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DIVISOR FUNCTION $\tau_3(\omega)$ IN ARITHMETIC PROGRESSION

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Dedicated to Professors Zoltán Daróczy and Imre Kátai on their 75th anniversary

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Abstract. We constructed the asymptotic formula over the ring of the Gaussian integers for summatory function of the divisor function $d_3(\omega)$ in an arithmetic progression $N(\omega) \equiv \ell \pmod q$ which is non-trivial for $q \leq x^{\frac{2}{7}-\varepsilon}$

1. Introduction

Let $\mathbb{Z}[i]$ be the ring of Gaussian integers and $k \geq 2$, $k \in \mathbb{N}$. We define the divisor function $\tau_k(w)$, $w \in \mathbb{Z}[i]$ as the coefficient of $N(w)^s$ in the Dirichlet series

$$Z^k(s) = \sum_{w}{}^{*} \frac{\tau_k(w)}{N(w)^s}, \ \Re s > 1.$$

(Here, as usually, $N(w) = |w|^2$ is a norm of a Gaussian integer w, $Z(s)$ is the Hecke Z-function).

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The main point of this paper is to consider the case $k=3$ and to construct an asymptotic formula of the sum

$$D_k(x;\ell,q)=\sum_{\substack{N(w)\equiv\ell \pmod q,\\ N(w)\le x}}\tau_k(\alpha),\ \ (1\le\ell<q,\ (\ell,q)=1)$$

in particular to investigate the ranges of q and x for which this formula is non-trivial.

The similar problem was considered over $\mathbb{Z}$ in the works of Deshoulieres and Iwaniec[3] and Heath-Brown[4]. Over the ring of integer elements of the quadratic extension of $\mathbb{Q}$ in [6] it has been obtained an asymptotic formula for

$$T_k(x,w_0,\gamma)=\sum_{\substack{w\equiv w_0 \pmod\gamma,\\ |N(w)|\le x}}\tau_k(w),\ \ (w_0,\gamma\in\mathbb{Q}(\sqrt{d}),\ d\in\mathbb{Z}),$$

where γ is a fix integer from $\mathbb{Q}(\sqrt{d})$.

In rational case Deshoulieres and Iwaniec and also Heath-Brown used Deligne's bound for the k-fold Kloosterman sum to estimate $D_3(x;\ell,q)$.

In our paper we use the norm Kloosterman sums for which it is obtained a non-trivial estimate (see, section 3). This let us to construct an asymptotic formula for $D_3(x;\ell,q)$ over $\mathbb{Z}[i]$ that can be applied for investigation of asymptotic behavior for the sum

$$\sum_{N(w)\le x}\tau(N(w))\tau_3(N(w)).$$

Notations. We denote $G:=\mathbb{Z}[i]$ the ring of the Gaussian integers

$$G=\left\{a+bi\middle|a,b,\in\mathbb{Z},i^2=-1\right\}.$$

For the designation of the Gaussian integers we shall use the Greek letters α, β, γ, ξ, η; a Gaussian prime number denote through $\mathfrak{p}$ if $\mathfrak{p}\notin\mathbb{Z}$. For $\alpha\in\mathbb{Z}[i]$ we put $Sp(\alpha)=\alpha+\overline{\alpha}=2\Re(\alpha)$, $N(\alpha)=\alpha\cdot\overline{\alpha}$, where $\overline{\alpha}$ denotes a complex conjugate with α; $Sp(\alpha)$ and $N(\alpha)$ we name a trace and a norm (respectively) of α from $\mathbb{Q}(i)$ into $\mathbb{Q}$.

The writing $a\in\mathbb{Z}_q$(respectively, $a\in G_\gamma$) denotes that $a\in\mathbb{Z}$(respectively, $\alpha\in G$) and a (respectively, α) runs a complete residue system modulo q (modulo γ). Analogous, $a\in\mathbb{Z}_q^*$(respectively, $a\in G_\gamma^*$) denotes $a\in\mathbb{Z}$(respectively, $\alpha\in G$) and runs a reduced residue system modulo q (respectively, modulo γ).

The writing $\sum\limits_{S(C)}$ denotes that the summation runs over the region C which describe extra.

For $A\in\mathbb{N}$ (or $\alpha\in G$) put $\nu_p(A)=a$, (or $\nu_{\mathfrak{p}(\alpha)}=a$) if $p^a\|A$ (or $\mathfrak{p}^a\|\alpha$).

Moreover, $\exp(z) = e^z$, $e_q(z) = e^{2\pi i \frac{z}{q}}$ for $q \in \mathbb{N}$; the Vinogradov symbol as in $f(x) \ll g(x)$ means that $f(x) = O(g(x))$. The abbreviations "$e_q(\cdot)$" and "$e\left(\frac{2\pi i \ldots}{q}\right)$" are equal and use depend on the length of certain formula.

For $z \in \mathbb{Z}$ (respectively, $z \in G$), $(z,p) = 1$ let z^{-1} be the multiplicative inverse modulo p^m.

The signs $\sum^*$ and $\prod^*$ mean that the summation (product) conducts by all the non-associated integer (respectively, prime) Gaussian numbers.

2. Preliminary results

We begin this section with a few background definitions and facts.

Let q be a positive integer, $q > 1$, and let χ_q be a Dirichlet character modulo q. Determine the function χ on the ring of Gaussian integers G as $\chi(\alpha) : G \to \mathbb{C}$, $\chi(\alpha) := \chi_q(N(\alpha))$ for any $\alpha \in G$. It is clear that χ is a character of the group G_q^*.

Lemma 1. *Let ℓ, q be the positive integers, $q > 1$, $\ell \not\equiv 0 \pmod q$, and let $J(\ell, q)$ denote the number of the solutions of $x^2 + y^2 \equiv \ell \pmod q$. Then we have*

$$J(\ell,q) = E(\ell,q)q \prod_{\substack{p^a \| q \\ p \text{ is odd}}} \left\{1 - \frac{(\chi_4(p))^{\nu_p((\ell,p^a))+1} - \chi_4(p^{a+1})}{p} + \right.$$
$$\left. + \left(1 - \frac{1}{p}\right) \sum_{b=a-\nu_p((\ell,p^a))}^{a-1} \chi_4(p^{a-b}) \right\},$$

where

$$E(\ell,q) = \begin{cases} 1 & if \quad q \text{ is odd or } q \equiv 2 \pmod 4; \\ 1 & if \quad q \equiv 0 \pmod 4,\ \nu_2(\ell) > \nu_2(q) - 2; \\ 2 & if \quad q \equiv 0 \pmod 4,\ \ell \cdot 2^{-\nu_2(\ell)} \equiv 1 \pmod 4; \\ 0 & if \quad q \equiv 0 \pmod 4,\ \ell \cdot 2^{-\nu_2(\ell)} \equiv 3 \pmod 4, \end{cases}$$

moreover, the sign $'$ in production $\prod$ denotes that if $\nu_p(\ell) \geq \nu_p(q)$ then an appropriate multiplier in this production have to be substituted on $1 + \left(1 - \frac{1}{p}\right) \times$ $\times \sum_{b=0}^{a-1} \chi_4(p^{a-b})$.

Proof. The statement of Lemma follows by multiplication at q of the function $J(\ell, q)$ and the equation

$$J(\ell, p^a) = \frac{1}{p^a} \sum_{x,y \in \mathbb{Z}_{p^a}} \sum_{z=1}^{p^a} e_{p^a}(z(x^2 + y^2 - \ell))$$

with the subsequent application the formula for the square of Gaussian sum $G(h, p^a) = \sum_{x \in \mathbb{Z}_{p^a}} e_{p^a}(hx)$. ■

Lemma 2. *A non-principal character χ_q produces a non-principal character χ if $q \not\equiv 0 \pmod 4$. For $q \equiv 0 \pmod 4$ it has only one non-principal character χ_q producing the principal character χ_0 of group G_q^*.*

Proof. Using the standard representation of a Dirichlet character, it's enough to prove only the case $q = p^n$, p is a prime number. Every rational number a has a norm in $\mathbb{Q}(i)$ that equal to a^2, and hence, $\chi(a) = \chi_q(a^2)$, and therefore the equation $\chi(a) = 1$, $(a, q) = 1$ can be hold only if $\chi_q(a) = \pm 1$. Consequently, only the real characters χ_q can produce the principal character χ_0. The basic modulo of the Dirichlet character is an equal to $2^m P$, where $m = 0, 1, 2, 3$, and $P = 1$ or is a square-free odd number. So we should consider only the case of real characters with the basic modulus 2, 4, 8, p. If $p > 2$ be a prime number, for every $b \in \mathbb{Z}$, b is a quadratic non-residue mod p, there exists $\alpha \in G$ such that $N(\alpha) \equiv b \pmod p$ (see, Lemma 1), and hence, for only one non-principal real character $\chi_p(x) := \left(\frac{x}{p}\right)$ ($\left(\frac{x}{p}\right)$ is the Legendre symbol) we have $\chi(\alpha) = \chi_p(b) = -1$, i.e. χ is a non-principal character.

For the modulo 2 we have only the principal character. For the modulo 4 a non-principal character χ_4 generates the principal character χ (because $N(\alpha) = 1 \pmod 4$ if $(N(\alpha), p) = 1$). At last, modulo 8 we have only one the non-principal character χ_8 which inducing χ_4. So, if $q \not\equiv 0 \pmod 4$, only the principal character $\chi_{q,0}$ produces the principal character χ_0, and for $q \equiv 0 \pmod 4$ we have only one character χ_q inducing the principal character χ_0. That completes the proof of lemma. ■

Lemma 3. *Let $p > 2$ be a prime number, $a_1, a_2, b_1, b_2, c, m \in \mathbb{Z}$, $(a_1, b_1, a_2, b_2, p) = 1$, $m \geq 1$. Then for the exponential sum*

$$S := \sum_{x,y \in \mathbb{Z}_{p^m}^*} e_{p^m}(a_1 x + a_2 y + b_1 x^2 + b_2 y^2 + pcxy)$$

we have

$$S = \begin{cases} p^m e_{p^m}(A_1 a_1 + A_2 a_2 + B_1 b_1 + B_2 b_2 + \overline{C}_1 c) & if \quad m \text{ is even}, \\ p^{\frac{m+1}{2}} e_{p^m}(C_1 a_1 + C_2 a_2 + D_1 b_1 + D_2 b_2 + \overline{C}_2 c) & if \quad m \text{ is odd} \end{cases}$$

where $A_i, B_i, C_i, D_i, \overline{C}_i \in \mathbb{Z}$, $i = 1, 2$.

Proof. First, we suppose that $m = 2n$. Putting

$$x = x_0 + p^n u,\ y = y_0 + p^n v,\ x_0, y_0 \in \mathbb{Z}^*_{p^n},\ u, v \in \mathbb{Z}_{p^n}$$

we infer

$$S = \sum_{x_0, y_0 \in \mathbb{Z}^*_{p^n}} e_{p^{2n}}(a_1 x_0 + a_2 y_0 + b_1 x_0^2 + b_2 y_0^2 + pcx_0 y_0) \times$$
$$\times \left\{ \sum_{u, v \in \mathbb{Z}_{p^n}} e_{p^n}(a_1 u + a_2 v + 2(b_1 u x_0 + b_2 v y_0) + cp(u y_0 + v x_0)) \right\}.$$

The summation over u, v shows that the inner sum vanishes if at least one of congruence

$$(2.1)\qquad \begin{aligned} a_1 + 2b_1 x_0 + pcy_0 &\equiv 0 \pmod{p^n} \\ a_2 + 2b_2 y_0 + pcx_0 &\equiv 0 \pmod{p^n} \end{aligned}$$

violates.

Since $(a_1, a_2, b_1, b_2, p) = 1$ we have only one pair $(x_0, y_0) \in {\mathbb{Z}^*_{p^m}}^2$ such that (2.1) holds.

Thus we have

$$S = p^{2n} e_{p^{2n}}(a_1 x_0 + a_2 y_0 + b_1 x_0^2 + b_2 y_0^2 + pcx_0 y_0),$$

moreover, x_0, y_0 are linear combinations at a_1, a_2, b_1, b_2 with the coefficients from $\mathbb{Z}_{p^{2n}}$.

Now let $m = 2n + 1$. Putting

$$x = x_0 + p^n u,\ y = y_0 + p^n v,\ x_0, y_0 \in \mathbb{Z}^*_{p^n},\ u, v \in \mathbb{Z}_{p^{n+1}},$$

modulo p^{2n+1} we have

$$\begin{aligned} x^2 &\equiv x_0^2 + 2p^n x_0 u + p^{2n} u^2, \\ y^2 &\equiv y_0^2 + 2p^n y_0 v + p^{2n} v^2, \\ xy &\equiv x_0 y_0 + p^n (x_0 v + y_0 u) + p^{2n} uv, \end{aligned}$$

and hence,

$$S = \sum_{x_0, y_0 \in \mathbb{Z}^*_{p^n}} e^{\frac{a_1 x_0 + a_2 y_0 + b_1 x_0^2 + b_2 y_0^2 + pcx_0 y_0}{p^{2n}}} \times$$
$$\times \left\{ \sum_{u, v = 0}^{p^{n+1}} e^{\frac{a_1 u + a_2 v + 2(b_1 x_0 u + b_2 y_0 v) + p^n (b_1 u^2 + b_2 v^2) + pc(x_0 v + y_0 u)}{p^{n+1}}} \right\}.$$

The inner sum does not turn into zero only if

$$\begin{aligned} &pcx_0 + a_2 + 2b_2y_0 \equiv 0 \pmod{p^n}, \\ &pcy_0 + a_1 + 2b_1x_0 \equiv 0 \pmod{p^n}. \end{aligned} \tag{2.2}$$

From (2.2) it follows that there is only one pair $(x_0, y_0) \in \mathbb{Z}^2_{p^{n-1}}$ for which

$$\begin{aligned} x_0 &\equiv -a_1(2b_1)' + pX_0, \\ y_0 &\equiv -a_2(2b_2)' + pY_0 \end{aligned} \pmod{p^n}.$$

In such case the inner sums over u and v are the Gaussian sums $\pmod{p^{n+1}}$. Consequently, we obtain

$$S = p^{\frac{m+1}{2}} e_{p^m}(D_1x_0 + D_2y_0 + E_1x_0^2 + E_2y_0^2),$$

where D_j, E_j are the linear combinations at a_1, a_2, b_1, b_2 with the coefficients from $\mathbb{Z}_{p^{2n+1}}$. ■

Now, we remind the some necessary information about the Hecke Z-function.

Let $\delta_0, \delta_1 \in \mathbb{Q}(x)$ and let $s \in \mathbb{C}$, $\Re s > 1$. We define the Hecke Z-function by the absolute convergent series

$$Z_m(s; \delta_0, \delta_1) = \sum_{\omega} \frac{e^{4mi \arg \omega}}{N(\omega + \delta_0)^s} e^{2\pi i \Re(\delta_1 \omega)},$$

where $\delta_0, \delta_1 \in \mathbb{Q}(i)$, $m \in \mathbb{Z}$.

In the case $m = 0$, $\delta_0 = \delta_1 = 0$ we write $Z(s)$ instead $Z(s; 0, 0)$.

Lemma 4. *For $m \neq 0$ or $m = 0$ and δ_1 is not a Gaussian integer, $Z_m(s; \delta_0, \delta_1)$ is an entire function. If $m = 0$ and $\delta_1 \in G$, the function Z_m is a holomorphic except as $s = 1$, where it has simple pole with a residue π. Moreover, the functional equation*

$$\pi^{-s}\Gamma(2|m|+s)Z_m(s; \delta_0, \delta_1) = \pi^{-(1-s)}\Gamma(2|m|+1-s)Z_m(1-s; \delta_1, -\delta_0)e^{-2\pi i \Re(\delta_0, \delta_1)}$$

holds in all the cases.

For the proof in the case $\delta_0 = \delta_1 = 0$ see[5]. The proof in other cases is similar.

Corollary 1. *In the domain $-\frac{1}{4} \leq \Re s = \sigma \leq 2$, $|\Im s| = |t| \geq 3$ the following estimates hold:*

$$Z_m(s; 0, 0) \ll \begin{cases} \log^4(m^2 + t^2) & if \quad 1 \leq \sigma \leq 2; \\ (m^2 + t^2)^{\frac{1-\sigma}{3}} \log^4(m^2 + t^2) & if \quad \frac{1}{2} \leq \sigma \leq 1; \\ (m^2 + t^2)^{\frac{19-32\sigma}{18}} \log^4(m^2 + t^2) & if \quad 0 \leq \sigma \leq \frac{1}{2}; \\ (m^2 + t^2)^{1-2\sigma} & if \quad -\frac{1}{4} \leq \sigma \leq 0. \end{cases}$$

This assertion follows from Lemma 4, an estimate $Z_m(s,0,0)$ on half-line (see, (6.16), [1]) and the application of the Phragmen-Lindelöf theorem for an analytic functions in the strip $-\frac{1}{4} \le \Re s \le \frac{1}{2}$.

3. Norm Kloosterman sums

Let $\alpha_1, \dots, \alpha_n \in G$, $q > 1$ be a positive integer.
Denote,

$$\widetilde{K}_n := \widetilde{K}_n(\alpha_0, \alpha_1, \dots, \alpha_n; q) = \sum_{S(C)} e^{2\pi i \Re\left(\frac{\alpha_0\xi_0 + \cdots + \alpha_n\xi_n}{q}\right)}, \tag{3.1}$$

where $C := \{\xi_j \in G_q,\ i = 0, 1, \dots, n,\ N(\xi_0 \cdots \xi_n) \equiv 1 \pmod q\}$.

$\widetilde{K}_n$ we will call the n-fold norm Kloosterman sum.

It's easy to check that for $q_1 q_2 = q$, $(q_1, q_2) = 1$

$$\widetilde{K}_n(\alpha_0, \dots, \alpha_n; q) = \widetilde{K}_n(\alpha_0\overline{q}_2', \dots, \alpha_n\overline{q}_2'; q_1) \cdot \widetilde{K}_n(\alpha_0\overline{q}_1', \dots, \alpha_n\overline{q}_1'; q_2), \tag{3.2}$$

where $q_1\overline{q}_1' \equiv 1 \pmod{q_2}$, $q_2\overline{q}_2' \equiv 1 \pmod{q_1}$.

This property reduces our problem to compute $\widetilde{K}_n(a_0, \dots, a_n; q)$ for prime power modulus $q = p^m$.

For a prime number p and the Gaussian integer α denote

$$r_\alpha = \max_r(p^r \| \alpha) = \min(\nu_p(\Re\alpha), \nu_p(\Im\alpha)). \tag{3.3}$$

In order to estimate $\widetilde{K}_n(\alpha_0, \dots, \alpha_n; p^m)$ it is sufficient to study the case

$$\min(r_{\alpha_0}, \dots, r_{\alpha_n}) = 0,$$

i.e. at least one from $\alpha_0, \dots, \alpha_n$ does not divide on p. But then, putting $\alpha_j = a_j + ib_j$, $j = 0, 1, \dots, n$, we have $GCD(a_0, \dots, a_n, b_0, \dots, b_n, p) = 1$.

We consider the case $n = 2$.

First, we assume that $m = 1$.

Let $p \equiv 1 \pmod 4$.

Denote $\xi_j = x_j + iy_j$, $j = 0, 1, 2$. Then from (3.1) it follows

$$\widetilde{K}_2 = \sum_{S(C)} e_p(a_0x_0 + a_1x_1 + a_2x_2 - b_0y_0 - b_1y_1 - b_2y_2), \tag{3.4}$$

where $C := \left\{ x_j, y_j \in \mathbb{Z}_p :\ \prod_{j=0}^{2} (x_j^2 + y_j^2) \equiv 1 \pmod p \right\}$.

Let ε_0 be a solution of congruence $x^2 \equiv -1 \pmod p$.

Put for every $j = 0, 1, 2$

$$u_j = x_j + \varepsilon_0 y_j,\ v_j = x_j - \varepsilon_0 y_j.$$

In view of $\xi_j \in G_p^*$ it follows that $N(\xi_j) = x_j^2 + y_j^2 \not\equiv 0 \pmod p$ and thus $u_j v_j \equiv x_j^2 + y_j^2 \not\equiv 0 \pmod p$. Hence, $u_j, v_j \in \mathbb{Z}_p^*$.

Furthermore, $x_j = 2'(u_j + v_j)$, $y_j = -2'\varepsilon_0(u_j - v_j)$, where $2 \cdot 2' \equiv 1 \pmod{p^m}$.

Consider three cases:

(i) $\alpha_j \not\equiv 0 \pmod p$, $j = 0, 1, 2$;

(ii) $\alpha_0 \equiv 0 \pmod p$, $\alpha_1 \not\equiv 0$, $\alpha_2 \not\equiv 0 \pmod p$;

(iii) $\alpha_0 \equiv \alpha_1 \equiv 0 \pmod p$, $\alpha_2 \not\equiv 0 \pmod p$.

For the first case we have

$$\begin{aligned}
&\widetilde{K}_2(\alpha_0, \alpha_1, \alpha_2; p) = \sum_{S(C)} e_p\left(\sum_{j=0}^{2}(A_j u_j + B_j v_j)\right), \\
&C := \left\{u_j, v_j \in \mathbb{Z}_p^*,\ j = 0, 1, 2;\ \prod_{j=0}^{2} u_j v_j \equiv 1 \pmod p\right\}.
\end{aligned} \tag{3.5}$$

It is obvious that $A_j \equiv 2'(a_j + \varepsilon_0 b_j) \pmod p$, $B_j \equiv 2'(a_j - \varepsilon_0 b_j) \pmod p$, and hence, $A_j B_j \equiv 4'(a_j^2 + b_j^2) \equiv 4' N(\alpha_j) \pmod p$. But then $(A_j, p) = (B_j, p) = 1$.

By (3.5) and the estimate of P. Deligne[2], we have

$$|\widetilde{K}| \le 4p^{\frac{5}{2}}. \tag{3.6}$$

For the second case we obtain

$$\widetilde{K}_2(\alpha_0, \alpha_1, \alpha_2; p) = \sum_{u_0, v_0 \in \mathbb{Z}_p^*} \sum_{S(C)} e_p\left(\sum_{j=1}^{2}(A_j u_j + B_j v_j)\right),$$

where

$$C := \left\{u_j, v_j \in \mathbb{Z}_p^*,\ j = 1, 2;\ \prod_{j=1}^{2} u_j v_j \equiv (u_0 v_0)' \pmod p\right\}.$$

For every collection of (u_0, v_0) (it is $(p-1)^2$ such collections at all) we have the norm Kloosterman sum $\widetilde{K}_1$ (which has been estimated in [9]). Thus, in the case $\alpha_0 \equiv 0 \pmod p$, we obtain

$$|\widetilde{K}_2(0,\alpha_1,\alpha_2;p)| \le 2(p-1)^2 p^{\frac{3}{2}} \le 2p^{\frac{7}{2}}. \tag{3.7}$$

At last, in the third case, we find at once

$$|\widetilde{K}_2(0,0,\alpha_2;p)| \le (p-1)^4 \max_{a\in\mathbb{Z}_p^*} \left| \sum_{\substack{x\in G_p^* \\ N(x)\equiv a \pmod p}} e_p\left(\Re\frac{x}{p}\right)\right| \le 2p^{\frac{9}{2}}. \tag{3.8}$$

Let now $p \equiv 3 \pmod 4$.

In this case G_p is a finite field $\mathbb{F}_{p^2}$, $|G_p| = p^2$.

Moreover, for any $u, v \in \mathbb{Z}$

$$(u+iv)^p \equiv u^p - iv^p \equiv u - iv \pmod p,$$

and so $2\Re(u+iv) \equiv Tr(u+iv) \pmod p$, where $Tr(\cdot)$ is a trace from $\mathbb{F}_{p^2}$ into $\mathbb{F}_p$.

So, the investigated sum $\widetilde{K}_2$ coincide with the sum

$$S_1(V,\alpha) = \sum_{X\in V_1} e_p\left(m\frac{Tr(\alpha\cdot X)}{p}\right),\ (m,p)=1,$$

where $\alpha = (\alpha_0,\alpha_1,\alpha_2) \in \mathbb{F}_{p^2}^3$, $X = (x_1,x_2,x_3) \in \mathbb{F}_p$, $\alpha \cdot X = \sum_{i=0}^{2} \alpha_i x_i$, V_1 is an algebraic manyfold produced by a polynomial $x_0x_1x_2 - 1$ over $\mathbb{F}_{p^2}$.

Hence, applying to above the result of P. Deligne, we obtain

$$|S_2(V,\alpha)| \le (n+1-\ell)(p^2)^{\frac{5}{2}+\ell},$$

where ℓ is a number of α_i under condition $\alpha_i \equiv 0 \pmod p$.

Moreover, from the representation $S_2(V,\alpha)$ by way of characteristic roots of the Riemann zeta-function of algebraic manifold over finite field we conclude $S_1^2(V,\alpha) \sim 3S_2(V,\alpha)$ and thus we have

$$|\widetilde{K}_2| \le 3p^{\frac{5}{2}+\ell}. \tag{3.9}$$

The case $p=2$, $m=1$ is trivial.

Let $m > 1$.

There is no loss of generality in assuming that $\alpha_0 \not| p$. Thus it follows that $(a_0, b_0, p) = 1$.

Let $\xi_0 = x_0 + iy_0$. By the definition of $\widetilde{K}_2$, we may write

$$\begin{aligned}\widetilde{K}_2 &= \sum_{\substack{\xi_j \in G_{p^m}\\ j=0,1,2}} e_{p^m}\left(\sum_{j=0}^{2} \Re(\alpha_j \xi_j) \frac{1}{p^m} \sum_{k=0}^{p^m-1} e_{p^m}(k(N(\xi_0,\xi_1\xi_2)-1))\right) = \\ &= \frac{1}{p^m}\sum_{S(C)} e_{p^m}\left(k(N(\xi_0\xi_1\xi_2)-1) + \sum_{j=0}^{2}(a_j \Re\xi_j - b_j \Im\xi_j)\right),\end{aligned} \tag{3.10}$$

where

$$C := \{k \in \mathbb{Z}_{p^m},\ \xi_j \in G_{p^m},\ j = 0,1,2\}.$$

Note, that the summation over k in (3.10) gives zero if $N(\xi_1)$, $N(\xi_2)$ are not coprimes to p.

Therefore, we have

$$\widetilde{K}_2 = \frac{1}{p^m} \sum_{\xi_1,\xi_2 \in G^*_{p^m}} \sum_{k=0}^{p^m-1} \sum_{x_0,y_0 \in \mathbb{Z}_{p^m}} e^{2\pi i \frac{k((x_0^2+y_0^2)N(\xi_1)N(\xi_2)-1)+\sum_{j=0}^{2}(a_j\Re\xi_j - b_j\Im\xi_j)}{p^m}}.$$

Now, taking into account that the summation over x_0 (or y_0) gives zero if $k \equiv 0 \pmod p$, we deduce

$$\begin{aligned}\widetilde{K}_2 =& \frac{1}{p^m} \sum_{k \in \mathbb{Z}^*_{p^m}} e_{p^m}(-k) \sum_{\xi_1,\xi_2 \in G^*_{p^m}} e_{p^m}\left(\Re(\alpha_1\xi_1 + \alpha_2\xi_2)\right) \times \\ &\times \sum_{x_0,y_0 \in \mathbb{Z}_p} e_{p^m}(F(k,\xi)),\end{aligned} \tag{3.11}$$

where $F(k,\xi) = kN(\xi_1\xi_2)(x_0^2 + y_0^2) + a_0x_0 + b_0y_0$.

The inner sum over x_0, y_0 in (3.11) is the product of two classical Gaussian sums, and, hence

$$\begin{aligned}&\sum_{x_0,y_0 \in \mathbb{Z}_{p^m}} e_{p^m}(kN(\xi_1\xi_2)(x_0^2+y_0^2) + a_0x_0 - b_0y_0) = \\ &= e_{p^m}(-(4kN(\xi_1\xi_2))' \cdot (a_0^2 + b_0^2))(-1)^{\frac{p-1}{2}} p^m.\end{aligned} \tag{3.12}$$

(Here, as always A' denotes the multiplicate inversive for A modulo p^m if $(A,p) = 1$).

We continue the calculation of $\widetilde{K}_2$.

From (3.11), (3.12) we infer

$$\widetilde{K}_2 = \sum_{k \in \mathbb{Z}_{p^m}} (-1)^{\frac{p-1}{2}} e_{p^m}(-k) \sum_{\xi_1,\xi_2 \in G^*_{p^m}} e_{p^m}(F_1(k,\xi)), \tag{3.13}$$

where

$$F_1(k,\xi)=\sum_{j=1}^{2}(a_j\Re\xi_j-b_j\Im\xi_j)-4'k'(a_0^2+b_0^2)N(\xi_1\xi_2)'.$$

Put $m_1=\left[\frac{m}{2}\right]$, $\xi_j=\eta_j+p^{m_1}\zeta_j$, $\eta_j\in G^*_{p^m}$, $\zeta_j\in G_{p^{m-m_1}}$,

$$\eta_j=x_j+iy_j,\ \zeta_j=u_j+iv_j,\ x_j,y_j\in\mathbb{Z}_{p^{m_1}},\ u_j,v_j\in\mathbb{Z}_{p^{m-m_1}},\ j=1,2. \tag{3.14}$$

Then

$$N(\xi_j)'=N(\eta_j)'(1-2p^{m_1}(x_ju_j+y_jv_j)N(\zeta_j)'). \tag{3.15}$$

Consequently, by (3.13)-(3.15), we obtain

$$\begin{aligned}\widetilde{K}_2=\sum_{k\in\mathbb{Z}^*_{p^m}}e_{p^m}(-k)\prod_{j=1}^{2}\Bigg(&\sum_{\substack{x_j,y_j\in\mathbb{Z}_{p^{m_1}}\\(x_j^2+y_j^2,p)=1}}e_{p^{m_1}}(\Re(\alpha_j\eta_j))\times\\&\times\sum_{u_j,v_j\in\mathbb{Z}_{p^{m-m_1}}}e_{p^m}(F_2(k,x,y))\Bigg),\end{aligned} \tag{3.16}$$

where

$$\begin{aligned}F_2(k,x,y)&=H_0+p^{m_1}H_1,\\H_0&=4'k'(a_0^2+b_0^2)((x_1^2+y_1^2)(x_2^2+y_2^2))':=4'k'(a_0^2+b_0^2)D,\\H_1&=\sum_{j=1}^{2}(-2'k'(a_0^2+b_0^2)D(x_ju_j+y_jv_j)+a_ju_j-b_jv_j).\end{aligned} \tag{3.17}$$

The summation over u_j, v_j in (3.16) gives zero if it disturbs at least one of the congruences

$$\begin{cases}2ka_j-D(a_0^2+b_0^2)x_j\equiv 0 \pmod{p^{m-m_1}}\\2kb_j+D(a_0^2+b_0^2)y_j\equiv 0 \pmod{p^{m-m_1}}\end{cases},\ j=1,2 \tag{3.18}$$

If (3.18) holds, then the summation over u_j, v_j gives $p^{2(m-m_1)}$.

Let $\min(r_{\alpha_1},r_{\alpha_2})=0$ (see, notation (3.3)). In such case we have $(a_1,a_2,b_1,b_2,p)=1$. Let $(a_1,p)=1$. From the congruence

$$ka_1-D(a_0^2+b_0^2)x_1\equiv 0 \pmod p$$

it follows that $(x_1,p)=1$ and $D(a_0^2+b_0^2)\equiv ka_1x_1 \pmod{p^{m-m_1}}$. Thus from (3.18) we infer

$$x_2\equiv a_1'a_2x_1,\ y_1\equiv -a_1'b_1x_1,\ y_2\equiv -a_1'b_2x_1 \pmod{p^{m-m_1}}. \tag{3.19}$$

By (3.16), (3.17), (3.19) and Lemma 1, we lead the estimate for $\widetilde{K}_2$ to an exponential sum over k.

$$|\widetilde{K}_2| \le p^{2m} \left| \sum_{k\in\mathbb{Z}^*_{p^m}} e_{p^m}(-k+Ak') \right| \le 2p^{\frac{5m}{2}}. \tag{3.20}$$

In the case $\min(r_{\alpha_1}, r_{\alpha_2}) = r > 0$ the system (3.18) has no solutions under assemption $(x_j^2+y_j^2, p) = 1$, $j = 1, 2$ and, consequently, the sum $\widetilde{K}_2$ is zero.

The same estimates hold for $\widetilde{K}_2$ if $p = 1+i$.

So, we proved our statement.

Lemma 5. *Let α_0, α_1, α_2 be the Gaussian integers,* $\min(r_{\alpha_0}, r_{\alpha_1}, r_{\alpha_2}) = 0$, *and $q > 1$ be a positive integer, $q = q_1q_2$, $(q_1, q_2) = 1$, q_1 is square-free, q_2 is square-full.*
Then

$$\widetilde{K}_2(\alpha_0, \alpha_1, \alpha_2; q) = \widetilde{K}_2(\alpha_0\overline{q}_2, \alpha_1\overline{q}_2, \alpha_2\overline{q}_2; q_1) \cdot \widetilde{K}_2(\alpha_0\overline{q}_1, \alpha_1\overline{q}_1, \alpha_2\overline{q}_1; q_2),$$

moreover,

$$|\widetilde{K}_2(\alpha_0\overline{q}_2, \alpha_1\overline{q}_2, \alpha_2\overline{q}_2; q_1)| \le 2^{\omega(q_1)} \prod_{p|q_1} p^{\frac{5}{2}+\ell_p(\alpha_1,\alpha_2)},$$

where

$$\ell_p(\alpha_1, \alpha_2) = \begin{cases} 0 & if \quad r_{\alpha_1} = r_{\alpha_2} = 0; \\ 1 & if \quad r_{\alpha_1} = 0,\ r_{\alpha_2} = 1; \\ 2 & if \quad r_{\alpha_1} = r_{\alpha_2} = 1; \end{cases}$$

$$|\widetilde{K}_2(\alpha_0\overline{q}_1, \alpha_1\overline{q}_1, \alpha_2\overline{q}_1; q_2)| \le \begin{cases} 3^{\omega(q_2)} q_2^{\frac{3}{2}} & \textit{if } \min(r_{\alpha_1}, r_{\alpha_2}) = 0 \\ & \textit{for all } p|q_2; \\ 0 & \textit{else}. \end{cases}$$

4. Main results

First we will suppose that $(\ell, q) = 1$ and ℓ is a norm residue mod$q$.

For $\Re s > 1$ we denote

$$F(s; \ell; q) := \sum_{\substack{\alpha\in G \\ S(C)}} \frac{\tau_3(\alpha)}{N(\alpha)^s}, \tag{4.1}$$

where $C : \{\alpha \in G : \; N(\alpha) \equiv \ell \pmod q), \; N(\alpha) \le x\}$;

$$F^*(s;\ell,q) = F(s;\ell,q) - \frac{A(\ell)}{\ell^s}, \tag{4.2}$$

where

$$A(\ell) = \sum_{\substack{\alpha \\ N(\alpha)=\ell}} \tau_3(\alpha).$$

It is clear, that

(4.3)

$$F(s;\ell,q) = \sum_{\substack{\alpha_1,\alpha_2,\alpha_3 \in G_q^* \\ N(\alpha_1,\alpha_2,\alpha_3)\equiv \ell\,(q)}} Z\left(s;\frac{\alpha_1}{q},0\right) Z\left(s;\frac{\alpha_2}{q},0\right) Z\left(s;\frac{\alpha_3}{q},0\right) N(q)^{-3s}.$$

Hence, be a simple generalization of the well-known Perron's formula for the Dirichlet series on an arithmetic progression (see,[8]) we have

$$\begin{aligned} T(x;\ell,q) := \sum_{\substack{N(\alpha)\equiv \ell \pmod q \\ N(\alpha)\le x}} \tau_3(\alpha) = \frac{1}{2\pi i} \int\limits_{c-iT}^{c+iT} F^*(s;\ell,q) \frac{x^s}{s} ds + \\ + O\left(\frac{x^c}{Tq(c-1)^3}\right) + O(x^\varepsilon), \end{aligned} \tag{4.4}$$

where $c > 1$, $1 < T \le x$ are the parameters to be chosen later.

Now we obtain by moving the path of integration to the line $\Re s = -b$, $b > 0$,

$$\begin{aligned} T(x;\ell,q) = \sum_{\substack{N(\alpha)\equiv \ell \pmod q \\ N(\alpha)\le x}} \tau_3(\alpha) = \left(\operatorname*{res}_{s=1} + \operatorname*{res}_{s=0}\right)\left(F^*(s;\ell,q)\frac{x^s}{s}\right) + \\ + \frac{1}{2\pi i}\int\limits_{-b-iT}^{-b+iT} F^*(s;\ell,q)\frac{x^s}{s}ds + \frac{1}{2\pi i}\int\limits_{-b+iT}^{c+iT} F^*(s;\ell,q)\frac{x^s}{s}ds - \\ - \frac{1}{2\pi i}\int\limits_{-b-iT}^{c-iT} F^*(s;\ell,q)\frac{x^s}{s}ds + O\left(\frac{x^c}{Tq(c-1)^3}\right) + O\left(x^\varepsilon\right). \end{aligned} \tag{4.5}$$

For the calculation of integrals in (4.5) we consider the function $F^*(s;\ell,q)$ in the strip $-\varepsilon \le \Re s \le c$.

It is obviously that

$$|F^*(c+it;\ell,q)| \le \sum_{\substack{N(\alpha)\equiv \ell \pmod q \\ N(\alpha)>q}} \frac{\tau_3(\alpha)}{N(\alpha)^c} \le q^{-1+\varepsilon}, \; (\varepsilon > 0). \tag{4.6}$$

On the line $\Re s = -\varepsilon$ we apply the functional equation for $Z(s;\delta,0)$ and then obtain

(4.7)
$$F^*(s;\ell,q) = \pi^{-3}\left(\frac{\pi}{q}\right)^{6s}\frac{\Gamma^3(1-s)}{\Gamma^3(s)}\times$$
$$\times \sum_{\substack{\alpha_1,\alpha_2,\alpha_3\in G_q\\ N(\alpha_1\alpha_2\alpha_3)\equiv\ell\ (q)}}\ \sum_{\omega_1,\omega_2,\omega_3\in G} e_q\left(\Re\left(\sum_{j=1}^{3}\alpha_j\omega_j\right)\right)N(\omega_1\omega_2\omega_3)^{-1+s} - \frac{A(\ell)}{\ell^s}.$$

By the absolute convergence of the series at ω_1, ω_2, ω_3 we may write

$$F^*(s;\ell,q) = \pi^{3}\left(\frac{\pi}{q}\right)^{6s}\frac{\Gamma^3(1-s)}{\Gamma^3(s)}\times$$

(4.8)
$$\times\sum_{\substack{\delta\in G\\ \delta|q}}\sum_{\substack{\omega\in G\\ (\omega,q)=\delta}} N(\omega)^{-1+s}\sum_{\substack{\omega_1,\omega_2,\omega_3\in G\\ \omega_1\omega_2\omega_3=\omega}}\widetilde{K}_2(\omega_1,\omega_2,u_0\omega_3;q),$$

where $N(u_0)\equiv\ell \pmod q$.

Putting $\delta=\delta_1\delta_2\delta_3$ and $\omega=\delta\omega'=\delta_1\omega_1'\delta_2\omega_2'\delta_3\omega_3'$ and applying Lemma 6 we have for $s=-\varepsilon+it$:

$$F^*(s;\ell,q)\ll q^{-6\varepsilon+\frac{3}{2}}2^{\nu(q)}\left|\frac{\Gamma^3(1-s)}{\Gamma^3(s)}\right| \ll q^{\frac{3}{2}-6\varepsilon}2^{\nu(q)}(|t|+3)^{3+6\varepsilon}. \tag{4.9}$$

Hence, by (4.6)-(4.9) and the Phragmen-Lindelöf theorem we infer for $-\varepsilon\le$ $\le\Re s\le c$, $|\Im s|\ge 3$

$$F^*(s;\ell,q)\ll q^{\frac{3c-5\sigma-5\varepsilon}{2(c+\varepsilon)}}|t|^{\frac{3(c-\sigma)}{c+\varepsilon}}. \tag{4.10}$$

Thus trivially we have

$$\int\limits_{-b\pm iT}^{c\pm iT} F^*(s;\ell,q)\frac{x^s}{s}ds \ll q^{\frac{3}{2}-6b}2^{\nu(q)}T^{2+6b}+\frac{x}{Tq^{1-b}}. \tag{4.11}$$

Moreover, by Stirling's formula and a simple transformation, we obtain

$$\tfrac{1}{2\pi i}\int\limits_{-b-iT}^{-b+iT} F^*(s;\ell,q)\tfrac{x^s}{s}ds =$$

(4.12)
$$=\sum_{\substack{\delta|q\\ \delta\in G}}\sum_{\substack{\omega\in G\\ (\omega,q)=\delta}} N(\omega)^{-1}\Phi(\omega)\left(\Im(\omega)+O\left(T^{2+6b}\left(\tfrac{xN(\omega)}{q^6}\right)^{\frac{-b}{6}}\right)\right),$$

where

$$\Phi(\omega) = \sum_{\omega_1\omega_2\omega_3=\omega} \widetilde{K}_2(\omega_1,\omega_2,u_0\omega_3;q),\ \ (N(u_0)\equiv \ell \pmod q)$$

$$\begin{aligned}\mathfrak{I}(\omega) &= \frac{y^{-b}}{\sqrt{2\pi}}\times\\ &\times \frac{1}{2\pi i}\int\limits_{T_0}^{T} t^{2+6b}\left(y^{-\frac{it}{6}}e^{it(\log t-1)} + y^{\frac{it}{6}}e^{-it(\log t-1)}\right)g(t)dt+\\ &+O\left(T_0^{3+6b}y^{-b}\right),\end{aligned} \tag{4.13}$$

where $y=\frac{\pi^6}{q^6}(xN(\omega))$, $T_0>1$ and will select later.

In the last integral put $t=y^{\frac16}t_1$, $f(t)=t(\log\frac{t}{y^{\frac16}}-1)=y^{\frac16}t_1(\log t_1-1)$.

Since, $f'(t_1)=0$ only if $t_1=1$, and $f''(1)=y^{\frac16}>0$, we conclude, that the integral $\mathfrak{I}(\omega)$ has only one stationary point $t_1=1$ if $1\in\left[y^{-\frac16}T_0, y^{-\frac16}T\right]$.

Hence, for $N(\omega)>\frac{T^6q^6}{\pi^6x}$, an integration by parts yields estimate

$$\mathfrak{I}(\omega)\ll y^{-b}\min\left(T^{\frac52+6b}, T^{2+6b}\log\left(\frac{N(\omega)x}{T^6q^6}\right)^{-1}\right)\log T. \tag{4.14}$$

For $N(\omega)\le\frac{T^6q^6}{\pi^6x}$, by the method of stationary phase (see, [7]: Theorem 1.4, p.162) we deduce

$$\begin{aligned}\mathfrak{I}(\omega) &= \frac{1}{\sqrt{2\pi}}\cdot\frac{1}{\pi}\sin\left(\frac{\pi}{4}-y^{\frac16}\right)\left(y^{\frac{5}{12}}+a_1y^{\frac{3}{12}}+a_2y^{\frac{1}{12}}+O\left(y^{-\frac12}\right)\right)+\\ &+O\left(T^{2+6b}\right)+O\left(T_0^{3+6b}\right)\end{aligned} \tag{4.15}$$

with the absolute constants a_1, a_2 in symbol "O".

So, to sum up, we have obtained from (4.4), (4.5), (4.11)-(4.15) the following relation

$$\begin{aligned}T(x;\ell,q) &= (\operatorname*{res}_{s=1}\operatorname*{res}_{s=0})\left(F^*(s;\ell,q)\frac{x^s}{s}\right)+\sum_{\substack{\delta\in G\\ \delta|q}}\sum_{\substack{(\omega,q)=\delta\\ N(\omega)\le X}}\left(\frac{\Phi(\omega)N(\omega)^{-1}}{\pi\sqrt{2\pi}}\cdot\right.\\ &\left.\cdot\sin\left(\frac{\pi}{4}-\frac{\pi\sqrt[6]{xN(\omega)}}{q}\right)\left(\frac{\pi^{\frac52}}{q^{\frac52}}(xN(\omega))^{\frac{5}{12}}\right)\right)+\\ &+O\left(\left(\frac{xN(\omega)}{q^6}\right)^{\frac{3}{12}}\right)+O\left(q^{\frac52}T^{\frac52}x^{\varepsilon}\log T\right)+O\left(q^{\frac52+\varepsilon}T_0^{3+6b}\log T\right),\end{aligned} \tag{4.16}$$

where $X = \frac{T^6 q^6}{\pi^6 x}$.

After all this preliminary work, it is straight-forward to prove the main result of this paper.

Theorem 1. *Let ℓ, q be the positive integers, $1 \le \ell < q$, $(\ell, q) = 1$. Then for $x \to \infty$ we have*

$$\sum_{\substack{N(\omega)\equiv \ell \pmod q \\ N(\omega)\le x}} \tau_3(\alpha) = \frac{x}{q^2}\Im(\ell,q)\prod_{\mathfrak{p}|q}\left(1-\frac{1}{N(\mathfrak{p})}\right)^2 P_2(\log x)+ \tag{4.17}$$
$$+\frac{x}{q^2}\Im(\ell,q)\prod_{\mathfrak{p}|q}\left(1-\frac{1}{N(\mathfrak{p})}\right)P_1(\log x)+$$
$$+\frac{12x}{q^2}\Im(q,\ell)+O\left(x^{\frac{5}{7}+\varepsilon}\right),$$

where $\Im(q,\ell)$ is determined by Lemma 1 and $P_j(u)$ are the polynomials of j^{th}-degree with the computable coefficients, moreover these coefficients and the constant in the error term do not depend on x, ℓ, q.

Proof. Let χ be an arbitrary character and let χ_0 be a principal character, both from the group of characters $\widehat{G}_q$ modulo q. The Hecke Z-function with a character χ is defined by the series

$$Z(s,\chi) = \sum \frac{\chi(\omega)}{N(\omega)^s},\ (\Re s > 1).$$

For $\chi = \chi_0$

$$Z(s,\chi_0) = \prod_{\mathfrak{p}\in G}{}^{*}\left(1-\frac{\chi_0(\mathfrak{p})}{N(\mathfrak{p})^s}\right)^{-1} = \prod_{\mathfrak{p}|q}\left(1-N^{-s}(\mathfrak{p})\right)Z(s).$$

Since, $Z(s) = \zeta(s)L(s,\chi_4)$, we have

$$Z(s,\chi_0) = \frac{\pi\overline{\varphi}(q)}{4q^2}\cdot\frac{1}{s-1} + \frac{\pi\overline{\varphi}(q)}{4q^2}\sum_{\mathfrak{p}|q}{}^{*}\frac{\log N(\mathfrak{p})}{N(\mathfrak{p})-1}+$$
$$+\frac{\overline{\varphi}(q)}{q^2}\left(\frac{\pi}{4}E + L'(1,\chi_4)\right) + a_1'(s-1)+\cdots =$$
$$= \frac{\pi\overline{\varphi}(q)}{4q^2}\cdot\frac{1}{s-1} + b_{0,q}(\chi_0) + a_1'(s-1)+\cdots,$$

where E is the Euler's constant, $L(s,\chi_4)$ is L-function of Dirichlet with the non-principal character $\chi \pmod 4$,

$$\overline{\varphi}(q) = q^2 \prod_{\mathfrak{p}|q} \left(1 - \tfrac{1}{N(\mathfrak{p})}\right),$$

$$b_{0,q}(\chi_0) = \tfrac{\pi\overline{\varphi}(q)}{4q^2} \left(E + \tfrac{L'(1,\chi_4)}{L(1,\chi_4)} + \sum_{\mathfrak{p}|q}{}^{*} \tfrac{\log N(\mathfrak{p})}{N(\mathfrak{p})-1}\right).$$

For an arbitrary character $\chi \in \widehat{G}_q$ we have

$$Z(s,\chi) = \frac{\varepsilon(\chi)}{s-1} + b_{0,q}(\chi) + b_1(\chi)(s-1) + \cdots, \tag{4.18}$$

where

$$\varepsilon(\chi) = \begin{cases} \frac{\pi\overline{\varphi}(q)}{4q^2} & if \quad \chi = \chi_0 \\ 0 & if \quad \chi \neq \chi_0. \end{cases}$$

Next,

$$q^{-2s} Z\left(s, \frac{\alpha_1}{q}, 0\right) = (\overline{\varphi}(q))^{-1} \sum_{\chi \in G_q} \overline{\chi}(\alpha_1) \sum_{\alpha} \frac{\chi(\alpha)}{N(\alpha)^s} =$$
$$= \frac{4}{\overline{\varphi}(q)} \sum_{\overline{\chi} \in \widehat{G}_q} \overline{\chi}(\alpha_1) Z(s,\chi).$$

Thus, by (4.18)

$$q^{-2s} Z\left(s, \frac{\alpha_1}{q}, 0\right) = \frac{\pi}{q^2} \cdot \frac{1}{s-1} + \frac{4}{\overline{\varphi}(q)} \sum_{\chi \in \widehat{G}_q} b_{0,q}(\chi)\overline{\chi}(\alpha_1) + \cdots.$$

So,

$$\operatorname*{res}_{s=1} \left(F^*(s) \tfrac{x^s}{s}\right) =$$

$$= \operatorname*{res}_{s=1} \left\{ \sum_{N(\alpha_1\alpha_2\alpha_3) \equiv \ell \ (q)} q^{-6s} \tfrac{x^s}{s} \prod_{j=1}^{3} Z\left(s, \tfrac{\alpha_j}{q}, 0\right) \right\} =$$

$$= \sum_{\substack{\alpha_1,\alpha_2,\alpha_3 \in G_q^* \\ N(\alpha_1\alpha_2\alpha_3) \equiv \ell \ (q)}} \left\{ \tfrac{\pi^3}{q^6} x P_2 \log x + \tfrac{\pi^2 x}{q^4 \overline{\varphi}(q)} x P_1 \log x \sum_{\chi \in G_q} b_{0,q}(\chi) \left(\sum_{j=1}^{3} \overline{\chi}(\alpha_j)\right) + \right.$$

$$\left. + \tfrac{4\pi}{q^2 \overline{\varphi}^2(q)} x \sum_{\chi_1,\chi_2 \in G_q} b_{0,q}(\chi_1) b_{0,q}(\chi_2) \left(\sum_{\substack{i=\overline{1,2},\ j=\overline{2,3}, \\ i \neq j}} \overline{\chi}(\alpha_i)\overline{\chi}(\alpha_j) \right) \right\}.$$

Taking into account that a summation over α_1, α_2, α_3 shows the terms with $\chi_i \neq \chi_0$ gives 0, we obtain

$$\underset{s=1}{\operatorname{res}}\left(F^*(s)\frac{x^s}{s}\right) = \frac{\pi^3\overline{\varphi}(q)^2\Im(q,\ell)}{q^6}xP_2\log x + \frac{3\pi^2\Im(q,\ell)}{q^4}xP_1\log x + \frac{12\pi\Im(q,\ell)}{q^2\overline{\varphi}(q)}x, \tag{4.19}$$

where $\Im(q,\ell)$ is determined by Lemma 1, $\overline{\varphi}(q)$ be totient Euler function in $\mathbb{Z}[i]$.

Moreover,

$$\underset{s=0}{\operatorname{res}}\left(F^*(s)\frac{x^s}{s}\right) = (\zeta(0)L(0,\chi_4))^3 - 1 = C_0 \ \textit{is constant}. \tag{4.20}$$

Now, we see from (16), (18), (19) that the assertion of the Theorem is proved by choosing $T_0 = T^{\frac{1+6b}{2+6b}}$, $b = \frac{1}{\log x} < \varepsilon$, $T = x^{\frac{2}{7}}q^{-1}$. ■

The asymptotic formula (4.17) is non-trivial if $q \leq x^{\frac{2}{7}+\varepsilon}$. Similar asymptotic formula for the divisor function $\tau_2(\alpha)$ under $N(\alpha) \equiv \ell \pmod q$ have been obtained in [10] a non-trivial for $q \leq x^{\frac{5}{16}+\varepsilon}$. We claim that with a growth of k the divisor function $\tau_k(\alpha)$ in an arithmetic progression $N(\alpha) \equiv \ell \pmod q$ has an asymptotic formula that is a non-trivial in still smaller range of a change q.

The case $(\ell, q) > 1$ can be leaded easily to considered $(\ell, q) = 1$.

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