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ON THE CHARACTERIZATION OF GENERALIZED DHOMBRES EQUATIONS HAVING NON CONSTANT LOCAL ANALYTIC OR FORMAL SOLUTIONS

Ludwig Reich (Graz, Austria)

Jörg Tomaschek (Ville de Luxembourg, Grand Duchy of Luxembourg)

Dedicated to Professors Zoltán Daróczy and Imre Kátai on their 75*th anniversary*

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Abstract. We discuss solvability conditions of the generalized Dhombres functional equation in the complex domain. Our focus lies on the so-called 'infinity' case, but also the z_0–case is investigated. That means that we consider solutions of a generalized Dhombres equation with initial value $f(\infty) = w_0$, $w_0 \neq 0$, or $f(\infty) = \infty$, or $f(z_0) = 1$ for $z_0 \neq 0, \infty$. For both situations we give a characterization of the generalized Dhombres equations which are solvable.

1. Introduction

For a complex variable the generalized Dhombres equation is given by

$$f(zf(z)) = \varphi(f(z)) \tag{1.1}$$

where the function φ is known and the function f is unknown. The equation was introduced by J. Dhombres in [2] in the year 1975. The complex case was first

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considered by L. Reich, J. Smítal, M. Štefánková in [3] in 2005. The authors started computing solutions f of (1.1) with $f(0) = 0$. This was followed by further works in the subsequent years. Nowadays formal solutions f of (1.1) where $f(0) = 0$, $f(0) = w_0 \neq 0$ and $f(z_0) = 1$, $z_0 \neq 0$ are well described. Recently the situations with $f(\infty) = w_0 \in \mathbb{C} \setminus \{0\}$ and $f(\infty) = \infty$ were considered.

In this note we are interested in the problem to find a necessary and sufficient condition (in terms of the given function φ) for the existence of non constant local analytic or formal solutions of (1.1) assuming certain initial values.

These considerations started with a formula, namely (32) in [3] and were continued in [5] with one theorem which deals with the universal solvability of an equation of the form (1.1) with $f(0) = w_0 \in \mathbb{C} \setminus \{0\}$, see Theorem 1.1 below. In this article we will mainly consider the remaining cases, namely $f(z_0) = 1$, $z_0 \neq 0$ and $f(\infty) = w_0 \in \mathbb{C} \setminus \{0\}$ or $f(\infty) = \infty$. We start with the so called 'infinity' case.

Before we do this we want to recall the existence theorem already mentioned, therefore we need some well-known transformations. In [4] the generalized Dhombres functional equation (1.1) was transformed to the equivalent linear functional equation

$$(w_0 + z^k)U(z) = U(\psi(u)) \tag{1.2}$$

by $f(z) = w_0 + g(z) = w_0 + T(z)^k$, $g(0) = 0$, $U(z) = T^{-1}(z)$, $\varphi(z) = w_0 + {}+\tilde{\varphi}(y - w_0)$, $\tilde{\varphi}(0) = 0$ and $\psi(z)^k = \tilde{\varphi}(z^k)$. The condition that a local analytic or formal solution f of (1.1) is non constant is then transformed to the equivalent condition that the linear equation (1.2) has a solution U different from 0. In [5] the following theorem was shown.

Theorem 1.1. *The functional equation* (1.2) $(w_0 + z^k)U(z) = U(\psi(z))$ *with* $\psi(z) \in z\mathbb{C}[\![z^k]\!]$ *has a solution* U *which is different from zero if and only if there exists a function* $\tilde{g}_0$ *with* $\tilde{g}_0(z) = z + \dots$ *such that*

$$\tilde{\varphi}(z) = \tilde{g}_0\left((w_0 + z)^k \tilde{g}_0^{-1}(z)\right). \tag{1.3}$$

Hence (1.1) *has a non constant solution* f *with* $f(0) = w_0$ *if and only if there exists* $\tilde{g}_0(z) = z + \dots$ *and* $k \in \mathbb{N}$ *such that* (1.3) *holds.*

In this article we want to derive, where it is possible, analogous characterizations of the solvability of generalized Dhombres equations as we have in Theorem 1.1. As we will see, this is possible in the 'infinity' case, in the case where $f(z_0) = 1$ and $z_0 \neq 0$ we get a somewhat different general condition for solvability. The difference lies in the fact that in the 'infinity' case a generator $\tilde{g}_0$ plays the main role, while there is no generator in the 'z_0' case. The main reason for that is that each possible solution of (1.1) with $f(z_0) = 1$ (for some $z_0 \neq 0$) is of the form $f(z) = 1 + g(z)$ with $g(z_0) = 0$, $g'(z_0) \neq 0$, in a neighbourhood of $z = z_0$.

For an introduction to formal power series we refer the reader to [1], by $\mathbb{E}$ we denote the roots of one. The ring of formal power series in one variable with the ususal addition, multiplication and substitution is given by

$$\mathbb{C}[\![z]\!] := \big\{F \;\big|\; F(z) = \sum_{\nu \geq 0} a_\nu z^\nu,\ a_\nu \in \mathbb{C}\big\}.$$

We will use the following definition

$$z\mathbb{C}[\![z^k]\!] := \big\{G \;\big|\; G(z) = \sum_{\nu \geq 0} a_{\nu k+1} z^{\nu k+1}\big\}.$$

2. The case $f(\infty) = w_0 \in \mathbb{C} \setminus \{0\}$ or $f(\infty) = \infty$

We consider the generalized Dhombres functional equation

$$f(zf(z)) = \varphi(f(z)) \tag{1.1}$$

with $f(\infty) = w_0 \in \mathbb{C} \setminus \{0\}$. We need to rewrite this equation, therefore we recall some transformations recently done in [6].
Let f be holomorphic in a neighbourhood of $z = \infty$ and let $f(\infty) = w_0 \in \in \mathbb{C} \setminus \{0\}$. We write $z = \frac{1}{u}$ where u is in a neighbourhood of zero and we get

$$f\left(\frac{1}{u} f\left(\frac{1}{u}\right)\right) = \varphi\left(f\left(\frac{1}{u}\right)\right)$$

or equivalently

$$f\left(\frac{\frac{1}{1}}{\frac{u}{f\left(\frac{1}{u}\right)}}\right) = \varphi\left(f\left(\frac{1}{u}\right)\right).$$

We define the new function $\hat{f}$ by $\hat{f}(u) = f\left(\frac{1}{u}\right)$, $\hat{f}$ is holomorphic in a neighbourhood of $u = 0$. Using $\hat{f}$ leads to

$$\hat{f}\left(\frac{u}{\hat{f}(u)}\right) = \varphi\left(\hat{f}(u)\right).$$

Since $\hat{f}(0) = w_0$ it is possible to write $\hat{f}(u) = w_0 + g(u)$ where g is holomorphic in a neighbourhood of zero and $g(0) = 0$. We get

$$w_0 + g\left(\frac{u}{w_0 + g(u)}\right) = \varphi(w_0 + g(u)).$$

For $u = 0$ we have $\varphi(w_0) = w_0$ and hence we can write φ as $\varphi(y) = w_0 + \tilde{\varphi}(y - -w_0)$ with $\tilde{\varphi}(0) = 0$. So we get the so-called transformed generalized Dhombres functional equation

$$g\left(\frac{u}{w_0 + g(u)}\right) = \tilde{\varphi}(g(u)) \tag{2.1}$$

which is different from the situation where $f(0) = w_0 \in \mathbb{C}$. We refer to the developed theory on generalized Dhombres equations and therefore we write $g(u) = T(u)^k$ for a $k \in \mathbb{N}$ and ord $T = 1$. Hence we have

$$T\left(\frac{u}{w_0 + T(u)^k}\right)^k = \tilde{\varphi}(T(u)^k),$$

or if we substitute $T^{-1}(u)$ for u

$$T\left(\frac{T^{-1}(u)}{w_0 + u^k}\right)^k = \tilde{\varphi}(u^k).$$

Taking the k-th root (see Lemma 2.1 below) and comparing the first coefficients on both sides leads to

$$T\left(\frac{T^{-1}(u)}{w_0 + u^k}\right) = \psi(u)$$

where $\psi(u)^k = \tilde{\varphi}(u^k)$. Here, like in (2.1), in the brackets a fraction occurs, this is different to the classical theory. We set $U = T^{-1}$ and so we obtain the linear functional equation

$$(w_0 + u^k)^{-1} U(u) = U(\psi(u)). \tag{2.2}$$

We need the following lemma which dates back to Lemma 2 in [4], nevertheless we want to give a straighter proof. Therefore we need the following definition.

Definition 2.1. Let $B(z) \in \mathbb{C}[\![z]\!]$. By $(B(z))^{\frac{1}{k}} \in \mathbb{C}[\![z]\!]$ we understand a (formal) series such that $\left(B(z)^{\frac{1}{k}}\right)^k = B(z)$, if such a series $(B(z))^{\frac{1}{k}}$ exists.

Lemma 2.1. *Let $A(z) \in \mathbb{C}[\![z]\!]$, ord $A = 1$ and $k \in \mathbb{N}$. There exists exactly k different determinations of $\left(A(z^k)\right)^{\frac{1}{k}}$, uniquely determined by the choice $a_1^{\frac{1}{k}}$ of their coefficient of z. Furthermore $\left(A(z^k)\right)^{\frac{1}{k}} \in z\mathbb{C}[\![z^k]\!]$.*

Proof. Let $C(z) \in \mathbb{C}[\![z]\!]$ such that

$$A(z^k) = C(z)^k. \tag{2.3}$$

Since $\operatorname{ord} A(z^k) = k$, we get $\operatorname{ord} C = 1$, hence $C(z) = \gamma_1 z(1 + \tilde{C}(z))$ with $\gamma_1 \neq 0$, $\tilde{C}(0) = 0$. Write $A(z) = a_1 z(1 + \tilde{A}(z))$, $\tilde{A}(0) = 0$, then

$$(2.3) \Leftrightarrow a_1 z^k(1 + \tilde{A}(z^k)) = \gamma_1^k z^k (1 + \tilde{C}(z))^k,$$

and hence

$$(2.3) \Leftrightarrow \begin{cases} a_1 = \gamma_1^k, \\ 1 + \tilde{A}(z^k) = (1 + \tilde{C}(z))^k, \tilde{C}(0) = 0. \end{cases}$$

By the implicit function theorem we obtain that $\tilde{C}(z)$ exists and is uniquely determined. Applying the binomial series $(1 + y)^{\frac{1}{k}} = 1 + \sum_{\nu \geq 1} \binom{\frac{1}{k}}{\nu} y^\nu$ and substituting $\tilde{A}(z^k)$ for y we get

$$\begin{aligned} 1 + \tilde{C}(z) &= (1 + \tilde{A}(z^k))^{\frac{1}{k}} \\ &= 1 + \sum_{\nu \geq 1} \binom{\frac{1}{k}}{\nu} \tilde{A}(z^k)^\nu \in \mathbb{C}[\![z^k]\!]. \end{aligned}$$

We have k different possibilities of $\gamma_1 = a_1^{\frac{1}{k}}$ and furthermore

$$C(z) = \gamma_1 z(1+\tilde{C}(z)) \in z\mathbb{C}[\![z^k]\!]. \qquad \blacksquare$$

To show the uniqueness of $\tilde{C}$ it is also possible to use other mehtods than the ones in the previous proof, therefore we have the following alternative proof.

Proof of the uniqueness of $\tilde{C}$. We start with

$$1 + \tilde{A}(z^k) = (1 + \tilde{C}(z))^k. \tag{2.4}$$

Differentiating this leads to

$$\tilde{A}'(z^k) k z^{k-1} = k(1 + \tilde{C}(z))^{k-1} \tilde{C}'(z)$$

which is equivalent to

$$\tilde{A}'(z^k) z^{k-1} = (1 + \tilde{C}(z))^k \frac{\tilde{C}'(z)}{1 + \tilde{C}(z)}.$$

By substituting (2.4) we obtain

$$\tilde{A}'(z^k) z^{k-1} = (1 + \tilde{A}(z^k)) \frac{\tilde{C}'(z)}{1 + \tilde{C}(z)}$$

or

$$\frac{\tilde{A}'(z^k)z^{k-1}}{1+\tilde{A}(z^k)} = \frac{\tilde{C}'(z)}{1+\tilde{C}(z)}, \tag{2.5}$$

with $\tilde{C}(0) = 0$. By Cauchy's Theorem there exists a unique $\tilde{C}(z)$ in a neighbourhood of 0. Then

$$\tilde{A}'(z^k)kz^{k-1} = k(1+\tilde{C}(z))^{k-1}\tilde{C}'(z),$$

implies

$$\frac{d}{\mathrm{d}z}((1+\tilde{A}(z^k)) - (1+\tilde{C}(z))^k) = 0.$$

Therefore we get

$$((1+\tilde{A}(z^k)) - (1+\tilde{C}(z))^k) = \delta \in \mathbb{C},$$

where substituting $z = 0$ leads to $\delta = 0$. This proves that $\tilde{C}$ satisfies (2.4). ∎

In [4] it was shown that a series $F \in \mathbb{C}[\![z^k]\!]$ if and only if $F \circ L_\eta = F$ where $L_\eta(z) = \eta z$ and η is a root of one of order k. We will use this in the proof of the following lemma.

Lemma 2.2. *Let $F \in z\mathbb{C}[\![z^k]\!]$ with $F(z) = \sum_{\nu=l}^{\infty} \beta_{\nu k+1} z^{\nu k+1}$. Then we have $F(z)^k \in \mathbb{C}[\![z^k]\!]$.*

Proof. We write

$$F(z) = \beta_{lk+1} z^{lk+1} \left(1 + \sum_{\nu=1}^{\infty} \tilde{\beta}_{\nu k} z^{\nu k}\right).$$

Let η be a root of one of order k, then using $(\eta^{lk+1})^k = (\eta^{lk}\eta)^k = \eta^k = 1$ we have

$$\begin{aligned} F(\eta z)^k &= \beta_{lk+1}^k \left(\eta^{lk+1}\right)^k \left(z^{lk+1}\right)^k \left(1 + \sum_{\nu=1}^{\infty} \tilde{\beta}_{\nu k} \eta^{\nu k} z^{\nu k}\right)^k = \\ &= \beta_{lk+1}^k \left(z^{lk+1}\right)^k \left(1 + \sum_{\nu=1}^{\infty} \tilde{\beta}_{\nu k} z^{\nu k}\right)^k = \\ &= F(z)^k \end{aligned}$$

and hence $F(z)^k \in \mathbb{C}[\![z^k]\!]$. ∎

Since the following theorem depends on a special representation of the solutions of a generalized Dhombres functional equation, it is necessary to consider the next lemma.

Lemma 2.3. *Let $F(z) \in \mathbb{C}[\![z^k]\!]$, $F(z) = c_k z^k + \dots$, $c_k \neq 0$ for some $k \in \mathbb{N}$. Then there exists a series $G \in \mathbb{C}[\![z]\!]$ such that $G(c_k z^k) = F(z)$.*

Proof. We have

$$\begin{aligned} F(z) &= \sum_{\nu \geq 1} c_{\nu k} z^{\nu k} = \\ &= c_k z^k (1 + \sum_{\nu \geq 2} c_{\nu k} c_k^{-\nu} (c_k z^k)^{\nu - 1}) = \\ &= G(c_k z^k). \end{aligned}$$

■

Together with the previous lemmata we can prove our main theorem. This theorem describes the solvability of a generalized Dhombres functional equation. The proof is done in a similar way as it was done in [5]. Note that in this characterization we use the so-called generator $\tilde{g}_0$.

Theorem 2.1. *Let $w_0 \in \mathbb{C} \setminus \{0\}$. Then the transformed generalized Dhombres functional equation*

$$g\left(\frac{u}{w_0 + g(u)}\right) = \tilde{\varphi}(g(u)) \tag{2.1}$$

has a non constant solution g if and only if there exists a function $\tilde{g}_0$, $\tilde{g}_0(u) = u + \dots$ and a $k \in \mathbb{N}$ such that

$$\tilde{\varphi}(u) = \tilde{g}_0\left(\frac{\tilde{g}_0^{-1}(u)}{(w_0 + u)^k}\right) \tag{2.6}$$

holds.
Hence (1.1) has a non constant solution f with $f(\infty) = w_0 \notin \{0, \infty\}$ if and only if there exists $\tilde{g}_0(z) = z + \dots$ and $k \in \mathbb{N}$ such that (2.6) holds.

Proof. Assume that there exists a $\tilde{g}_0$, $\tilde{g}_0(u) = u + \dots$ such that $\tilde{\varphi}(y) = \tilde{g}_0\left(\frac{\tilde{g}_0^{-1}(y)}{(w_0+y)^k}\right)$ holds. Then $\tilde{g}_0$ is invertible and (2.6) becomes equivalent to

$$\tilde{\varphi}(\tilde{g}_0(u)) = \tilde{g}_0\left(\frac{u}{(w_0 + \tilde{g}_0(u))^k}\right).$$

We apply the holomorphic transformation $y = c_k u^k$, see [1], and hence we get

$$\tilde{\varphi}(\tilde{g}_0(c_k u^k)) = \tilde{g}_0\left(\frac{c_k u^k}{(w_0 + \tilde{g}_0(c_k u^k))^k}\right) = \tilde{g}_0\left(c_k\left(\frac{u}{(w_0 + \tilde{g}_0(c_k u^k))}\right)^k\right).$$

We define $g(u) := \tilde{g}_0(c_k u^k)$ and therefore we obtain

$$\tilde{\varphi}(g(u)) = g\left(\frac{u}{(w_0 + g(u))}\right). \tag{2.1}$$

In [6] we have shown that this equation has non constant solutions g.
Let us assume that the equation $g\left(\frac{u}{w_0+g(u)}\right) = \tilde{\varphi}(g(u))$ has non constant solutions g with ord $g = k$. Then it was shown before, that this equation can be transformed to the linear functional equation

$$(w_0 + u^k)^{-1} U(u) = U(\psi(u)). \tag{2.2}$$

So we have to show that if (2.2) is solvable, then there exists a function $\tilde{g}_0$ such that (2.1) holds. From [6] and by Lemma 2.1 the function $\psi(u) = \frac{1}{w_0} u + \ldots \in$ $\in u\mathbb{C}[\![u^k]\!]$. We write $U(u) = u_1 u U^\star(u)$, $U^\star(u) = 1 + \ldots$ and hence we obtain the equivalent equation

$$\frac{1}{w_0(1 + w_0^{-1} u^k)} u_1 u U^\star(u) = u_1 \frac{1}{w_0} u \psi^\star(u) U^\star(\psi(u))$$

or

$$\frac{1}{(1 + w_0^{-1} u^k)\psi^\star(u)} U^\star(u) = U^\star(\psi(u))$$

where $\psi^\star(u) = 1 + \ldots \in \mathbb{C}[\![u^k]\!] \cap \Gamma_1$. Hence

$$\frac{1}{(1 + w_0^{-1} u^k)\psi^\star(u)} = 1 + \cdots$$

which allows us to use the formal logarithm, we set

$$\begin{aligned} A(u) &:= \operatorname{Ln}\left(\frac{1}{(1 + w_0^{-1} u^k)\psi^\star(u)}\right) \in \mathbb{C}[\![u^k]\!] \\ X^\star(u) &:= \operatorname{Ln} U^\star(u), \quad \text{ord } X^\star > 0. \end{aligned}$$

We obtain

$$A(u) + X^\star(u) = X^\star(\psi(u)).$$

Since $A(u) \in \mathbb{C}[\![u^k]\!]$ we decompose $X^\star$ in the following way, we write $X^\star(u) =$ $= X_1^\star(u) + X_2^\star(u)$ where

$$X_1^\star(u) = \sum_{\substack{\nu \geq 1 \\ \nu \equiv 0 \pmod{k}}} \xi_\nu u^\nu, \qquad X_2^\star(u) = \sum_{\substack{\nu \geq 1 \\ \nu \not\equiv 0 \pmod{k}}} \xi_\nu u^\nu.$$

Then we have

$$A(z) = X_1^\star(\psi(u)) - X_1^\star(u) + X_2^\star(\psi(u)) - X_2^\star(u)$$

where $X_1^\star(\psi(u)) - X_1^\star(u) \in \mathbb{C}[\![u^k]\!]$ and $X_2^\star(\psi(u)) - X_2^\star(u) \notin \mathbb{C}[\![u^k]\!]$, if this expression is not zero. This leads to the system

$$\begin{cases} A(u) = X_1^\star(\psi(u)) - X_1^\star(u), \\ \quad 0 = X_2^\star(\psi(u)) - X_2^\star(u). \end{cases} \tag{2.7}$$

The system (2.7) has a solution $X_1^\star \in \mathbb{C}[\![u^k]\!]$ (see [6]). Therefore $U(u) = = u_1 u \exp(X_1^\star(u)) \in u\mathbb{C}[\![u^k]\!]$ and after reversing our calculations also $T(u) \in \in u\mathbb{C}[\![u^k]\!]$. By Lemma 2.2 the function $g(u) = T(u)^k \in \mathbb{C}[\![u^k]\!]$ with $g(u) = = \frac{1}{u_1^k} u^k + \cdots$ and hence by Lemma 2.3 there exists a function $\tilde{g}_0$, $\tilde{g}_0(y) = y + \ldots$ such that $g(c_k u^k) = \tilde{g}_0(u)$ which proves this theorem. ■

The situation where $f(\infty) = \infty$ reduces to the situation $f(0) = 0$, see [6], and hence it is covered by [3].

3. The case $f(z_0) = 1$, $z_0 \neq 0$

As it is said in the introduction, in this situation we do not obtain such a general condition for solvability like in the previous one. We will use this section to show why this is even so. In fact the difference as it was said before lies in the missing of a generator $\tilde{g}_0$.

In this case we also need some transformations, these are the following ones. Let f be a holomorphic solution of the generalized Dhombres functional equation (1.1) with $f(z_0) = 1$, $z_0 \neq 0$ then we obtain the equation

$$g(z_0 g(\zeta) + \zeta + \zeta g(\zeta)) = \tilde{\varphi}(g(\zeta)) \tag{3.1}$$

where $f(z) = w_0 + \tilde{g}(z)$, $\tilde{g}(z_0) = 0$ and $\tilde{g}(z) = \tilde{g}(z_0 + \zeta) =: g(\zeta)$, $g(\zeta) = c_k\zeta^k + \cdots$, $k \geq 1$, $c_k \neq 0$ and ζ is defined as $\zeta = z - z_0$. Furthermore we use $\varphi(w) = \varphi(w_0) + \tilde{\varphi}(\omega)$ with $\omega = w - w_0$ and $\tilde{\varphi}(w_0) = 0$ and $\tilde{\varphi}(y) = d_m y^m + \cdots$ with $d_m \neq 0$, for the details we refer the reader to [5] Section 4.1. If we use $T(z)^k = g(z)$ and $U = T^{-1}$ equation (3.1) becomes

$$z_0\zeta^k + (1 + \zeta^k)U(\zeta) = U(\hat{\varphi}(\zeta)) \tag{3.2}$$

with $\hat{\varphi}(\zeta)^k = \tilde{\varphi}(\zeta^k)$ and $\tilde{\varphi}(\zeta) = d_m\zeta^m + \cdots$ as we will see later.
We have the following theorem.

Theorem 3.1. *Let $z_0 \in \mathbb{C}$, $z_0 \neq 0$, $\tilde{\varphi}(y) \in \mathbb{C}[\![y]\!]$ with $\tilde{\varphi}(0) = 0$. Let $g \in \mathbb{C}[\![\zeta]\!]$, $g \neq 0$ and ord $g = k \geq 1$ be a solution of*

$$g(z_0 g(\zeta) + \zeta + \zeta g(\zeta)) = \tilde{\varphi}(g(\zeta)). \tag{3.1}$$

Then ord $g = k = 1$.

Proof. We assume that $k \geq 2$. Let $g(\zeta) = c_k\zeta^k + \ldots$, $c_k \neq 0$, then $g(z_0g(\zeta) + +\zeta+\zeta g(\zeta)) = c_k\zeta^k + \ldots$ and hence ord $g(z_0g(\zeta)+\zeta+\zeta g(\zeta)) = k$. Equation (3.1) implies $\tilde{\varphi} \neq 0$ and ord $\tilde{\varphi} \geq 1$, we have $k =$ ord $\tilde{\varphi} \cdot k$ which leads to ord $\tilde{\varphi} = 1$ and therefore we write $\tilde{\varphi}(y) = d_1y + d_2y^2 + \ldots$, $d_1 \neq 0$. If we compare the coefficients of ζ^k in (3.1) we obtain $c_k = d_1c_k$ which gives us $d_1 = 1$, because $c_k \neq 0$. So we obtain $\tilde{\varphi}(y) = y + d_2y^2 + \ldots$.
We have $\tilde{\varphi}(y) \neq y$, because if we assume that $\tilde{\varphi}(y) = y$, (3.1) is given by

$$g(z_0g(\zeta) + \zeta + \zeta g(\zeta)) = g(\zeta).$$

We write $g(\zeta) = T(\zeta)^k$ with ord $T = 1$, this gives us

$$T(z_0T(\zeta)^k + \zeta + \zeta T(\zeta)^k)^k = T(\zeta)^k.$$

After comparing the coefficients of ζ^k we obtain

$$T(z_0T(\zeta)^k + \zeta + \zeta T(\zeta)^k) = T(\zeta).$$

We apply T^{-1} on both sides, hence

$$\zeta + z_0T(\zeta)^k + \zeta T(\zeta) = \zeta$$

or

$$(z_0 + \zeta)T(\zeta)^k = 0$$

which is a contradiction because $z_0 + \zeta \neq 0$ and $T(\zeta)^k \neq 0$. Hence $\tilde{\varphi}(y) = = y + d_ry^r + \ldots$, $r \geq 2$, $d_r \neq 0$.
Next we consider (3.1) with the transformation $g(\zeta) = T(\zeta)^k$, we have

$$T(z_0T(\zeta)^k + \zeta + \zeta T(\zeta)^k)^k = \tilde{\varphi}(T(\zeta)^k).$$

Substituting $T(\zeta)$ for ζ leads to

$$T(z_0\zeta^k + T^1(\zeta) + \zeta^kT^{-1}(\zeta))^k = \tilde{\varphi}(T(\zeta)^k).$$

Then, by Lemma 2.1 there exists a uniquely determined $\psi \in \mathbb{C}[\![z]\!]$ with $\tilde{\varphi}(\zeta^k) = = \psi(\zeta)^k$, $\psi(y) = y + \ldots$. Writing $U := T^{-1}$ we obtain

$$z_0\zeta^k + (1 + \zeta^k)U(\zeta) = U(\psi(\zeta))$$

or

$$z_0(1 + \zeta^k) + (1 + \zeta^k)U(\zeta) = z_0 + U(\psi(\zeta)).$$

This is the same as

$$(1 + \zeta^k)(z_0 + U(\zeta)) = z_0 + U(\psi(\zeta)),$$

which is equivalent to

$$(1+\zeta^k)(1+z_0^{-1}U(\zeta)) = 1+z_0^{-1}U(\psi(\zeta)).$$

We write $V(\zeta) := 1+z_0^{-1}U(\zeta)$, then we have $V(\zeta) = 1+\dots$ and

$$(1+\zeta^k)V(\zeta) = V(\psi(\zeta)). \tag{3.3}$$

Let V_1 and V_2 be two solutions of (3.3), with $V_j(\zeta) = 1+\dots$, $j = 1,2$. We define $W(\zeta) = \frac{V_1(\zeta)}{V_2(\zeta)}$. Equation (3.3) implies

$$W(\zeta) = W(\psi(\zeta)), \quad W(\zeta) = 1+\dots. \tag{3.4}$$

We have $\psi(\zeta) = \zeta+\dots$, $\psi(\zeta) \neq \zeta$, and hence let $(\psi_t)_{t\in\mathbb{C}}$ be the analytic embedding of ψ. Then (3.4) implies $W(\zeta) = W(\psi_t(\zeta))$ for all $t \in \mathbb{C}$, see [5]. We differentiate this equation with respect to t, hence we obtain

$$0 = \frac{\partial W}{\partial y}(\psi_t(\zeta)) \cdot \frac{\partial \psi_t(\zeta)}{\partial t}.$$

We set $t = 0$ and so we have $\frac{\partial W}{\partial y} = 0$. Therefore $W = 1$ and hence $V_1 = V_2$. We conclude that (3.3) has at most one solution V with $V(0) = 1$.
By Lemma 2.1 we know that $\psi(\zeta) \in \zeta\mathbb{C}[\![\zeta^k]\!]$, $\psi(\zeta) = \zeta+\dots$. Let $\epsilon = e^{\frac{2\pi i}{k}}$, we substitute $\epsilon\zeta$ for ζ in (3.3), hence, with $\psi(\epsilon\zeta) = \epsilon\psi(\zeta)$ we get

$$(1+\zeta^k)V(\epsilon\zeta) = V(\epsilon\psi(\zeta)).$$

Therefore $V(\epsilon\zeta)$ is also a solution of (3.3) with $V(\epsilon\cdot 0) = 1$. Hence $V(\zeta) = V(\epsilon\zeta)$, and so we have $V(\zeta) \in \mathbb{C}[\![\zeta^k]\!]$. We write $V(\zeta) = \tilde{V}(\zeta^k)$ with $\tilde{V}(y) \in \mathbb{C}[\![y]\!]$. We substitute this representation into (3.3), we get

$$(1+\zeta^k)\tilde{V}(\zeta^k) = \tilde{V}(\psi(\zeta)^k)$$

which is equivalent to

$$(1+\zeta^k)\tilde{V}(\zeta^k) = \tilde{V}(\tilde{\varphi}(\zeta^k)).$$

Therefore we have

$$(1+y)\tilde{V}(y) = \tilde{V}(\tilde{\varphi}(y))$$

in $\mathbb{C}[\![y]\!]$ with $\tilde{V}(0) = 1$. We define $\operatorname{Ln}\tilde{V} =: X$ and hence we get

$$\operatorname{Ln}(1+y) = X(\tilde{\varphi}(y)) - X(y)$$

with ord $X \geq 1$. We have $\operatorname{Ln}(1+y) = y+\cdots$ and therefore ord $(\operatorname{Ln}(1+y) = y+\cdots) = 1$. Let ord $X = l \geq 1$, then ord $(X(\tilde{\varphi}(y)) - X(y)) > l \geq 1$ because $\tilde{\varphi}(y) = y+\dots$ which is a contradiction. Hence $k = 1$. ∎

We want to prove the following theorem in a different way than it was done in [5], because then we can see the dependence of the solutions on the initial value z_0.

Theorem 3.2. *Let $\tilde{\varphi}(\zeta) \in \mathbb{C}[\![\zeta]\!]$, $\tilde{\varphi}(\zeta) = d_1\zeta + \ldots$, $d_1 \neq 0$. If d_1 is no root of one, then* (3.2) *has a unique solution.*

Proof. We consider the functional equation (3.2) for $k = 1$, computing both sides of (3.2), together with $U(\zeta) = u_1\zeta + u_2\zeta + \ldots$ leads us to

$$\begin{aligned}(z_0 + u_1)\zeta + (u_1 + u_2)\zeta^2 + (u_2 + u_3)\zeta^3 + \ldots + (u_{n-1} + u_n)\zeta^n + \cdots =\\ = u_1 d_1\zeta + (u_1 d_2 + u_2 d_1^2)\zeta^2 + \ldots + [P_n(u_1, \ldots, u_{n-1}) + u_n d_1^n]\,\zeta^n + \cdots .\end{aligned}$$

Hence we obtain

$$\begin{aligned} u_1 &= (d_1 - 1)^{-1} z_0 \\ u_2 &= P_2(d_1, d_2, z_0) \\ &\vdots \\ u_n &= P_n(d_1, \ldots, d_n, z_0) \\ &\vdots \end{aligned}$$

So the coefficients only depend on the coefficients of $\tilde{\varphi}$ and on the value z_0. ■

If d_1 is a root of unity, we have to distinguish between the case where $\tilde{\varphi}$ is linearizable and where it is not. We will not do this in this article. After the transformations which leads to equation (3.3) we see, that we can immediately apply the formal logarithm to equation (3.3) and we see, that there is no parameter like u_1 (compare with the proof of Theorem 2.1) which cancels before – only z_0 was removed. Exactly this is the difference between this case and the case before. We have no generator $\tilde{g}_0$ such that we can represent all our solutions by $\tilde{g}_0$. Nevertheless in the sense of our solvability condiditons we can state the following two theorems. Note that k is always equal to 1.

Theorem 3.3. *Let $z_0 \in \mathbb{C} \setminus \{0\}$ and let $\tilde{\varphi}(\zeta) \in \mathbb{C}[\![\zeta]\!]$, $\tilde{\varphi}(\zeta) = d_1\zeta + \ldots$, $d_1 \neq 0$. Then the transformed generalized Dhombres functional equation*

$$g(z_0 g(\zeta) + \zeta + \zeta g(\zeta)) = \tilde{\varphi}(g(\zeta)) \tag{3.1}$$

is solvable if and only if there exists an invertible $g \in \mathbb{C}[\![\zeta]\!]$ such that

$$\tilde{\varphi}(\zeta) = g(z_0\zeta + g^{-1}(\zeta) + \zeta g^{-1}(\zeta)). \tag{3.5}$$

In more detail we can show that

1. *if* $g(\zeta) = c_1\zeta + \cdots$ *such that* $c_1 z_0 + 1 \notin \mathbb{E}$ *and* (3.5) *holds, then* (3.1) *has a unique solution,*

2. *if* $g(\zeta) = c_1\zeta + \cdots$ *such that* $c_1 z_0 + 1 \in \mathbb{E}$, $\tilde{\varphi}(\zeta) = (c_1 z_0 + 1)\zeta + \cdots$ *is not linearizable and* (3.5) *holds, then* (3.1) *has a unique solution,*

3. *if* $g(\zeta) = c_1\zeta + \cdots$ *such that* $c_1 z_0 + 1 \in \mathbb{E}$, $\tilde{\varphi}(\zeta) = (c_1 z_0 + 1)\zeta + \cdots$ *is linearizable and* (3.5) *holds, then* (3.1) *has a solution.*

We omit the proof of *2.* and *3.* because of the length of it.

Theorem 3.4. *Let* $z_0 \in \mathbb{C} \setminus \{0\}$ *and let* $\tilde{\varphi}(\zeta) \in \mathbb{C}[\![\zeta]\!]$, $\tilde{\varphi}(\zeta) = d_m\zeta^m + \cdots$, $d_m \neq 0$, $m \geq 2$. *Then the transformed generalized Dhombres functional equation*

$$g(z_0 g(\zeta) + \zeta + \zeta g(\zeta)) = \tilde{\varphi}(g(\zeta)) \tag{3.1}$$

is solvable if and only if there exists an invertible $g \in \mathbb{C}[\![\zeta]\!]$ *such that*

$$\tilde{\varphi}(\zeta) = g(z_0\zeta + g^{-1}(\zeta) + \zeta g^{-1}(\zeta)). \tag{3.6}$$

Proof. We want to give an idea of the proof. Let the transformed generalized Dhombres equation be solvable. Then we transform it to

$$z_0\zeta + (1+\zeta)U(\zeta) = U(\hat{\varphi}(\zeta)), \tag{3.2}$$

where we see that $u_1 = -z_0$ and for $2 \leq \nu \leq m$ we have $u_\nu = (-1)^\nu z_0$. For ζ^m we obtain $u_m + u_{m-1} = -z_0 d_m$. We proceed by induction. Reversing our calculations show that ord $g = 1$.
On the other hand let $\tilde{\varphi}$ be given by (3.6). Then we use the invertibility of g such that we obtain the transformed generalized Dhombres equation. This equation is solvable. ■

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L. Reich
Institut für Mathematik und Wissenschaftliches Rechnen
Karl-Franzens-Universität
Graz
Austria
ludwig.reich@uni-graz.at

J. Tomaschek
Mathematics Research Unit
Université du Luxembourg
Ville de Luxembourg
Grand Duchy of Luxembourg
joerg.tomaschek@uni.lu