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ON WRIGHT- BUT NOT JENSEN-CONVEX FUNCTIONS OF HIGHER ORDER

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Dedicated to the 75th birthday of
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Abstract. In this paper, we construct a general class of real functions whose members, for odd n, are nth-order Jensen-convex but not nth-order Wright-convex. This implies, for odd n, that the class of nth-order Jensen-convex functions is strictly bigger than that of nth-order Wright-convex functions while the analogous problem for even n remains unsolved.

1. Introduction

In the theory of convex functions three basic classes of convexity properties are traditionally considered. Given a nonempty real interval I, a function

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$f : I \to \mathbb{R}$ is called *convex*, *Wright-convex*, and *Jensen-convex* if f satisfies the following inequalities

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) \quad (x, y \in I,\ t \in [0,1]),$$
$$f(tx + (1-t)y) + f((1-t)x + ty) \le f(x) + f(y) \quad (x, y \in I,\ t \in [0,1]),$$
$$f\left(\tfrac{1}{2}x + \tfrac{1}{2}y\right) \le \tfrac{1}{2}f(x) + \tfrac{1}{2}f(y) \quad (x, y \in I),$$

respectively. Obviously, convex functions are always Wright-convex and Wright-convex functions are always Jensen-convex. If f is continuous, more generally f is upper bounded on a set of positive measure or on a set of second Baire category then these convexity properties are equivalent to each other (cf. [4], [9], [10]).

One can easily see that beyond convex functions, also additive functions are Wright-convex. Thus discontinuous additive functions are Wright-convex but not convex (because convex functions are continuous at interior points of I). Hence the class of Wright-convex functions is strictly larger than the class of convex functions. The exact connection between the notions of convexity and Wright-convexity was established by C. T. Ng [6] in 1987 in the following result.

Theorem A. *Let $I \subseteq \mathbb{R}$ be an open interval and $f : I \to \mathbb{R}$. Then f is Wright-convex if and only if there exists a convex function $g : I \to \mathbb{R}$ and an additive function $A : \mathbb{R} \to \mathbb{R}$ such that $f = g + A|_I$.*

In view of Rodé's generalization of the Hahn–Banach Theorem [11], Jensen-convex functions can also be described in terms of additive functions.

Theorem B. *Let $I \subseteq \mathbb{R}$ be an open interval and $f : I \to \mathbb{R}$. Then f is Jensen-convex if and only if there exists a family $\{A_\gamma\}_{\gamma\in\Gamma}$ of real additive functions and a family of real constants $\{a_\gamma\}_{\gamma\in\Gamma}$ such that $f = \sup_{\gamma\in\Gamma}(A_\gamma|_I + a_\gamma)$.*

As a consequence of this theorem, we can easily obtain that $|A| = \max(A, -A)$ is a Jensen-convex function provided that A is a real additive function. To demonstrate that there exist Jensen-convex but not Wright-convex functions, we show that $|A|$ is Wright-convex if and only if $A(x) = cx$ holds for some real constant c. Indeed, if $|A|$ is Wright-convex then we have that

$$\begin{aligned} |A|(tx + (1-t)y) &\le |A|(tx + (1-t)y) + |A|((1-t)x + ty) \\ &\le |A|(x) + |A|(y) \qquad (x, y \in \mathbb{R},\ t \in [0,1]). \end{aligned}$$

Therefore, A is bounded on any compact interval $[x, y]$. By the classical theorem of Bernstein and Doetsch [1], it follows that A is a continuous additive function, i.e., $A(x) = cx$ for some constant c.

In what follows, we recall the higher-order generalizations of the above notions and formulate analogous problems. Given a natural number n, a function $f : I \to \mathbb{R}$ is called *nth-order convex (or simply n-convex)*, *nth-order Wright-convex (or simply n-Wright-convex)*, and *nth-order Jensen-convex (or simply n-Jensen-convex)* (cf. [2], [3], [4], [8], [9]), if f satisfies the following inequalities

$$\begin{aligned}
[x_0, \dots, x_{n+1}; f] \geq 0 \quad & \big(x_0, \dots, x_{n+1} \in I,\, x_i \neq x_j \, (i \neq j)\big), \\
\big(\Delta_{h_1} \cdots \Delta_{h_{n+1}} f\big)(x) \geq 0 \quad & \big(h_1, \dots, h_{n+1} \in \mathbb{R}_+,\, x \in I \cap (I - (h_1 + \cdots + h_{n+1}))\big), \\
\big(\Delta_h^{n+1} f\big)(x) \geq 0 \quad & \big(h \in \mathbb{R}_+,\, x \in I \cap (I - (n+1)h)\big),
\end{aligned}$$

respectively. Here Δ_h stands for the difference operator defined by $\big(\Delta_h f\big)(x) := := f(x+h) - f(x)$ and $[x_0, \dots, x_{n+1}; f]$ denotes the $(n+1)$th-order divided difference of f defined for pairwise distinct elements $x_0, \dots, x_{n+1} \in I$ by

$$[x_0, \dots, x_{n+1}; f] := \sum_{i=0}^{n+1} \frac{f(x_i)}{\prod_{j=0, \, j \neq i}^{n+1} (x_i - x_j)}.$$

Obviously, n-Wright-convex functions are always n-Jensen-convex. On the other hand, the implication that n-convex functions are always n-Wright-convex easily follows from the identity

$$\big(\Delta_{h_1} \cdots \Delta_{h_n} f\big)(x) = h_1 \cdots h_n \sum_{(i_1, \dots, i_n)} [x, x + h_{i_1}, \dots, x + h_{i_1} + \cdots + h_{i_n}; f],$$

where the summation is taken over all permutations $(i_1, \dots, i_n)$ of the set $\{1, \dots, n\}$ (see [2]).

One can also see that, in the particular case $n = 1$, the notions of 1-convexity, 1-Wright-convexity, and 1-Jensen-convexity are equivalent to that of convexity, Wright-convexity, and Jensen-convexity, respectively. Indeed, taking $x, y \in$ with $x < y$ and $t \in]0,1[$, and, for $n = 1$, substituting $x_0 := 1$, $x_1 := tx + (1-t)y$, $x_2 := y$; $h_1 := t(y-x)$, $h_2 := (1-t)(y-x)$; and $h := \frac{1}{2}(y-x)$ in the inequalities defining the notions of 1-convexity, 1-Wright-convexity, and 1-Jensen-convexity above, these inequalities turn out to be equivalent to those that define convexity, Wright-convexity, and Jensen-convexity, respectively.

It is now a natural problem is to characterize the classes n-convex, n-Wright-convex, and n-Jensen-convex functions and to show that these classes are different. The following characterization of n convexity is due to Popoviciu ([4, Thm. 15.8.5], [8], [9]).

Theorem C. *Let $n \geq 2$, $I \subseteq \mathbb{R}$ be an open interval and $f : I \to \mathbb{R}$. Then f is n-convex if and only if f is $(n-1)$ times continuously differentiable and $f^{(n-1)}$ is convex.*

As a consequence of this theorem, it follows that polynomials of degree n are always n-convex.

The description of n-Wright-convex functions was obtained by Maksa and Páles in [5].

Theorem D. *Let $n \geq 1$, $I \subseteq \mathbb{R}$ be an open interval and $f : I \to \mathbb{R}$. Then f is n-Wright-convex if and only if there exists a unique n-convex function $g : I \to \mathbb{R}$ such that $f|_{\mathbb{Q}\cap I} = g|_{\mathbb{Q}\cap I}$ and $f - g$ is a polynomial function of nth degree, i.e., there exists $A_0 \in \mathbb{R}$ and, for each $k \in \{1, \dots, n\}$, there exists a symmetric k-additive function $A_k : \mathbb{R}^k \to \mathbb{R}$ such that*

$$f(x) = g(x) + A_n(x, \dots, x) + \cdots + A_1(x) + A_0 \qquad (x \in I).$$

Thus, polynomial functions of nth degree are always n-Wright-convex. Theorem D clearly implies that the class of n-Wright-convex functions is strictly bigger than that of n-convex functions. What concerns n-Jensen-convex functions, there is no known characterization of this class of functions. Furthermore, for even n it is not known if there exists an n-Jensen-convex function which is not n-Wright-convex. For odd n, Nikodem, Rajba and Wąsowicz [7] succeeded to construct a function which is n-Jensen-convex but not n-Wright-convex. More precisely, they showed that, for some discontinuous additive function $A : \mathbb{R} \to \mathbb{R}$, the function $f := |A|^n$ is n-Jensen-convex but not n-Wright-convex. In view of the main result of this paper, it will easily follow that this conclusion remains valid for all discontinuous additive functions A. The main tool of our approach is the use of the above decomposition theorem of Maksa and Páles.

2. Main results

Given a natural number n, a function $f : \mathbb{R} \to \mathbb{R}$ is called nth-order positively $\mathbb{Q}$-homogeneous if the identity

$$f(rx) = |r|^n f(x) \qquad (x \in \mathbb{R},\ r \in \mathbb{Q}) \tag{2.1}$$

holds.

The main result of this paper is stated in the following theorem.

Theorem 1. *Let n be an odd natural number and let $f : \mathbb{R} \to \mathbb{R}$ be a nonnegative nth-order positively $\mathbb{Q}$-homogeneous function. Then the following statements are equivalent.*

(i) f *is continuous;*

(ii) f *is of the form* $f(x) = c|x|^n$ *for some constant* $c \geq 0$*;*

(iii) f *is nth-order convex;*

(iv) f *is nth-order Wright-convex.*

Proof. Assume that f is continuous. Putting $x = 1$ in (2.1), we have that $f(r) = |r|^n f(1)$ for all $r \in \mathbb{Q}$. The continuity of f yields that $f(x) = |x|^n f(1)$ for all $x \in \mathbb{R}$. Thus (ii) holds with $c = f(1) \geq 0$.

Assume that (ii) holds. If $n = 1$, then $f(x) = c|x|$, hence f is obviously convex, i.e., 1-convex. Now assume that n is odd and $n > 1$. Then $n \geq 3$. By Popoviciu's characterization theorem of higher-order convexity (cf. [9], [4, Thm. 15.8.5]), in order to prove that f is nth-order convex, it is equivalent to showing that f is $(n-1)$ times continuously differentiable and $f^{(n-1)}$ is convex. Using (ii) and the oddness of n, a simple computation yields that $f^{(n-1)}(x) = cn!|x|$. Hence f is indeed $(n-1)$ times continuously differentiable and $f^{(n-1)}$ is convex resulting that f is nth-order convex.

If f is nth-order convex, then f is also nth-order Wright-convex (cf. [2]), i.e., (iii) trivially implies (iv).

Finally, assume that f is nth-order Wright-convex. Then, by Theorem D, there exists a continuous nth-order convex function $g : \mathbb{R} \to \mathbb{R}$ and an nth degree polynomial function $P : \mathbb{R} \to \mathbb{R}$ such that

$$f(x) = g(x) + P(x) \qquad (x \in \mathbb{R}) \qquad \text{and} \qquad P(r) = 0 \qquad (r \in \mathbb{Q}). \tag{2.2}$$

The polynomiality of P results that it is of the form

$$P(x) = A_n(x, \dots, x) + \dots + A_1(x) + A_0 \qquad (x \in \mathbb{R}), \tag{2.3}$$

where, for $k \in \{1, \dots, n\}$, $A_k : \mathbb{R}^k \to \mathbb{R}$ is an i-additive function and A_0 is a constant. Substituting $x = r \in \mathbb{Q}$, into the first equality in (2.2), it follows that

$$g(r) = f(r) - P(r) = f(r) = |r|^n f(1) \qquad (r \in \mathbb{Q}).$$

Thus, by the continuity of g, we get that $g(x) = |x|^n f(1)$ for all $x \in \mathbb{R}$. Combining this with (2.2) and (2.3), we obtain that

$$f(x) = |x|^n f(1) + A_n(x, \dots, x) + \dots + A_1(x) + A_0 \qquad (x \in \mathbb{R}).$$

Replacing x by rx and using the kth-order $\mathbb{Q}$-homogeneity of k-additive functions, we get

$$|r|^n f(x) = |r|^n |x|^n f(1) + r^n A_n(x, \dots, x) + \dots + rA_1(x) + A_0 \quad (x \in \mathbb{R},\ r \in \mathbb{Q}),$$

which, by a continuity argument, yields that

$$|y|^n f(x) = |y|^n |x|^n f(1) + y^n A_n(x,\dots,x) + \dots + yA_1(x) + A_0 \quad (x,y \in \mathbb{R}).$$

For positive y, both sides of this equation are polynomials of y. By comparing the coefficients of y^k, it follows that $A_k = 0$ for $k \in \{0,1,\dots,n-1\}$. Thus we get

$$|y|^n f(x) = |y|^n |x|^n f(1) + y^n A_n(x,\dots,x) \qquad (x,y \in \mathbb{R}).$$

Substituting $y = 1$ and $y = -1$, by the oddness of n, it follows that

$$f(x) = |x|^n f(1) + A_n(x,\dots,x) \qquad \text{and}$$
$$-f(x) = -|x|^n f(1) + A_n(x,\dots,x) \qquad (x \in \mathbb{R}).$$

Hence A_n is also identically zero and we get

$$f(x) = |x|^n f(1) \qquad (x \in \mathbb{R}),$$

which shows the continuity of f, i.e., the validity of (i). ■

Corollary. *Let n be an odd natural number and let $A_1,\dots,A_k : \mathbb{R}^n \to \mathbb{R}$ be symmetric n-additive functions. Then the function $f : \mathbb{R} \to \mathbb{R}$ defined by*

$$(2.4) \qquad f(x) := |A_1(x,\dots,x)| + \dots + |A_k(x,\dots,x)| \qquad (x \in \mathbb{R})$$

is nth-order Wright-convex if and only if $A_1,\dots,A_k$ are continuous.

Proof. If the symmetric n-additive functions $A_1,\dots,A_k$ are continuous, then they are of the form

$$A_i(x_1,\dots,x_n) = c_i x_1 \cdots x_n \qquad (x_1,\dots,x_n \in \mathbb{R})$$

for some constants $c_i \in \mathbb{R}$ (see [4, Thm. 13.4.3]). Therefore, for all $x \in \mathbb{R}$ we have that $f(x) = (|c_1| + \dots + |c_k|)|x|^n$. Obviously f is a nonnegative nth-order positively $\mathbb{Q}$-homogeneous which satisfies condition (ii) of the Theorem 1 with $c = |c_1| + \dots + |c_k|$. Thus f is also nth-order Wright-convex.

To prove the converse, let $A_1,\dots,A_k : \mathbb{R}^n \to \mathbb{R}$ be symmetric n-additive functions and let f be defined by (2.4). By the $\mathbb{Q}$-homogeneity property of n-additive functions, we immediately have that f is a nonnegative nth-order positively $\mathbb{Q}$-homogeneous. If f is nth-order Wright-convex, then, in view of the equivalence of conditions (iv) and (i) of the Theorem 1, it follows that f is continuous. Then it is continuous at the origin and hence for $\varepsilon = 1$ there exists $\delta > 0$ such that $f(x) < \varepsilon$ whenever $|x| < \delta$. This implies that $|A_k(x,\dots,x)| < \varepsilon$ for $|x| < \delta$ and $k \in \{1,\dots,n\}$. Hence, for all $k \in \{1,\dots,n\}$, the nth degree polynomial function $x \mapsto A_k(x,\dots,x)$ is bounded on the open interval $]-\delta,\delta[$. This yields that $A_1,\dots,A_n$ are continuous. ■

By taking a discontinuous additive function A in the subsequent theorem, we obtain that the class of nth-order Jensen-convex functions is strictly bigger than the class of nth-order Wright-convex functions provided that n is an odd natural number. The analogous statement for even n is conjectured and has been an open problem.

Theorem 2. *Let $A : \mathbb{R} \to \mathbb{R}$ be an additive function and n be an odd natural number. Then the function $f := |A|^n$ is nth-order Jensen-convex. The function f is nth-order Wright-convex if and only if A is continuous.*

Proof. The function $g(y) = |y|^n$ is $(n-1)$ times continuously differentiable on $\mathbb{R}$, and by the oddness of n, we have that its $(n-1)$ derivative $g^{(n-1)}(y) = n!|y|$ is convex. Thus, by Popoviciu's characterization theorem of nth-order convexity (cf. [9], [4, Thm. 15.8.5]), it follows that g is nth-order convex. Therefore, it is also nth-order Jensen-convex. This yields that, for all $y \in \mathbb{R}$ and $h \geq 0$, we have that

$$0 \leq \big(\Delta_h^{n+1} g\big)(y) = \sum_{k=0}^{n+1} (-1)^{n+1-k} \binom{n+1}{k} g(y+kh) \tag{2.5}$$

By the evenness of $n+1$ we obtain the identity

$$\big(\Delta_{-h}^{n+1} g\big)(y) = (-1)^{n+1} \big(\Delta_h^{n+1} g\big)(y-(n+1)h) = \big(\Delta_h^{n+1} g\big)(y-(n+1)h),$$

which shows that (2.5) is also valid for all $y \in \mathbb{R}$ and $h \leq 0$.

Now observe that $f = g \circ A$, and hence, for $x, u \in \mathbb{R}$,

$$\begin{aligned}
\big(\Delta_u^{n+1} f\big)(x) &= \sum_{k=0}^{n+1} (-1)^{n+1-k} \binom{n+1}{k} f(x+ku) = \\
&= \sum_{k=0}^{n+1} (-1)^{n+1-k} \binom{n+1}{k} g\big(A(x+ku)\big) = \\
&= \sum_{k=0}^{n+1} (-1)^{n+1-k} \binom{n+1}{k} g\big(A(x)+kA(u)\big) = \\
&= \big(\Delta_{A(u)}^{n+1} g\big)\big(A(x)\big) \geq 0,
\end{aligned}$$

which completes the proof of the nth-order Jensen-convexity of f.

Finally, assume that f is nth-order Wright-convex. Then, with the n-additive function $A_1(x_1, \ldots, x_n) := A(x_1) \cdots A(x_n)$ we have that f is of the form (2.4) (where $k = 1$), hence, by the Corollary, f is nth-order Wright-convex if and only if the n-additive function A_1 is continuous. However, this can only happen if the additive function A is continuous.

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