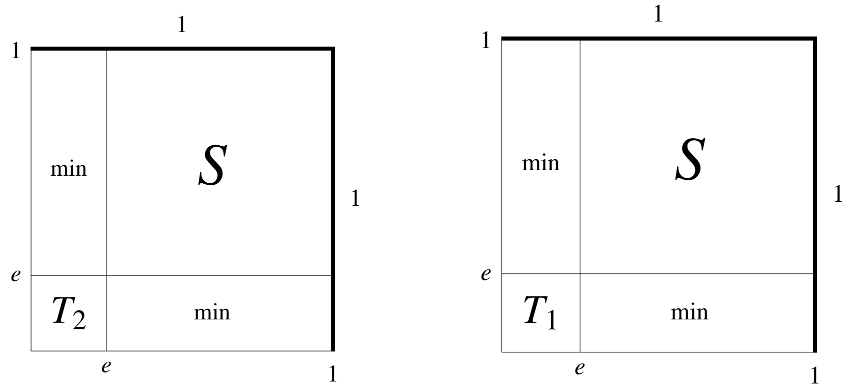
Corollary 2.1. *If U_5 is a uninorm with neutral value e, then T is an ordinal sum (see Figure 3b) of two t-norms, T_1 and T_2, i.e.*

$$T(x,y) = \begin{cases} a \cdot T_1\left(\frac{x}{a}, \frac{y}{a}\right), & \text{if } (x,y) \in [0,a]^2; \\ a + (e-a) \cdot T_2\left(\frac{x-a}{e-a}, \frac{y-a}{e-a}\right), & \text{if } (x,y) \in [a,e]^2; \\ min(x,y), & \text{otherwise.} \end{cases} \tag{2.10}$$

Corollary 2.2. *U_5' and U_5'' defined below are also uninorms:*

$$U_5'(x,y) = \begin{cases} eT_2\left(\frac{x}{e}, \frac{y}{e}\right), & (x,y) \in [0,e]^2; \\ e + (1-e)S\left(\frac{x-e}{1-e}, \frac{y-e}{1-e}\right), & (x,y) \in [e,1]^2; \\ 1, & x = 1 \text{ or } y = 1; \\ min(x,y), & \text{otherwise.} \end{cases} \tag{2.11}$$

$$U_5''(x,y) = \begin{cases} eT_1\left(\frac{x}{e}, \frac{y}{e}\right), & (x,y) \in [0,e]^2; \\ e + (1-e)S\left(\frac{x-e}{1-e}, \frac{y-e}{1-e}\right), & (x,y) \in [e,1]^2; \\ 1, & x = 1 \text{ or } y = 1; \\ min(x,y), & \text{otherwise.} \end{cases} \tag{2.12}$$



(a) U_5' (b) U_5''

Figure 4

Corollary 2.3. *From Proposition 2.1 and Corollary 2.2 it follows immediately, that if U_5 is a uninorm, then S must be dual to a positive t-norm.*

Proposition 2.10. *U_5 is a uninorm if and only if S is dual to a positive t-norm.*

Proof. The necessity of this condition is the statement of Corollary 2.3. Now we prove that is is also sufficient. Note that U_5 differs from U_{min} only at points (x, y), where $a \leq x < e$ and $y = 1$, or $a \leq y < e$ and $x = 1$. Since the associativity of U_{min} is already known, we only need to concentrate on these regions.

Since we examine the associativity equation

$$U_5(x, U_5(y, z)) = U_5(U_5(x, y), z), \tag{2.13}$$

we need to take the following possibilities into consideration:

1. $a \leq x < e$ and $U(y, z) = 1$. From $U(y, z) = 1$ by the dual-positivity of S we get
   (a) $y = 1$ and $z \geq a$, or
   (b) $y \geq a$ and $z = 1$.

   In case 1a both sides of the associative equation in (2.13) are trivially 1.

   In case 1b the left hand side of (2.13) is 1. In this region

$$U_5(x, y) = \begin{cases} \min(x, y), & y > e, y \neq 1; \\ a + (e-a) \cdot T_2\left(\frac{x-a}{e-a}, \frac{y-a}{e-a}\right) & a \leq y \leq e; \\ 1, & y = 1, \end{cases}$$

which means that $U_5(x,y) \geq a$, and therefore the right hand side of (2.13) is also 1.

2. $x = 1$ and $a \leq U_5(y,z) < e$. From the second condition it follows immediately that $a \leq y \leq e$ and $a \leq z \leq e$ hold and therefore both sides of the associativity equation in (2.13) is 1.

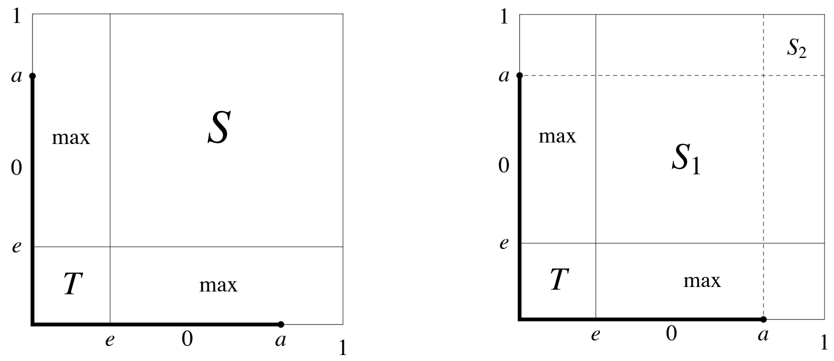
3. $z = 1$ and $a \leq U_5(x,y) < e$. The proof is similar to case 2.

4. $U_5(x,y) = 1$ and $a \leq z < e$. The proof is similar to case 1.

Now let us define a function U_7 the following way.

(2.14)
$$U_7(x,y) = \begin{cases} eT\left(\frac{x}{e}, \frac{y}{e}\right), & (x,y) \in [0,e]^2; \\ e + (1-e)S\left(\frac{x-e}{1-e}, \frac{y-e}{1-e}\right), & (x,y) \in [e,1]^2; \\ 0, & x = 0 \textit{ and } y \leq a \textit{ or } y = 0 \textit{ and } x \leq a; \\ \max(x,y), & \textit{otherwise}, \end{cases}$$

where T is a t-norm, S is a t-conorm, $e \in]0,1[$, $a \in]e,1[$ (see Figure 3a).



(a) U_7 (b)

Figure 5: U_7

We consider the conditions under which U_7 can be a uninorm. Suppose that U_7 is a uninorm with neutral element e.

Proposition 2.11. *If U_7 is a uninorm with neutral element e, then $U_7(a,a) = a$.*

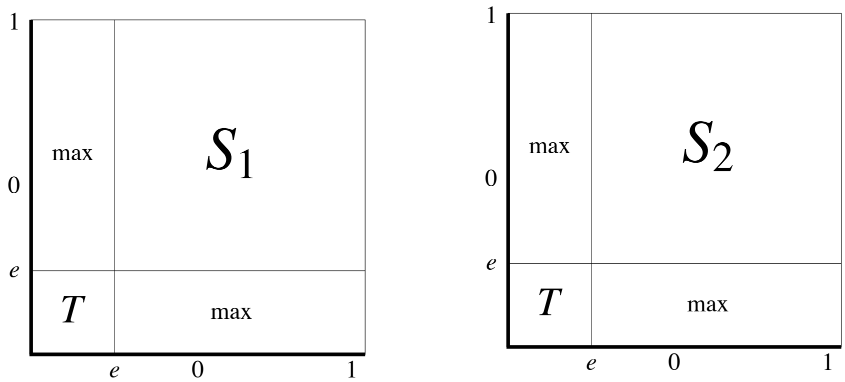
Proof. From the disjunctive property of t-conorms it follows immediately that $U_5(a,a) \geq a$. Suppose $U_7(a,a) > a$. Then by definition, $U_7(0, U_7(a,a)) = a$, on the other hand by associativity $U_7(0, U_7(a,a)) = U_7(U_7(0,a), a)) = 0$, a contradiction.

Corollary 2.4. *If U_7 is a uninorm with neutral value e, then S is an ordinal sum of two t-conorms, S_1 and S_2 (see Figure 5b).*

Corollary 2.5. *U_7' and U_7'' are also uninorms.*

$$U_7'(x,y) = \begin{cases} eT\left(\frac{x}{e}, \frac{y}{e}\right), & (x,y) \in [0,e]^2; \\ e + (1-e)S_1\left(\frac{x-e}{1-e}, \frac{y-e}{1-e}\right), & (x,y) \in [e,1]^2; \\ 0, & x = 1 \text{ or } y = 1; \\ max(x,y), & otherwise, \end{cases} \tag{2.15}$$

$$U_7''(x,y) = \begin{cases} eT\left(\frac{x}{e}, \frac{y}{e}\right), & (x,y) \in [0,e]^2; \\ e + (1-e)S_2\left(\frac{x-e}{1-e}, \frac{y-e}{1-e}\right), & (x,y) \in [e,1]^2; \\ 0, & x = 1 \text{ or } y = 1; \\ max(x,y), & otherwise. \end{cases} \tag{2.16}$$



(a) U_7' (b) U_7''

Figure 6

Corollary 2.6. *From Proposition 2.1 and Corollary 2.5 it follows immediately, that if U_7 is a uninorm, then T must be a positive t-norm.*

Proposition 2.12. *U_7 is a uninorm if and only if T is a positive t-norm.*

Proof. The necessity of this condition is the statement of Corollary 2.6. Now we prove that is is also sufficient. Note that U_7 differs from U_{max} only at points (x, y), where $e < x \leq a$ and $y = 0$, or $e < y \leq a$ and $x = 0$. Since the associativity of U_{max} is already known, we only need to concentrate on these regions.

Since we examine the associativity equation

$$U_7(x, U_7(y, z)) = U_7(U_7(x, y), z), \tag{2.17}$$

we need to take the following possibilities into consideration:

1. $e < x \leq a$ and $U(y, z) = 0$. From $U(y, z) = 0$ by the positivity of T we get
   (a) $y = 0$ and $z \leq a$, or
   (b) $y \leq a$ and $z = 0$.

   In case 1a both sides of the associative equation in (2.17) are trivially 0.

   In case 1b the left hand side of (2.17) is 0. In this region

   $$U_7(x, y) = \begin{cases} e + (1-e) \cdot S\left(\frac{x-a}{e-a}, \frac{y-a}{e-a}\right), & e \leq y \leq a; \\ \max(x, y), & 0 < y < e, y \neq 1; \\ 0, & y = 0, \end{cases}$$

   which means that $U_7(x, y) \leq a$, and therefore the right hand side of (2.17) is also 0.
2. $x = 0$ and $e < U_7(y, z) \leq a$. From the second condition it follows immediately that $e \leq y \leq a$ and $e \leq z \leq a$ hold and therefore both sides of the associativity equation in (2.13) is 0.
3. $z = 0$ and $e < U_7(x, y) \leq a$. The proof is similar to case 2.
4. $U_7(x, y) = 0$ and $e < z \leq a$. The proof is similar to case 1.

3. Conclusion

In this paper, some new construction methods of uninorms with fixed values along the borders were discussed and sufficient and necessary conditions were presented.

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