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# ON CONDITIONAL FUNCTIONAL EQUATIONS WITH APPLICATIONS ON HYPERGROUPS

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*Dedicated to Professors Zoltán Daróczy and Imre Kátai on their 75th anniversary*

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**Abstract.** We characterize additive and exponential functions on some two-point support hypergroups using regularity theorems concerning conditional functional equations.

## 1. Introduction

The concept of DJS–hypergroup (according to the initials of C. F. Dunkl, R. I. Jewett and R. Spector) can be introduced using different axiom systems. We use Lasser's axiom system, which is due to R. Lasser. Here we omit the details. For the definition of hypergroups and basic facts the reader should refer to [1]. Nevertheless, here we give a heuristic introduction.

The base set of a hypergroup is a locally compact Haussdorff space $K$, and we suppose that there is a continuous mapping from $K \times K$ to the set of

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all compactly supported probability measures on $K$. This mapping is called *convolution*. Furthermore, there is an involutive homeomorphism from $K$ to $K$, called *involution*, and there is a fixed element of $K$, called *identity*. The Dirac–measure $\delta_x$ has a relevant role on hypergroups. The axiom system of a hypergroup contains the relations and operations between Dirac–measures, identity, convolution and involution. An important subclass of hypergroups is the class of commutative hypergroups. We say that a hypergroup is *commutative* or *Hermitian*, if the convolution of Dirac–measures is commutative.

Using the convolution one introduces translation operators, which are defined for every continuous complex valued function on $K$. If the hypergroup is not commutative we make a distinction between *right* and *left translation operators*. The definition of translation is as follows:

$$\tau_y f(x) = \int_K f(t)\; d(\delta_x * \delta_y)(t) \qquad (x, y \in K).$$

For the sake of simplicity we use the following suggestive notation

$$f(x * y) = \tau_y f(x) \qquad (x, y \in K).$$

In the following we focus on commutative hypergroups. The classical functional equations on hypergroups are introduced in the following way.

A continuous complex valued function $m$ on the hypergroup $K$ is called *exponential*, if it is not identically zero and

$$\tau_y m(x) = m(x)m(y)$$

holds for each $x, y$ in $K$.

A complex valued function $a$ on the hypergroup $K$ is called *additive*, if

$$\tau_y a(x) = a(x) + a(y)$$

holds for each $x, y$ in $K$.

Exponential and additive functions play a basic role in the theory of functional equations, in particular, in spectral synthesis and its applications. For basic information on these topics the reader should consult with [3], [7]. Recently new results concerning spectral analysis and spectral synthesis have been established on some types of hypergroups (see [5], [8], [9], [11]). There are also on-going investigations in connection with functional equations on hypergroups (see [4], [6], [12]). In this paper we give the description of exponential and additive functions on some types of two-point support hypergroups. These hypergroups are studied in [1]. Our problem leads us to the study of some conditional functional equations. Conditional functional equations play an important role in several applications of functional equations. For example, a recent volume (see [10]) is devoted to characterization problems in probability theory and statistics, where the results heavily depend on the solution of different conditional functional equations. While dealing with regular solutions of conditional functional equations the most powerful tools are the so-called regularity results, which play a fundamental role in this paper, too. Concerning such type of results, we refer to [2].

## 2. Conditional d'Alembert–type functional equations

**Theorem 2.1.** *Let* $f:[0,1]\to\mathbb{C}$ *be a continuous function satisfying* $f(0)=1$ *and*

$$f(x+y)+f(x-y)=2f(x)f(y)\,, \tag{1}$$

*whenever* $0\le y\le x$ *and* $x+y\le 1$. *Then there exists a complex number* $\lambda$ *such that* $f$ *has the form:*

$$f(x)=\cosh\lambda x \tag{2}$$

*for each* $x\ge 0$.

**Proof.** Let $\varphi=\Re f$ and $\psi=\Im f$, then $\varphi,\psi:[0,1]\to\mathbb{R}$ are continuous functions and we have

$$\varphi(x+y)+\varphi(x-y)=2\varphi(x)\varphi(y)-2\psi(x)\psi(y) \tag{3}$$

$$\psi(x+y)+\psi(x-y)=2\varphi(x)\psi(y)+2\psi(x)\varphi(y) \tag{4}$$

whenever $0\le y\le x$ and $x+y\le 1$, further $\varphi(0)=1$ and $\psi(0)=0$. Let $0<a\le 1$ such that $\varphi(x)>0$ for $0\le x\le a$.

Suppose first that $\varphi$ and $\psi$ are linearly dependent, that is, $\psi=c\varphi$ holds on $[0,1]$ for some real $c$. By $\varphi(0)=1$ and $\psi(0)=0$ it follows $c=0$, hence $\psi=0$, which implies that $f=\varphi$ is real valued. Let $A=\frac{a}{4}, B=\frac{a}{2}, C=0, D=\frac{a}{4}$, then we can apply Remark 22.12. in [2] for the functional equation (5) on the intervals $]A,B[$ and $]C,D[$ to infer that $f=\varphi$ is $\mathcal{C}^\infty$ on the interval $]0,\frac{a}{4}[$.

If $\varphi$ and $\psi$ are linearly independent, then with the same choice of $A,B,C,D$ we can apply the same result as above for the functional equation (3) on the intervals $]A,B[$ and $]C,D[$ to infer that $\varphi$ and $\psi$ are $\mathcal{C}^\infty$ on the interval $]0,\frac{a}{4}[$.

It follows that in any case $f$ is $\mathcal{C}^\infty$ on some interval $]0,K[$, where $0<K<1$.

Let $m=\min\left\{\frac{5K}{4},1\right\}$. Suppose that $\frac{3K}{4}<t<m$, then, by (5), the substitution $x=t-\frac{K}{4}, y=\frac{K}{4}$ gives $0\le y\le x\le 1$, $0\le x+y\le 1$ and

$$f(t)=2f\Big(t-\frac{K}{4}\Big)f\Big(\frac{K}{4}\Big)-f\Big(t-\frac{K}{2}\Big). \tag{5}$$

As $t-\frac{K}{4}$ and $t-\frac{K}{2}$ is in $]0,K[$, the right hand side is $\mathcal{C}^\infty$ on $\Big]\frac{3K}{4},m\Big[$. It follows that $f$ is $\mathcal{C}^\infty$ on $\Big]0,m\Big[$. If $\frac{5K}{4}\ge 1$, then we have that $f$ is $\mathcal{C}^\infty$ on $]0,1[$. If $\frac{5K}{4}<1$, then replacing $K$ by $\frac{5K}{4}$ and repeating the above argument after some steps we get that $f$ is $\mathcal{C}^\infty$ on $]0,1[$.

Differentiating (5) twice with respect to $y$ and then substituting $y=0$ we obtain that

$$f''(x)=cf(x) \tag{6}$$

for each $x$ in $]0,1[$ with $c=f''(0)$. As $f(0)=1$, our statement follows. ■

**Theorem 2.2.** *Let $f:[0,+\infty[\to\mathbb{C}$ be a continuous function satisfying $f(0)=$ $=1$ and*

$$f(x+y)+f(x-y)=2f(x)f(y)\,, \tag{7}$$

*whenever $0\le y\le x$. Then there exists a complex number $\lambda$ such that $f$ has the form:*

$$f(x)=\cosh\lambda x \tag{8}$$

*for each $x\ge 0$.*

**Proof.** The proof is similar to that of the previous theorem. ■

The following corollaries are easy consequences.

**Corollary 2.1.** *Let $f:[0,1]\to\mathbb{R}$ be a continuous function satisfying $f(0)=1$ and*

$$f(x+y)+f(x-y)=2f(x)f(y)\,, \tag{9}$$

*whenever $0\le y\le x$ and $x+y\le 1$. Then there exists a real number $\lambda$ such that $f$ has the form:*

$$f(x)=\cosh\lambda x \quad \text{or} \quad f(x)=\cos\lambda x \tag{10}$$

*for each $x$ in $[0,1]$.*

**Corollary 2.2.** *Let* $f : [0, +\infty[ \to \mathbb{R}$ *be a continuous function satisfying* $f(0) =$ $= 1$ *and*

$$f(x+y) + f(x-y) = 2f(x)f(y)\,, \tag{11}$$

*whenever* $0 \le y \le x$. *Then there exists a real number* $\lambda$ *such that* $f$ *has the form:*

$$f(x) = \cosh \lambda x \quad or \quad f(x) = \cos \lambda x \tag{12}$$

*for each* $x$ *in* $[0, +\infty[$.

## 3. Conditional square-norm equations

**Theorem 3.1.** *Let* $f : [0, 1] \to \mathbb{C}$ *be a continuous function satisfying*

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)\,, \tag{13}$$

*whenever* $0 \le y \le x$ *and* $x + y \le 1$. *Then there exists a complex number* $\lambda$ *such that* $f$ *has the form:*

$$f(x) = \lambda x^2 \tag{14}$$

*for each* $x$ *in* $[0, 1]$. *Moreover,* $f$ *is real if and only if* $\lambda$ *is real.*

**Proof.** Clearly $f(0) = 0$. Let $\varphi = \Re f$ and $\psi = \Im f$, then $\varphi, \psi : [0, 1] \to \mathbb{R}$ are continuous functions and we have

$$\varphi(x+y) + \varphi(x-y) = 2\varphi(x) + 2\varphi(y) \tag{15}$$

$$\psi(x+y) + \psi(x-y) = 2\psi(x) + 2\psi(y) \tag{16}$$

whenever $0 \le y \le x$ and $x + y \le 1$, further $\varphi(0) = \psi(0) = 0$. This means that the real and imaginary parts of $f$ satisfy the same functional equation (13), hence we may suppose that $f$ itself is real valued.

If $f$ and 1 are linearly dependent, then $f$ is constant, $f = 0$, hence our statement follows.

Assume that the functions $f$ and 1 are linearly independent. Then, similarly as in the proof of Theorem 2.1, with the same choice of $A, B, C$ and $D$, we infer that $f$ in $C^\infty$ on some interval $]0, K[$ , with a certain K in $]0, 1[$ .

Differentiating (13) three times with respect to $y$, then substituting $y = 0$ and differentiating again we obtain that

$$f'''(x) = 0 \tag{17}$$

for each $x$ in $]0, 1[$, which implies that $f$ is a quadratic polynomial on $[0, 1]$. Substituting into (13) our statement follows. The last assertion is obvious. ■

From this result the following theorem can be concluded. Its proof is rather similar to that of the previous theorem, therefore we will omit it.

**Theorem 3.2.** *Let $f : [0, +\infty[\to \mathbb{C}$ be a continuous function satisfying*

$$f(x + y) + f(x - y) = 2f(x) + 2f(y)\,, \tag{18}$$

*whenever $0 \leq y \leq x$. Then there exists a complex number $\lambda$ such that $f$ has the form:*

$$f(x) = \lambda x^2 \tag{19}$$

*for each $x \geq 0$. Moreover, $f$ is real if and only if $\lambda$ is real.*

## 4. Exponential and additive functions on the hypergroup $K_1 = ([0, 1], *)$

Let $K_1$ be the hypergroup on the interval $[0, 1]$ with the convolution defined by

$$\delta_x * \delta_y = \frac{1}{2}\delta_{x+y} + \frac{1}{2}\delta_{|x-y|}.$$

This is a one-dimensional compact hypergroup (see [1], Example 3.4.6 on p.191.). The characterizing equation of additive functions has the form

$$a(x + y) + a(|x - y|) = 2a(x) + 2a(y) \qquad (0 \leq x, y \leq 1)\,. \tag{20}$$

Our next theorem describes the additive functions on $K_1$.

**Theorem 4.1.** *Let $K_1$ be the hypergroup defined above. Then the continuous function $a : [0, 1] \to \mathbb{C}$ is an additive function on $K_1$ if and only if there exists a complex number $\lambda$ such that*

$$a(x) = \lambda x^2 \tag{21}$$

*holds for each $x$ in $[0, 1]$.*

**Proof.** If $a$ is additive on $K_1$ then it is continuous and satisfies equation (20), hence also equation (13). By Theorem 3.1 it has the given form.

Conversely, it is easy to check that any continuous function $a$ of the given form is an additive function on the $K_1$–hypergroup, hence the theorem is proved. ■

Using the convolution and the definition of exponential functions we have that the continuous function $m : [0,1] \to \mathbb{C}$ is an exponential on $K_1$ if and only if it satisfies

$$m(x+y) + m(x-y) = 2m(x)m(y) \qquad (0 \le y \le x,\ x+y \le 1)\,. \tag{22}$$

Using the above results we obtain the following statement.

**Theorem 4.2.** *Let $K_1$ be the two-point support hypergroup defined above. The continuous function $m : [0,1] \to \mathbb{C}$ is an exponential function on $K_1$ if and only if there exists a complex number $\lambda$ such that*

$$m(x) = \cosh \lambda x \tag{23}$$

*holds for each $x$ in $[0,1]$.*

**Proof.** If $m$ is an exponential on $K_1$ then it is continuous and satisfies equation (22), hence also equation (5). By Theorem 2.1 it has the given form.

Conversely, it is easy to check that any continuous function $m$ of the given form is an exponential function on the $K_1$–hypergroup, hence the theorem is proved. ■

## 5. Exponential and additive functions on the hypergroup $K_2 = ([0,+\infty[,*)$

The hypergroup $K_2$ is defined on the nonnegative reals $[0,+\infty[$ and the convolution is defined by

$$\delta_x * \delta_y = \frac{1}{2}\delta_{x+y} + \frac{1}{2}\delta_{x-y} \qquad (0 \le y < x).$$

This hypergroup $K_2 = ([0,+\infty[,*)$ is a noncompact one-dimensional hypergroup (see [1], Example 3.4.5 on p. 191.). On $K_2$ the characterizing equation of additive functions has the form

$$a(x+y) + a(x-y) = 2a(x) + 2a(y) \qquad (0 \le y < x)\,. \tag{24}$$

Now we exhibit the general form of additive functions on the hypergroup $K_2$.

**Theorem 5.1.** *Let $K_2$ be the two-point support hypergroup defined above. The continuous function $a : [0, +\infty[ \to \mathbb{C}$ is an additive function on $K_2$ if and only if there exists a complex number $\lambda$ such that*

$$a(x) = \lambda x^2 \tag{25}$$

*holds for each $x$ in $[0, +\infty[$.*

**Proof.** If $a$ is additive on $K_2$ then it is continuous and satisfies equation (24), hence also equation (13). By Theorem 3.2 it has the given form.

Conversely, it is easy to check that any function $a$ of the given form is an additive function on the $K_2$–hypergroup, hence the theorem is proved. ■

Using similar arguments and Theorem 2.2 we get the general form of exponential functions on $K_2$.

**Theorem 5.2.** *Let $K_2$ be the two-point support hypergroup defined above. The continuous function $m : [0, +\infty[ \to \mathbb{C}$ is an exponential function on $K_2$ if and only if there exists a complex number $\lambda$ such that*

$$m(x) = \cosh \lambda x \tag{26}$$

*holds for each $x$ in $[0, +\infty[$.*

## 6. Exponential and additive functions on the cosh–hypergroup

In the theory of hypergroups the Sturm–Liouville hypergroups represent an important subclass. A general introduction to this theory and examples can be found in [1], [6]. Sturm–Liouville hypergroups are generated by a Sturm–Liouville function. This function is continuous on the nonnegative reals and differentiable on the positive reals. Using the function cosh, we can build up a Sturm–Liouville hypergroup on the nonnegative reals, called the *cosh-hypergroup.*

Another way to introduce the cosh–hypergroup is the following. We consider the nonnegative reals as a base set and we introduce the convolution with the formula

$$\delta_x * \delta_y = \frac{\cosh(x+y)}{2\cosh x \cosh y}\delta_{x+y} + \frac{\cosh(|x-y|)}{2\cosh x \cosh y}\delta_{|x-y|}\,.$$

This hypergroup is also a special two-point support hypergroup, which is actually identical with the cosh-hypergroup (see [1]). We denote this hypergroup by $K_3 = ([0, +\infty[, *)$. Exponentials on this hypergroup satisfy the following equation:

$$\cosh(x+y)f(x+y) + \cosh(|x-y|)f(|x-y|) = 2\cosh x\, f(x)\cosh y\, f(y)\,.$$

The substitution $g(t) = \cosh t f(t)$ gives

$$g(x+y) + g(x-y) = 2g(x)g(y)$$

for $0 \le y \le x$, which shows the relation to the hypergroup $K_2$. Hence the following result is a consequence of the previous theorems.

**Theorem 6.1.** *Let $K_3$ be the* cosh*–hypergoup. Then the continuous function $m : [0, +\infty[\to \mathbb{C}$ is an exponential on $K_3$ if and only if there exists a complex number $\lambda$ such that*

$$m(x) = \frac{\cosh \lambda x}{\cosh x} \tag{27}$$

*holds for each $x$ in $[0, +\infty[$.*

The case of additive functions is a bit more complicated. We consider the equation of additive functions

$$\cosh(x+y)\, a(x+y) + \cosh(x-y)\, a(x-y) =$$

$$= 2\cosh x \cosh y\, a(x) + 2\cosh x \cosh y\, a(y) \quad (0 \le y \le x).$$

Differentiating this equation twice with respect to the variable $y$, then substituting $y = 0$ and $\lambda = a''(0)$ and using the properties $a(0) = 0$ and $a'(0) = 0$ we have

$$a''(x) + 2\frac{\sinh x}{\cosh x}\, a'(x) = \lambda\,.$$

This means that on the *cosh*–hypergroup an additive function is the solution of the previous equation. The next theorem describes the additive functions on $K_3$.

**Theorem 6.2.** *Let $K_3$ be the* cosh*–hypergoup. The continuous function $a : [0, +\infty[\to \mathbb{C}$ is an additive function on $K_3$ if and only if there exists a complex number $\lambda$ such that*

$$a''(x) + \frac{2\sinh x}{\cosh x} a'(x) = \lambda \tag{28}$$

*holds for each $x$ in $[0, +\infty[$.*

The solutions of equation (28) are special Bessel–functions.

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