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# SOME IDENTITIES WITH APPLICATIONS

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*Dedicated to Professors Zoltán Daróczy and Imre Kátai*
*on the occasion of their 75th birthday*

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**Abstract.** For positive integers $r$ computable formulas for the sums of the doubly infinite series $\sum_{k\in\mathbb{Z}} 1/(z-k\pi)^r$ and $\sum_{k\in\mathbb{Z}}(-1)^k/(z-k\pi)^r$ will be presented. As applications exact lower and upper bounds for the derivatives of the functions cot and $1/\sin$ will be also shown.

## 1. The sums of the series $\sum\limits_{k\in\mathbb{Z}} \frac{1}{(z-k\pi)^r}$ and $\sum\limits_{k\in\mathbb{Z}} \frac{(-1)^k}{(z-k\pi)^r}$

Let us denote the set of integers, positive integers and complex numbers by $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{C}$, respectively.

**1.1.** It is clear that the doubly infinite series $\sum_{k\in\mathbb{Z}} \frac{1}{(z-k\pi)^r}$ is absolutely convergent on the domain

$$D := \mathbb{C} \setminus \{k\pi \mid k \in \mathbb{Z}\}$$

and uniformly convergent in every compact subset of $D$, if $r = 2, 3, \ldots$. Its sum function

$$A_r(z) := \sum_{k=-\infty}^{+\infty} \frac{1}{(z-k\pi)^r} \qquad (z \in D)$$

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is a $\pi$-periodic meromorphic function on $\mathbb{C}$ with $r$th order poles at the points $k\pi$ $(k \in \mathbb{Z})$.

If $r = 1$, then the above doubly infinite series does not converge (see for example $z = -1/2$), but the sequence of its symmetric partial sums is absolutely convergent and

$$\lim_{n\to+\infty} \sum_{k=-n}^{n} \left( \frac{1}{z-k\pi} + \frac{1}{k\pi} \right) = \cot z \qquad (z \in D),$$

moreover the convergence is uniform in every compact subset of $D$, see [3, p. 310]. Therefore let us define the function $A_1(z)$ by the following way

$$A_1(z) := \lim_{n\to+\infty} \sum_{k=-n}^{n} \frac{1}{z-k\pi} = \cot z \qquad (z \in D). \tag{1}$$

By the partial fraction decomposition of the function $1/\sin^r$ explicit formulas can be obtained for $A_{2r}$ (see [7]). Using another method we give formulas for *all* functions $A_r$ $(r \in \mathbb{N})$.

**Theorem 1.** *Let $r = 2, 3, \ldots$. The function $A_r$ can be written in the following form*

$$A_r(z) = \sum_{k=-\infty}^{+\infty} \frac{1}{(z-k\pi)^r} = \frac{1}{\sin^r z} S_r(\cos z) \qquad (z \in D), \tag{2}$$

*where $S_r$ are algebraic polynomials of degree $\leq (r-2)$ ($S_r \in \mathcal{P}_{r-2}$ shortly). They satisfy the recursive relation:*

$$\begin{gathered} S_2(z) = 1, \\ S_{r+1}(z) = zS_r(z) + \frac{1-z^2}{r} S_r'(z) \\ (z \in \mathbb{C},\ r = 2, 3, \ldots). \end{gathered} \tag{3}$$

**Proof.** If $r = 2$ then

$$\sum_{k=-\infty}^{+\infty} \frac{1}{(z-k\pi)^2} = \frac{1}{\sin^2 z} = \frac{1}{\sin^2 z} S_2(\cos z) \qquad (z \in D)$$

(see [5, p. 246]). In general case, we prove the statement by induction. Suppose that for an $r \in \mathbb{N}$ we have

$$\sin^r z \cdot \sum_{k=-\infty}^{+\infty} \frac{1}{(z-k\pi)^r} = S_r(\cos z) \qquad (z \in D).$$

After derivation, we get

$$r \sin^{r-1} z \cdot \cos z \cdot \sum_{k=-\infty}^{+\infty} \frac{1}{(z-k\pi)^r} - r \sin^r z \cdot \sum_{k=-\infty}^{+\infty} \frac{1}{(z-k\pi)^{r+1}} = \\ = -S_r'(\cos z) \cdot \sin z,$$

thus

$$\sin^{r+1} z \cdot \sum_{k=-\infty}^{+\infty} \frac{1}{(z-k\pi)^{r+1}} = \cos z \cdot \sin^r z \cdot \sum_{k=-\infty}^{+\infty} \frac{1}{(z-k\pi)^r} + \\ + \frac{1}{r} S_r'(\cos z) \cdot \sin^2 z = \\ = \cos z \cdot S_r(\cos z) + \frac{1-\cos^2 z}{r} S_r'(\cos z) = S_{r+1}(\cos z),$$

which means that the statement is true for $(r+1)$, too. ■

The first few $S_r$ polynomials obtained from the above recursive relation are as follows:

$$\begin{aligned} S_2(z) &= 1, \\ S_3(z) &= z, \\ S_4(z) &= \tfrac{1}{3}(2z^2+1), \\ S_5(z) &= \tfrac{1}{3}(z^3+2z), \\ S_6(z) &= \tfrac{1}{15}(2z^4+11z^2+2), \\ S_7(z) &= \tfrac{1}{45}(2z^5+26z^3+17z), \\ S_8(z) &= \tfrac{1}{315}(4z^6+114z^4+180z^2+17), \\ S_9(z) &= \tfrac{1}{315}(z^7+60z^5+192z^3+62z), \\ S_{10}(z) &= \tfrac{1}{2835}(2z^8+247z^6+1452z^4+1072z^2+62). \end{aligned} \tag{4}$$

Consequently for $A_r$ we obtain the following formulas which are valid at all points $z \in D$. The convergence is absolute in every $z \in D$ and uniform in every compact subset of $D$.

$$A_2(z) = \sum_{k=-\infty}^{+\infty} \frac{1}{(z-k\pi)^2} = \frac{1}{\sin^2 z},$$

$$A_3(z) = \sum_{k=-\infty}^{+\infty} \frac{1}{(z-k\pi)^3} = \frac{1}{\sin^3 z} \cos z,$$

$$A_4(z) = \sum_{k=-\infty}^{+\infty} \frac{1}{(z-k\pi)^4} = \frac{1}{\sin^4 z} \left[\tfrac{1}{3} + \tfrac{2}{3}\cos^2 z\right],$$

$$A_5(z) = \sum_{k=-\infty}^{+\infty} \frac{1}{(z-k\pi)^5} = \frac{1}{\sin^5 z}\left[\tfrac{2}{3}\cos z + \tfrac{1}{3}\cos^3 z\right],$$

$$A_6(z) = \sum_{k=-\infty}^{+\infty} \frac{1}{(z-k\pi)^6} = \frac{1}{\sin^6 z}\left[\tfrac{2}{15} + \tfrac{11}{15}\cos^2 z + \tfrac{2}{15}\cos^4 z\right],$$

$$A_7(z) = \sum_{k=-\infty}^{+\infty} \frac{1}{(z-k\pi)^7} = \frac{1}{\sin^7 z}\left[\tfrac{17}{45}\cos z + \tfrac{26}{45}\cos^3 z + \tfrac{2}{45}\cos^5 z\right].$$

**1.2.** Now we consider the doubly infinite series $\sum_{k\in\mathbb{Z}} \frac{(-1)^k}{(z-k\pi)^r}$, which is absolutely convergent on $D$ and uniformly convergent in every compact subset of $D$ for every $r = 2, 3, \ldots$. The sum

$$A_r^{\pm}(z) := \sum_{k=-\infty}^{+\infty} \frac{(-1)^k}{(z-k\pi)^r} \qquad (z \in D, \quad r = 2, 3, \ldots).$$

is a $\pi$-periodic meromorphic function on $\mathbb{C}$ with $r$th order poles at the points $k\pi$ $(k \in \mathbb{Z})$.

If $r = 1$, then

$$\lim_{n\to+\infty} \sum_{k=-n}^{n} \frac{(-1)^k}{z-k\pi} = \frac{1}{\sin z} \qquad (z \in D),$$

and the convergence is absolute in $D$ and uniform in every compact subset of $D$, see [5, p. 246]. Let

$$A_1^{\pm}(z) := \lim_{n\to+\infty} \sum_{k=-n}^{n} \frac{(-1)^k}{z-k\pi} = \frac{1}{\sin z} \qquad (z \in D). \tag{5}$$

By the partial fraction decomposition of the function $1/\sin^r$ explicit formulas can be obtained for $A_{2r-1}^{\pm}$ (see [7]). Using another method we give formulas for *all* functions $A_r^{\pm}$ $(r \in \mathbb{N})$.

**Theorem 2.** *Let $r = 1, 2, \ldots$. The function $A_r^{\pm}$ can be written in the following form*

$$A_r^{\pm}(z) = \sum_{k=-\infty}^{+\infty} \frac{(-1)^k}{(z-k\pi)^r} = \frac{1}{\sin^r z} Q_r(\cos z) \qquad (z \in D),$$

*where $Q_r \in \mathcal{P}_{r-1}$, satisfies the recursive relation:*

$$\begin{aligned} &Q_1(z) = 1, \\ &Q_{r+1}(z) = zQ_r(z) + \frac{1-z^2}{r} Q_r'(z) \\ &\qquad (z \in \mathbb{C}, \quad r = 1, 2, \ldots). \end{aligned} \tag{6}$$

The proof of this statement is similar to the proof of Theorem 1, so we omit the details.

The first few $Q_r$ polynomials obtained from the above recursive relation are as follows:

$$
\begin{aligned}
&Q_1(z) = 1,\\
&Q_2(z) = z,\\
&Q_3(z) = \tfrac{1}{2}(z^2+1),\\
&Q_4(z) = \tfrac{1}{6}(z^3+5z),\\
&Q_5(z) = \tfrac{1}{24}(z^4+18z^2+5),\\
&Q_6(z) = \tfrac{1}{120}(z^5+58z^3+61z),\\
&Q_7(z) = \tfrac{1}{720}(z^6+179z^4+479z^2+61),\\
&Q_8(z) = \tfrac{1}{5040}(z^7+543z^5+3111z^3+1385z),\\
&Q_9(z) = \tfrac{1}{40320}(z^8+1636z^6+18270z^4+19028z^2+1385).
\end{aligned}
\tag{7}
$$

Consequently for $A_r^{\pm}$ we obtain the following formulas which are valid at all points $z \in D$. The convergence is absolute in every $z \in D$ and uniform in every compact subset of $D$.

$$A_1^{\pm}(z) = \sum_{k=-\infty}^{+\infty} \frac{(-1)^k}{(z-k\pi)} = \frac{1}{\sin z},$$

$$A_2^{\pm}(z) = \sum_{k=-\infty}^{+\infty} \frac{(-1)^k}{(z-k\pi)^2} = \frac{1}{\sin^2 z}\cos z,$$

$$A_3^{\pm}(z) = \sum_{k=-\infty}^{+\infty} \frac{(-1)^k}{(z-k\pi)^3} = \frac{1}{\sin^3 z}\left[\tfrac{1}{2}+\tfrac{1}{2}\cos^2 z\right],$$

$$A_4^{\pm}(z) = \sum_{k=-\infty}^{+\infty} \frac{(-1)^k}{(z-k\pi)^4} = \frac{1}{\sin^4 z}\left[\tfrac{5}{6}\cos z+\tfrac{1}{6}\cos^3 z\right],$$

$$A_5^{\pm}(z) = \sum_{k=-\infty}^{+\infty} \frac{(-1)^k}{(z-k\pi)^5} = \frac{1}{\sin^5 z}\left[\tfrac{5}{24}+\tfrac{3}{4}\cos^2 z+\tfrac{1}{24}\cos^4 z\right],$$

$$A_6^{\pm}(z) = \sum_{k=-\infty}^{+\infty} \frac{(-1)^k}{(z-k\pi)^6} = \frac{1}{\sin^6 z}\left[\tfrac{61}{120}\cos z+\tfrac{29}{60}\cos^3 z+\tfrac{1}{120}\cos^5 z\right],$$

$$A_7^{\pm}(z) = \sum_{k=-\infty}^{+\infty} \frac{(-1)^k}{(z-k\pi)^7} = \frac{1}{\sin^7 z}\left[\tfrac{61}{720}+\tfrac{479}{720}\cos^2 z+\tfrac{179}{720}\cos^4 z+\tfrac{1}{720}\cos^6 z\right].$$

## 2. Inequalities

In this section we shall consider the functions $A_r$ and $A_r^{\pm}$ on $\mathbb{R}$ and shall give exact lower and upper estimates.

**2.1.** Let us define the functions*:

$$U_r(x) := (\sin^r x) \cdot A_r(x) = \sum_{k=-\infty}^{+\infty} \frac{\sin^r x}{(x-k\pi)^r} \qquad (x \in \mathbb{R}, \quad r = 1, 2, \ldots). \tag{8}$$

Theorem 1 states that $U_r$ is an algebraic polynomial of the cos function:

$$U_r(x) = S_r(\cos x) \qquad (x \in \mathbb{R}, r = 1, 2, 3, \ldots),$$

where $S_r$ $(\in \mathcal{P}_{r-2})$ satisfies the recursive relation (3).

**Theorem 3.** (i) *Let $r$ be any positive and even integer. Then $U_r$ is a $\pi$-periodic even function and*

$$m(U_r) \le U_r(x) \le 1 \qquad (x \in \mathbb{R}), \tag{9}$$

*where*

$$m(U_r) = S_r(0) = \frac{2^r(2^r-1)}{r!}|B_r| \quad (r = 2, 4, 6, \ldots). \tag{10}$$

*The values of $S_r(0)$ $(r = 2, 4, 6, \ldots)$ can be computed using the recursive relation* (3)*, $B_r$ denotes the $r$th Bernoulli number. On the interval $[0, \pi/2]$ the upper* (*lower*) *bound is attained exactly at the point $x = 0$ $(x = \pi/2)$.*

(ii) *Let $r$ be any positive and odd integer. Then $U_r$ is a $2\pi$-periodic even function and*

$$-1 \le U_r(x) \le 1 \qquad (x \in \mathbb{R}).$$

*On the interval $[0, \pi]$ the upper* (*lower*) *bound is attained exactly at the point $x = 0$ $(x = \pi)$.*

**Remark 1.** By (4) the first few values of $S_r(0)$ are

$$S_2(0) = 1, \quad S_4(0) = \tfrac{1}{3}, \quad S_6(0) = \tfrac{2}{15}, \quad S_8(0) = \tfrac{17}{315}, \quad S_{10} = \tfrac{62}{2835}.$$

*At points $x \in \mathbb{R}$ for which the function is formally undefined but has a finite limit, it is defined to be its limit.

**Remark 2.** We recall that the Bernoulli numbers $B_n$ ($n \in \mathbb{N}_0$) satisfy the recurrence relation

$$B_0 = 1,$$
$$\binom{n}{0}B_0 + \binom{n}{1}B_1 + \binom{n}{2}B_2 + \cdots + \binom{n}{n-1}B_{n-1} = 0 \qquad (n = 2, 3, \ldots)$$

(see [9] or [6, I, p. 682]). The first few Bernoulli numbers $B_n$ are

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42}, \quad B_8 = -\frac{1}{30}$$

with $B_{2n+1} = 0$ for $n \in \mathbb{N} \setminus \{1\}$.

**Proof of Theorem 3.** Let $r$ be an arbitrary positive integer. Then by (1) and (2) we have $U_r(x) = S_r(\cos x)$ ($x \in \mathbb{R}$). From the recursive relation (3) it follows that (by induction)

- the polynomial $S_r$ is even (odd) if $r$ is even (odd),
- every coefficients of $S_r$ are nonnegative,
- $S_r(1) = 1$ and $S_r(-1) = (-1)^r$.

Using these facts we obtain every statements of Theorem 3. We show only the assertion with respect to the minimum of $U_r$, if $r = 2, 4, 6, \ldots$. Using (2) we have

$$\min_{x\in\mathbb{R}} U_r(x) = \min_{x\in\mathbb{R}} S_r(\cos x) = S_r\left(\cos\frac{\pi}{2}\right) = S_r(0) =$$
$$= \sum_{k=-\infty}^{+\infty} \frac{\sin^r \frac{\pi}{2}}{\left(\frac{\pi}{2} - k\pi\right)^r} = \frac{2^r}{\pi^r}\sum_{k=-\infty}^{+\infty}\frac{1}{(2k-1)^r}.$$

Since

$$\sum_{k=-\infty}^{+\infty}\frac{1}{(2k-1)^r} = \frac{2(2^r-1)}{2^r}\sum_{k=1}^{+\infty}\frac{1}{k^r} \qquad (r = 2, 4, 6, \ldots),$$

and (see [6, p. 684] or [9, (40)])

$$B_r = 2(-1)^{r/2-1}\frac{r!}{(2\pi)^r}\sum_{k=1}^{+\infty}\frac{1}{k^r} \qquad (r = 2, 4, 6, \ldots), \tag{11}$$

thus we have

$$\min_{x\in\mathbb{R}} U_r(x) = m(U_r) = S_r(0) = \frac{2^r(2^r-1)}{r!}|B_r| \qquad (r = 2, 4, 6, \ldots) \qquad \blacksquare$$

The first few values of $m(U_r)$ $(r = 2, 4, \ldots)$ are

$$m(U_2) = 1, \quad m(U_4) = \frac{1}{3}, \quad m(U_6) = \frac{2}{15}, \quad m(U_8) = \frac{17}{315}, \quad m(U_{10}) = \frac{62}{2835}.$$

Since for every $r = 2, 4, 6, \ldots$

$$0 < m(U_r) = \frac{2^r(2^r-1)}{r!}|B_r| = 2\,\frac{2^r-1}{\pi^r}\sum_{k=1}^{+\infty}\frac{1}{k^r} < 4\,\frac{2^r-1}{\pi^r},$$

thus we have

$$\lim_{r\to+\infty} m(U_r) = 0.$$

**Remark 3.** From (11) we obtain the exact values of the Rieman's zeta-function at positive even integers:

$$\zeta(r) = \sum_{k=1}^{+\infty}\frac{1}{k^r} = (-1)^{r/2-1}\frac{(2\pi)^r}{2\cdot r!}B_r \quad (r = 2, 4, 6, \ldots).$$

Thus $\zeta(r)$ can be computed recursively using the Bernoulli numbers or the values $S_r(0)$.

**Remark 4.** In the theory of wavelet analysis the exact lower and upper bounds for the functions $U_r(x)$ $(x \in \mathbb{R})$ have important applications (see [2], [4, p. 24]). For positive even integers the inequality (9) is known, see [2, p. 90]. There the following explicit form for the lower bound $m(U_r)$ is proved:

$$m(U_r) = \frac{1}{(r-1)!}\prod_{k=1}^{r/2-1}\frac{(1+\lambda_k)^2}{|\lambda_k|} \qquad (r = 2, 4, 6, \ldots),$$

where $\lambda_k$'s are the roots of the Euler–Frobenius polynomials. But the exact values of $\lambda_k$'s are not known. The explicit forms (10) are more simple and they are computable, moreover the above proof of (9) is also simpler than in [2].

**2.2.** Let

$$V_r(x) := (\sin^r x)\cdot A_r^{\pm}(x) = \sum_{k=-\infty}^{+\infty}(-1)^k\frac{\sin^r x}{(x-k\pi)^r} \tag{12}$$
$$(x \in \mathbb{R}, \quad r = 1, 2, \ldots).$$

Theorem 2 states that $V_r$ is an algebraic polynomial of the cos function:

$$V_r(x) = Q_r(\cos x) \qquad (x \in \mathbb{R}, r = 1, 2, 3, \ldots),$$

where $Q_r$ $(\in \mathcal{P}_{r-1})$ satisfies the recursive relation (6).

**Theorem 4.** (i) *Let $r$ be an arbitrary positive odd integer. Then $V_r$ is a $\pi$-periodic even function and*

$$m(V_r) \le V_r(x) = \sum_{k=-\infty}^{+\infty} (-1)^k \frac{\sin^r x}{(x-k\pi)^r} \le 1 \qquad (x \in \mathbb{R},\ r = 1,3,5,\ldots),$$

*where*

$$m(V_r) = Q_r(0) \quad (r = 1,3,5,\ldots). \tag{13}$$

*The values of $Q_r(0)$ $(r = 1,3,5,\ldots)$ can be computed by the recursive relations* (6). *On the interval $[0,\pi/2]$ the upper* (*lower*) *bound is attained exactly at the point $x=0$ $(x=\pi/2)$.*

(ii) *Let $r$ be any positive and even integer. Then $V_r$ is a $2\pi$-periodic even function and*

$$-1 \le V_r(x) \le 1 \qquad (x \in \mathbb{R},\ r = 2,4,6,\ldots).$$

*On the interval $[0,\pi]$ the upper* (*lower*) *bound is attained exactly at the point $x=0$ $(x=\pi)$.*

**Proof.** Theorem 2 states that $V_r(x) = Q_r(\cos x)$ $(x \in \mathbb{R})$, where $Q_r$ is an even (odd) algebraic polynomial if $r$ odd (even); every coefficients of $Q_r$ are nonnegative; $Q_r(1) = 1$ and $Q_r(-1) = (-1)^{r+1}$. From these facts we obtain every statements of Theorem 4. ■

By (7) and (13) the first few values of $m(V_r)$ are

$$m(V_1) = 1, \quad m(V_3) = \tfrac{1}{2}, \quad m(V_5) = \tfrac{5}{24}, \quad m(V_7) = \tfrac{61}{720}, \quad m(V_9) = \tfrac{277}{8064}. \tag{14}$$

**Remark 5.** Since the algebraic polynomial $Q_r$ is even if $r$ is odd, thus

$$\min_{x\in\mathbb{R}} V_r(x) = \min_{x\in\mathbb{R}} Q_r(\cos x) = Q_r\left(\cos \tfrac{\pi}{2}\right) = Q_r(0) =$$

$$= \sum_{k=-\infty}^{+\infty} (-1)^k \frac{\sin^r \frac{\pi}{2}}{\left(\frac{\pi}{2} - k\pi\right)^r} = \frac{2^r}{\pi^r} \sum_{k=-\infty}^{+\infty} \frac{(-1)^{k+1}}{(2k-1)^r} = \frac{2^{r+1}}{\pi^r} \sum_{k=1}^{+\infty} \frac{(-1)^{k+1}}{(2k-1)^r}.$$

Consequently we obtain the following exact values for the sum of the series $\sum_{k\in\mathbb{N}} (-1)^{k+1}/(2k-1)^r$:

$$F_r := \sum_{k=1}^{+\infty} \frac{(-1)^{k+1}}{(2k-1)^r} = \frac{\pi^r}{2^{r+1}} Q_r(0) \qquad (r = 1,3,5,\ldots).$$

By (14) we have

$$F_1 = \frac{1}{4}\pi, \quad F_3 = \frac{1}{32}\pi^3, \quad F_5 = \frac{5}{1536}\pi^3, \quad F_7 = \frac{61}{184320}\pi^7, \quad F_9 = \frac{277}{8257536}\pi^9.$$

## 3. Derivatives

Not so easy to find explicit forms for the derivatives of the cot or of the $1/\sin$ functions. Starting from their series representation different computable formulas may be given. Exact lower and upper bounds for the derivatives will be also presented.

**3.1.** Let us consider first the derivatives of cot. From the identity

$$\cot z = \lim_{n\to+\infty} \sum_{k=-n}^{n} \frac{1}{z-k\pi} \qquad (z \in D)$$

(here the convergence is uniform in every compact subset of $D$) we have

$$\cot^{(r)} z = \frac{d^r}{dz^r} \cot z = (-1)^r r! \sum_{k=-\infty}^{+\infty} \frac{1}{(z-k\pi)^{r+1}} \qquad (z \in D, \quad r = 1, 2, \ldots).$$

The idea is that, the above sum can be written as algebraic polynomials of some trigonometric functions. We recall the following known result, which states that $\cot^{(r)}$ can be expressed as a polynomial of the cot function.

**Theorem 5.** (See [1, p. 161], [4, p. 23].) *Let $r$ be a positive integer. Then we have*

$$\frac{d^r}{dz^r} \cot z = (-1)^r P_r(\cot z) \qquad (z \in D),$$

*where the algebraic polynomial $P_r$ of degree $(r+1)$ obeys the following recursive relation*

$$P_0(z) = z, \qquad P_{r+1}(z) = (1+z^2)P_r'(z) \qquad (z \in \mathbb{C}).$$

Consequently for every $z \in D$ we have

$$\begin{aligned} \cot' z &= -(\cot^2 z + 1), \\ \cot'' z &= 2\cot^3 z + 2\cot z, \\ \cot^{(3)} z &= -(6\cot^4 z + 8\cot^2 z + 2), \\ \cot^{(4)} z &= 24\cot^5 z + 40\cot^3 z + 16\cot z. \end{aligned}$$

From Theorem 1 it follows that the derivatives of the cot function can be expressed also by polynomials of the cos function.

**Theorem 6.** *Let $r$ be a positive integer. Then we have*

$$\frac{d^r}{dz^r}\cot z = (-1)^r\, r!\, A_{r+1}(z) = \frac{(-1)^r\, r!}{\sin^{r+1} z}\, S_{r+1}(\cos z) \qquad (z \in D), \tag{15}$$

*where the algebraic polynomials $S_r$ are given by the recursive relation* (3).

Using (4) we have for every $z \in D$

$$\begin{aligned}
\cot' z &= \frac{-1}{\sin^2 z},\\
\cot'' z &= \frac{2}{\sin^3 z}\cos z,\\
\cot^{(3)} z &= \frac{-6}{\sin^4 z}\left[\tfrac{1}{3} + \tfrac{2}{3}\cos^2 z\right] = \frac{-2}{\sin^4 z}\left[1 + 2\cos^2 z\right],\\
\cot^{(4)} z &= \frac{4!}{\sin^5 z}\left[\tfrac{2}{3}\cos z + \tfrac{1}{3}\cos^3 z\right] = \frac{8}{\sin^5 z}\left[2\cos z + \cos^3 z\right].
\end{aligned}$$

The main advantage of the above representation of $\cot^{(r)}$ is that the their exact lower and upper bounds can be obtained more easily then by using Theorem 5. Indeed, from (8) and (15) we have:

$$\begin{aligned}
\mathrm{SC_r}(x) := \sin^{r+1} x \cdot \cot^{(r)} x &= (-1)^r\, r!\, \sin^{r+1} x \cdot A_{r+1}(x) =\\
&= (-1)^r\, r!\, U_{r+1}(x) = (-1)^r\, r!\, S_{r+1}(\cos x)\\
&(x \in \mathbb{R},\ \ r = 1, 2, 3, \ldots).
\end{aligned}$$

Thus by Theorem 3 we have

**Theorem 7.** (i) *Let $r$ be an odd positive integer. Then the function $\mathrm{SC_r}$ is a $\pi$-periodic even function and*

$$0 < r!\, m(U_{r+1}) \le -\mathrm{SC_r}(x) = -\sin^{r+1} x \cdot \cot^{(r)} x \le r! \qquad (x \in \mathbb{R},\ r = 1, 3, 5, \ldots),$$

*where $m(U_{r+1})$ is given by* (10). *On the interval $[0, \pi/2]$ the upper* (*lower*) *bound is attained exactly at the point $x = 0$ ($x = \pi/2$).*

(ii) *Let $r$ be an even positive integer. Then the function $\mathrm{SC_r}$ is a $2\pi$-periodic even function and*

$$-r! \le \mathrm{SC_r}(x) = \sin^{r+1} x \cdot \cot^{(r)} x \le r! \qquad (x \in \mathbb{R},\ r = 2, 4, 6, \ldots).$$

*On the interval $[0, \pi]$ the upper* (*lower*) *bound is attained exactly at the point $x = 0$ ($x = \pi$).*

**3.2.** The derivatives of the $\csc := 1/\sin$ function can be obtained from the identity

$$\csc z = \frac{1}{\sin z} = \lim_{n\to+\infty} \sum_{k=-n}^{n} \frac{(-1)^k}{z-k\pi} \qquad (z \in D)$$

(here the convergence is uniform in every compact subset of $D$):

$$\csc^{(r)} z = \left(\frac{1}{\sin}\right)^{(r)}(z) = (-1)^r r! \sum_{k=-\infty}^{+\infty} \frac{(-1)^k}{(z-k\pi)^{r+1}}$$
$$(z \in D, \quad r = 1, 2, \dots).$$

From Theorem 2 it follows that the derivatives of the $1/\sin$ function can be expressed also by polynomials of the cos function.

**Theorem 8.** *Let $r$ be a positive integer. Then we have*

$$\csc^{(r)} z = (-1)^r\, r!\, A^{\pm}_{r+1}(z) = \frac{(-1)^r\, r!}{\sin^{r+1} z}\, Q_{r+1}(\cos z) \qquad (z \in D), \tag{16}$$

*where the algebraic polynomials $Q_r$ are given by the recursive relation* (6).

Using (7) we have for every $z \in D$

$$\begin{aligned}
\csc' z &= \frac{-1}{\sin^2 z} \cdot \cos z,\\
\csc'' z &= \frac{1}{\sin^3 z}[1+\cos^2 z],\\
\csc^{(3)} z &= \frac{-1}{\sin^4 z}\left[5\cos z + \cos^3 z\right],\\
\csc^{(4)} z &= \frac{1}{\sin^5 z}\left[5 + 18\cos^2 z + \cos^4 z\right].
\end{aligned}$$

From (12) and (16) we have:

$$\begin{aligned}
\mathrm{SCS_r}(x) := \sin^{r+1} x \cdot \csc^{(r)} x &= (-1)^r\, r!\, \sin^{r+1} x \cdot A^{\pm}_{r+1}(x) =\\
&= (-1)^r\, r!\, V_{r+1}(x) = (-1)^r\, r!\, Q_{r+1}(\cos x)
\end{aligned}$$
$$(x \in \mathbb{R}, \quad r = 1, 2, 3, \dots).$$

Thus by Theorem 4 we have

**Theorem 9.** (i) *Let $r$ be an even positive integer. Then the function* $\mathrm{SCS_r}$ *is a $\pi$-periodic even function and*

$$0 < r!\, m(V_{r+1}) \le -\mathrm{SCS_r}(x) = -\sin^{r+1} x \cdot \csc^{(r)} x \le r!$$
$$(x \in \mathbb{R}, \; r = 1, 3, 5, \dots),$$

*where* $m(V_{r+1})$ *is given by* (14). *On the interval* $[0, \pi/2]$ *the upper* (*lower*) *bound is attained exactly at the point* $x = 0$ ($x = \pi/2$).

(ii) *Let* $r$ *be an odd positive integer. Then the function* $\mathrm{SCS_r}$ *is a* $2\pi$*-periodic even function and*

$$-r! \leq \mathrm{SCS_r}\,(x) = \sin^{r+1} x \cdot \csc^{(r)} x \leq r! \qquad (x \in \mathbb{R},\ r = 2, 4, 6, \ldots).$$

*On the interval* $[0, \pi]$ *the upper* (*lower*) *bound is attained exactly at the point* $x = 0$ ($x = \pi$).

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