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# ON A CHARACTERIZATION OF ACHIEVED QUOTA WITH RESPECT TO GIVEN TARGET VALUES BY MEANS OF FUNCTIONAL EQUATIONS

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*Dedicated to Zoltán Daróczy and Imre Kátai*
*on occasion of their birthday-anniversaries*

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**Abstract.** Given a target value $t$ and a corresponding achieved value $a$ the achieved quota usually is computed by $\frac{a}{t}$ provided that $t$ (and $a$) are positive. In times of economic stress it also makes sense to have non-positive target values. Then the question arises how to measure the achieved quota $q(t, a)$. One possible solution for this problem is given here. It heavily rests upon a system of functional equations and its solution(s).

## 1. Introduction

In his job as a division director of a business segment in an internationally operating banking group the first author had to submit brief monthly reports illustrating the achievement of the target values.

As to positive target values $t$ and positive achieved values $a$ a reasonable and also obvious measure for achievement is the achieved quota $\frac{a}{t}$. Thus there is no measurement problem during periods of positive economic performance.

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But during times of economic stress it also makes sense to consider target values which are non-positive. Moreover it also may happen, even for positive target values, that the achieved value is negative or zero.

Some experiments dealing with the general setting led the authors to the following form for the achieved quota $q(t,a)$. (In fact, the case $t = 0$ became of interest only later.)

$$\text{(Q)} \qquad q(t,a) = \begin{cases} \frac{a}{t}, & \text{if } t > 0 \\ 1, & \text{for } t = 0 \\ 2 + \frac{a}{|t|}, & \text{if } t < 0. \end{cases}$$

This function behaves quite nicely for the applications it was designed for. But nevertheless one might wonder about its shape. So we looked for a kind of characterization of this function. It will be shown in a moment that this function is the only one which satisfies some conditions natural in this context.

## 2. The quota function as solution of a system of functional equations

Let $q: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a function. The value $q(t,a)$ will be interpreted as the achieved quota for given target value $t$ and given achieved value $a$. We consider the following conditions on $q$.

$$\text{(Q1)} \qquad q(\lambda \cdot t, \lambda \cdot a) = q(t,a) \text{ for all } t, a \in \mathbb{R}, \lambda > 0$$

$$\text{(Q2)} \qquad q(t,a_1) \le q(t,a_2) \text{ for all } t, a_1, a_2 \in \mathbb{R} \text{ such that } a_1 \le a_2$$

$$\text{(Q3)} \qquad q(t,a+h) - q(t,a) = q(t,a) - q(t,a-h) \text{ for all } t, a, h \in \mathbb{R}, h > 0$$

$$\text{(Q4)} \qquad q(t,t) = 1 \text{ for all } t \in \mathbb{R}$$

$$\text{(Q5)} \qquad q(t, t + |t|) = 2 \text{ for all } t \in \mathbb{R}, t \neq 0.$$

Condition (Q1) is in accordance with the interpretation of $q(t,a)$ as a quota when $t, a > 0$ — note that the expression $\frac{a}{t}$ used in this case without any discussion does of course have this property. This condition is also meaningful in all other cases since it means that the quota should not change when $t$ and $a$ simultaneously are, for example, doubled.

Condition (Q2) is very natural: For fixed target value larger values of the achieved values should increase the corresponding quota.

As we will see, (Q3) plays a crucial role when describing all solutions of the system (Q1)–(Q5). The meaning of this condition is: *An equal increase and decrease of the achieved value results in an equal increase and decrease of the corresponding quota.*

(Q4) and (Q5) are some (natural) normalization conditions: The quota should be 100%, if the target value and the achieved value coincide. If the target value is positive and if the achieved value is twice as much the quota should be 200%. For negative target value $t$ (Q5) also has this meaning: $q(t,0)=2$. The interpretation of both cases together is that the quota should be 2, if the achieved value exceeds the target value by $|t|$.

It turns out that there is one, and only one, function $q$ which satisfies the conditions (Q1)–(Q5).

**Theorem 2.1.** *A function $q:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ satisfies the conditions (Q1)–(Q5) if, and only if, $q$ is given by (Q).*

**Proof.** Since it is (almost) trivial to verify that $q$ given by (Q) satisfies (Q1)–(Q5), let us assume that $q:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ satisfies these conditions. We must show that $q$ is given by (Q).

Obviously (Q3) holds true for all $h$ (not only for $h>0$). Thus, given $u,v\in\mathbb{R}$, we put $x=\frac{u+v}{2}$, $h=\frac{v-u}{2}$. Then (Q3) reads as

$$\text{(J)} \qquad q\left(t,\frac{u+v}{2}\right)=\frac{1}{2}(q(t,u)+q(t,v)), u,v\in\mathbb{R}.$$

This implies that $q(t,\cdot)$ is a Jensen function. By [1, p. 43, 2.1.3 Jensen's Equation] we may conclude $q(t,a)=A_t(a)+\beta_t$ where $\beta_t\in\mathbb{R}$ and $A_t:\mathbb{R}\to\mathbb{R}$ is additive ($A_t(u+v)=A_t(u)+A_t(v)$ for all $u,v\in\mathbb{R}$). (Q2) implies that $A_t$ must be increasing and thus bounded on every bounded interval. Hence by [2, p. 15, Corollary 5] there is some real $\alpha_t$ such that $q(t,a)=\alpha_t a+\beta_t$ for all $t,a\in\mathbb{R}$. We may remark that $\alpha_t\geq 0$ since $q(t,\cdot)$ is increasing. By (Q4) we get $\alpha_t t+\beta_t=1$, or $\beta_t=1-\alpha_t t$. So $q(t,a)=\alpha_t(a-t)+1$. If $t>0$ (Q5) implies $2=\alpha_t(2t-t)+1$. Therefore $\alpha_t=\frac{1}{t}$ and $q(t,a)=\frac{a}{t}$ for all $a\in\mathbb{R}$ if $t>0$. In the case $t<0$ the same condition implies $2=\alpha_t(0-t)+1$ which means that $\alpha_t=\frac{1}{-t}=\frac{1}{|t|}$. So $q(t,a)=\frac{a-t}{|t|}+1=\frac{a}{|t|}-\frac{t}{|t|}+1=\frac{a}{|t|}+2$ in this case. Let, finally, $t=0$. Then $q(0,a)=\alpha_0 a+1$ and $q(0,2a)=\alpha_0(2a)+1$. Since $q(0,2a)=q(2\cdot 0,2\cdot a)=q(0,a)$ by (Q1) we get $2\alpha_0 a+1=\alpha_0 a+1$. This, for some $a\neq 0$, gives $\alpha_0=0$.

Thus $q$ is given by (Q). ∎

The proof shows that (Q1) is implied by (Q2)–(Q5) when $t\neq 0$. The result itself describes the solution of the problem completely and also in a satisfying manner with one exception: The function $q(0,\cdot)$ is constant, so, when the target value is zero — a quite reasonable target in many cases — the achieved value does not influence the quota at all!

## 3. A modified system of functional equations

To overcome the disadvantages described above one necessarily has to modify the system of functional equations. A first step into this direction is the following result.

**Theorem 3.1.** *A function $q: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies the conditions (Q2)–(Q5) if, and only if, there is some real $\alpha \geq 0$ such that*

(Q')
$$q(t,a) = \begin{cases} \frac{a}{t}, & \text{if } t > 0 \\ \alpha \cdot a + 1, & \text{for } t = 0 \\ 2 + \frac{a}{|t|}, & \text{if } t < 0. \end{cases}$$

The **proof** is part of the proof of Theorem 2.1.

Thus without (Q1) there are many quota functions which are non constant when the target value is zero. But also these solutions are not quite convincing since they are not continuous near the target value zero. To get functions continuous also when the target value approaches zero one thus has to weaken the condition even more. (Q5) might be a candidate for weakening. For fixed $\tau \geq 0$ we consider

($Q5_\tau$)
$$q(t, t + |t|) = 2 \text{ for all } t \in \mathbb{R}, |t| > \tau$$

Note that this condition is the same as (Q5) if $\tau = 0$. The result below might be of some interest also in practice if one takes into account the fact that variable salary components of managers may and in fact do depend on some measure relating target to achieved values.

**Theorem 3.2.** *Let $\tau > 0$ and let $q: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be given. Then $q$ satisfies (Q2)–(Q4) and ($Q5_\tau$) if, and only if, there is some function $\alpha: [-\tau, \tau] \to [0, \infty[$ such that*

(Q")
$$q(t,a) = \begin{cases} \frac{a}{t}, & \text{if } t > \tau \\ \alpha(t) \cdot (a - t) + 1, & \text{for } t \in [-\tau, \tau] \\ 2 + \frac{a}{|t|}, & \text{if } t < -\tau. \end{cases}$$

*This function is continuous if, and only if, $\alpha$ is continuous and if $\alpha(\pm\tau) = \frac{1}{\tau}$.*

*Moreover this function satisfies $q(\lambda \cdot t, \lambda \cdot a) = q(t,a)$ for all $t$ with $|t| > \tau$ and all $\lambda \geq 1$.*

**Proof.** The function $q$ defined by (Q") satisfies all conditions in question. Obviously this function $q$ is continuous on $[-\tau, \tau] \times \mathbb{R}$ if and only if $\alpha$ is continuous. The additional condition $\alpha(\pm\tau) = \frac{1}{\tau}$ then is equivalent to the continuity of $q$ on $\mathbb{R}\times\mathbb{R}$. (Note, that $\alpha(t) = \frac{q(t,a_2)-q(t,a_1)}{a_2-a_1}$ for $t \in [-\tau,\tau]$ and $a_1 \neq a_2$.) It is also clear that $q$ satisfies $q(\lambda\cdot t, \lambda\cdot a) = q(t,a)$ for all $|t| > \tau$ and $\lambda \geq 1$.

Let, finally, $q$ satisfy (Q2)–(Q4) and (Q5$_\tau$). Then, as in the proof of Theorem 2.1, we get that there is some $\alpha\colon\mathbb{R}\to[0,\infty[$ such that $q(t,a) = \alpha(t)\cdot(a-t)+1$ for all $(t,a) \in \mathbb{R}\times\mathbb{R}$. Moreover, again by the proof of the mentioned theorem, (Q5$_\tau$) implies $q(t,a) = \frac{a}{t}$ for $t > \tau$, $q(t,a) = \frac{a}{|t|} + 2$ for $t < -\tau$ and $q(t,a) = \alpha(t)(a-t)+1$ for all $\tau \in [-\tau,\tau]$. ∎

The following picture visualizes $q$ for $\tau = 1$ and the constant function $\alpha = 1 = \frac{1}{\tau}$.

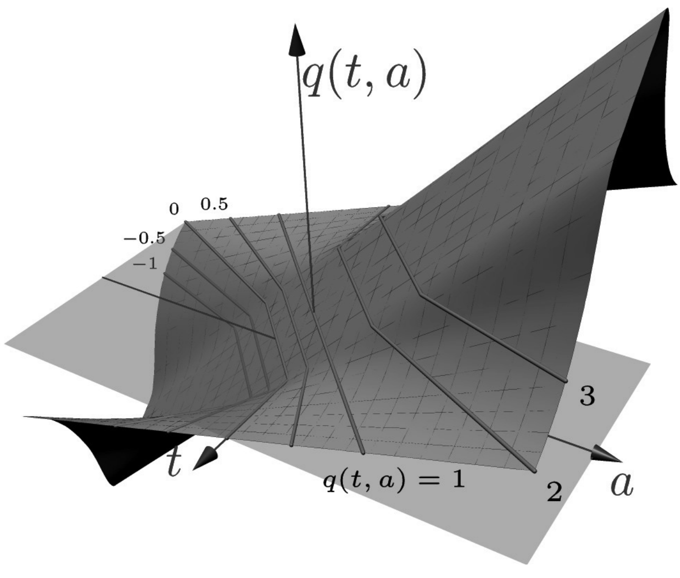


*Figure* 1. Visualization of $q$ for $\tau = 1$, $\alpha = \frac{1}{\tau} = 1$

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## References

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