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CHINI'S EQUATIONS IN ACTUARIAL MATHEMATICS

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Dedicated to Professors Zoltán Daróczy and Imre Kátai on their 75th anniversary

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Abstract. We deal with an equation mentioned by M. Chini in [3] and recalled by A. Guerraggio in [6]. This is a multiplicative form of the equation previously studied by T. Riedel, P. K. Sahoo and the second author (cf. [9]). We solve it completely in the case where sets of zeros of the unknown functions are nonempty, under very mild assumptions. In the remaining case, we reduce the problem to the one considered in [9].

1. Introduction

M. Chini in [3] and later A. Guerraggio in [6] mention the following functional equations stemming from actuarial mathematics:

$$f(x+y)+f(x+z)=cf(x+u(y,z)) \tag{1.1}$$

for $x,y,z\in\mathbb{R}$, and

$$f(x+y)f(x+z)=cf(x+u(y,z)) \tag{1.2}$$

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for $x, y, z \in \mathbb{R}$. Equation (1.1) has been considered by T. Riedel, M. Sablik and P. K. Sahoo in [9]. In the present paper we deal with (1.2), and we consider a slightly more general form of the equation, namely

$$f(x+y)f(x+z) = \psi(x+u(y,z)) \tag{1.3}$$

for $x, y, z \in \mathbb{R}$ or $x, y, z \in [0, \infty)$. Let us note that a similar equation

$$\varphi(rt)\varphi(st) = \varphi(c(r,s)t),\ r, s \geq 0,\ c(r,s) \geq 0,\ t \in \mathbb{R}$$

is the classical one defining strictly stable distributions (a random variable X is said to be strictly stable if for any independent copies X_1, X_2 of X and any constants a, b there exists a constant c such that $aX_1 + bX_2$ has got the same distribution as cX; cf. eg. W. Jarczyk and J. K. Misiewicz [7]).

We solve the equation (1.3) in the present paper. First, we do it for functions mapping $\mathbb{R}$ into $\mathbb{R}$, practically without admitting regularity assumptions.

Then we restrict ourselves to the (better corresponding the actuarial setting of the problem) case of variables x, y and z from the interval $[0, \infty)$. This makes a difference: we have to require assumption of continuity imposed on the function u.

In both cases we deal with the situation where (X stands for $\mathbb{R}$ or $[0, \infty)$)

$$Z_f := \{x \in X : f(x) = 0\}$$

is non-empty. If $Z_f = \emptyset$ then (roughly speaking) we can take logarithms on both sides of (1.3) and get

$$F(x+y) + F(x+z) = \Psi(x+u(y,z)), \tag{1.4}$$

which may be solved similarly as (1.1) was in [9], under assumptions analogous to ones applied there.

Results concerning the case $X = \mathbb{R}$ are presented in the following section, and those referring to the case $X = [0, \infty)$ in consecutive section.

2. Case $X = \mathbb{R}$

We introduce the following notation:

$$u_1(x,y) = u(x,y) - x,\ u_2(x,y) = u(x,y) - y, \tag{2.1}$$

for $x, y \in \mathbb{R}$, and we put

$$\gamma(y) = u(y,y) - y$$

for $y \in \mathbb{R}$.

Moreover, we assume that:

(**A**) $u_1(\mathbb{R}^2), u_2(\mathbb{R}^2)$ are <u>connected</u> subsets (or intervals) of $\mathbb{R}$, so $u_1(\mathbb{R}^2) = |l_1, r_1|, u_2(\mathbb{R}^2) = |l_2, r_2|$ for some $-\infty \le l_1 \le r_1 \le +\infty, -\infty \le l_2 \le r_2 \le +\infty$ (here $|$ means (or [or] or)).

Let us start with the following.

Lemma 2.1. *Let (f, u, ψ) be a triple solving (1.3) for all x, y, $z \in \mathbb{R}$. Under assumptions (**A**) and if $\emptyset \ne Z_f \ne \mathbb{R}$, there exist $\alpha, c, T \in \mathbb{R}$ such that one of the following conditions is satisfied*

$$\begin{cases} Z_f = (-\infty, \alpha], Z_\psi = (-\infty, \alpha + T], \\ u(y,z) = T + \min\{y, z\}, \\ f(x) = c \textit{ for } x > \alpha, \psi(x) = c^2 \textit{ for } x > \alpha + T, \end{cases}$$

$$\begin{cases} Z_f = (-\infty, \alpha), Z_\psi = (-\infty, \alpha + T), \\ u(y,z) = T + \min\{y, z\}, \\ f(x) = c \textit{ for } x \ge \alpha, \psi(x) = c^2 \textit{ for } x \ge \alpha + T, \end{cases}$$

$$\begin{cases} Z_f = [\alpha, \infty), Z_\psi = [\alpha + T, \infty), \\ u(y,z) = T + \max\{y, z\}, \\ f(x) = c \textit{ for } x < \alpha, \psi(x) = c^2 \textit{ for } x < \alpha + T, \end{cases}$$

$$\begin{cases} Z_f = (\alpha, \infty), Z_\psi = (\alpha + T, \infty), \\ u(y,z) = T + \max\{y, z\}, \\ f(x) = c \textit{ for } x \le \alpha, \psi(x) = c^2 \textit{ for } x \le \alpha + T. \end{cases}$$

Proof. We start with some transformations of equation (1.3). Putting in (1.3) $x \to x - y$, we get

$$f(x)f(x - y + z) = \psi(x + u_1(y,z)) \tag{2.2}$$

for all x, y, $z \in \mathbb{R}$. Similarly, putting in (1.3) $x \to x - z$, we have

$$f(x + y - z)f(x) = \psi(x + u_2(y,z)) \tag{2.3}$$

for every x, y, $z \in \mathbb{R}$. Taking in (2.2) $y = z$, we obtain

$$f(x)^2 = \psi(x + \gamma(y)) \tag{2.4}$$

for all $x,\ y \in \mathbb{R}$ and putting $x \to x - \gamma(y)$ in (2.4), we have

$$f(x - \gamma(y))^2 = \psi(x) \tag{2.5}$$

for all x, $y \in \mathbb{R}$.

Take arbitrary $x_0 \in Z_f, y, z \in \mathbb{R}$ and use equation (2.2)

$$0 = f(x_0)f(x_0 - y + z) = \psi(x_0 + u_1(y, z)), \tag{2.6}$$

in order to get $x_0 + u_1(y, z) \in Z_\psi$. Similarly one can show that $x_0 + u_2(y, z) \in Z_\psi$ for $y, z \in \mathbb{R}$. Therefore

$$x_0 + |l_1, r_1| \cup |l_2, r_2| \subseteq Z_\psi \tag{2.7}$$

for every $x_0 \in Z_f$.

Putting in (2.5) arbitrary $x_1 \in Z_\psi, y \in \mathbb{R}$, we get

$$f(x_1 - \gamma(y))^2 = \psi(x_1) = 0, \tag{2.8}$$

so

$$x_1 - \gamma(y) \in Z_f \tag{2.9}$$

for all $x_1 \in Z_\psi$ and $y \in \mathbb{R}$. From (2.7) and (2.9) follows that

$$x_0 + |l_1 - \gamma(y), r_1 - \gamma(y)| \cup |l_2 - \gamma(y), r_2 - \gamma(y)| \subseteq Z_f \tag{2.10}$$

for all $x_0 \in Z_f$, and $y \in \mathbb{R}$.

Observe that $\gamma(y) = u_1(y, y) = u_2(y, y)$, so $\gamma(y) \in |l_1, r_1| \cap |l_2, r_2|$ and $l_k - \gamma(y) \le 0 \le r_k - \gamma(y)$ for $k \in \{1, 2\}$ and every $y \in \mathbb{R}$.

If a $y_0 \in \mathbb{R}$ and a $k \in \{1, 2\}$ exist such that $l_k - \gamma(y_0) =: -\epsilon < 0$, then from (2.10) it follows that for every $x_0 \in Z_f$ we have $[x_0 - \frac{\epsilon}{2}, x_0] \subseteq Z_f$. Therefore we have $(-\infty, x_0] \subseteq Z_f$ for arbitrary $x_0 \in Z_f$.

Analogically, if there are a $y_0 \in \mathbb{R}$ and a $k \in \{1, 2\}$ such that $r_k - \gamma(y_0) > 0$, then $[x_0, +\infty) \subseteq Z_f$ for arbitrary $x_0 \in Z_f$.

However, from the assumption $Z_f \neq \mathbb{R}$ it follows that there do not exist $y_1, y_2 \in \mathbb{R}$, $i, k \in \{1, 2\}$ such that $l_i < \gamma(y_1)$ and $\gamma(y_2) < r_k$ (otherwise we would have $\mathbb{R} \subseteq Z_f$, as follows from the remarks above). Thus, exactly one of the following three cases holds:

1. there is a $k \in \{1, 2\}$ such that $l_k < r_1 = r_2 =: r$ and $\gamma(\mathbb{R}) = \{r\}$,
2. there is a $k \in \{1, 2\}$ such that $l := l_1 = l_2 < r_k$ and $\gamma(\mathbb{R}) = \{l\}$,
3. for every $y \in \mathbb{R}$ we have $\gamma(y) = l_1 = l_2 = r_1 = r_2$.

If the condition 3. is satisfied, then functions u_1, u_2 are constantly equal to l_1 and we have $u(y,z) = l_1 + y$ and $u(y,z) = l_1 + z$ for arbitrary $y, z \in \mathbb{R}$, which is impossible.

Let us consider the condition 1. (the argumentation in the case 2. is analogical).

Let $\alpha = \sup Z_f$. From 1. follows that $Z_f = (-\infty, \alpha)$ or $Z_f = (-\infty, \alpha]$. We consider the case

$$Z_f = (-\infty, \alpha] \tag{2.11}$$

(the consideration in the remaining case is similar).

We are going to determine Z_ψ. The condition (2.11) and $\{r\} = \gamma(\mathbb{R}) \subseteq$ $\subseteq u_1(\mathbb{R}^2) \cap u_2(\mathbb{R}^2)$ imply that $x_0 + r \in Z_\psi$ for every $x_0 \in Z_f$. Thus, from the relation $Z_f = (-\infty, \alpha]$ we get $(-\infty, \alpha + r] \subseteq Z_\psi$. Take arbitrary $x_1 \in Z_\psi$, $y \in \mathbb{R}$. From the condition (2.11) it follows that $x_1 - r \in Z_f$, which means that $x_1 - r \leq \alpha$. Thus $Z_\psi \subseteq (-\infty, \alpha + r]$. Finally, $Z_\psi = (-\infty, \alpha + r]$.

Our next goal is to describe preimages of some intervals under the function u. From (1.3) we have the equivalence

$$x + u(y,z) \notin Z_\psi \iff x + y, x + z \notin Z_f$$

for all x, y, $z \in \mathbb{R}$. Therefore, making also use of equalities $Z_f = (-\infty, \alpha]$, $Z_\psi = (-\infty, \alpha + r]$, we obtain

$$x + u(y,z) > \alpha + r \iff x + y,\ x + z > \alpha$$

for all x, y, $z \in \mathbb{R}$. Thus $u^{-1}((\alpha + r - x, \infty)) = (\alpha - x, \infty) \times (\alpha - x, \infty)$ for every $x \in \mathbb{R}$, so

$$u^{-1}((a, \infty)) = (a - r, \infty) \times (a - r, \infty) \tag{2.12}$$

for all $a \in \mathbb{R}$. Similarly, one can obtain

$$u^{-1}((-\infty, b]) = (-\infty, b - r] \times \mathbb{R} \cup \mathbb{R} \times (-\infty, b - r] \tag{2.13}$$

for all $b \in \mathbb{R}$.

We express the function u in an analytic way. Fix a $c \in \mathbb{R}$. Obviously, for y, $z \in \mathbb{R}$ we have

$$u(y,z) = c \iff (y,z) \in u^{-1}((-\infty, c]) \cap u^{-1}((a, \infty)) \quad \text{for every} \quad a < c.$$

Thus, from (2.12) and (2.13), we have

$$\begin{aligned} & u(y,z) = c \\ & \quad \iff (y,\ z > a - r \wedge (y \leq c - r \vee z \leq c - r) \quad \text{for every} \quad a < c) \\ & \quad \iff (c - r = y \leq z \vee c - r = z \leq y). \end{aligned}$$

Therefore $u(y,z) = y + r \le z + r$ or $u(y,z) = z + r \le y + r$. Equivalently

$$u(y,z) = \min\{y,z\} + r \tag{2.14}$$

for every y, $z \in \mathbb{R}$.

Define a function $g\colon \mathbb{R} \to \mathbb{R}$ by the formula $g(x) = f(x - r)$, $x \in \mathbb{R}$. Take arbitrary x, y, $z \in \mathbb{R}$. Putting $x - r_1$ in place of x in (1.3) and making use of (2.14), we obtain

$$\begin{aligned}\psi(x + \min\{y,z\}) &\\ &= \psi((x-r) + \min\{y,z\} + r) = \psi(x - r + u(y,z)) \\ &= f(x - r + y) f(x - r + z) = g(x+y) g(x+z).\end{aligned} \tag{2.15}$$

Taking $y = z$ in (2.15), we get $\psi(x+y) = g(x+y)^2$ for every x, $y \in \mathbb{R}$, so $\psi = g^2 \ge 0$. From the definition of g, and because of (2.7), we have $g(x) = 0$ if and only if $x \le \alpha + r$. Substituting $x > \alpha + r$, $y = 0$, $z \ge 0$ in (2.15), we obtain

$$g(x)^2 = g(x) g(x+z).$$

Thus $g(x) = g(x+z)$ for every $x > \alpha + r$, $z \ge 0$, so $g|_{(\alpha+r,\infty)} = c$ for some $c \in \mathbb{R}$. Since $\psi = g^2$, we have $\psi|_{(\alpha+r,\infty)} = c^2$, which completes considerations in the case (2.7). ■

Now we can formulate the main result of the present section.

Theorem 2.1. *Assume* (**A**). *A triple* (f, u, ψ) *is a solution of* (1.3) *for all* x, y, $z \in \mathbb{R}$ *if and only if one of the following holds.*

(1) *functions* f *and* ψ *are constantly equal to* 0, u *is arbitrary,*

or

(2) *there exist* $\alpha, c, T \in \mathbb{R}$ *such that*
$$\begin{cases} Z_f = (-\infty, \alpha], Z_\psi = (-\infty, \alpha + T], \\ u(y,z) = T + \min\{y,z\}, \\ f(x) = c \text{ for } x > \alpha, \psi(x) = c^2 \text{ for } x > \alpha + T \end{cases},$$

or

(3) *there exist* $\alpha, c, T \in \mathbb{R}$ *such that*
$$\begin{cases} Z_f = (-\infty, \alpha), Z_\psi = (-\infty, \alpha + T), \\ u(y,z) = T + \min\{y,z\}, \\ f(x) = c \text{ for } x \ge \alpha, \psi(x) = c^2 \text{ for } x \ge \alpha + T \end{cases},$$

or

(4) *there exist* $\alpha, c, T \in \mathbb{R}$ *such that*

$$\begin{cases} Z_f = [\alpha, \infty), Z_\psi = [\alpha + T, \infty), \\ u(y,z) = T + \max\{y, z\}, \\ f(x) = c \text{ for } x < \alpha, \psi(x) = c^2 \text{ for } x < \alpha + T \end{cases},$$

or

(5) *there exist* $\alpha, c, T \in \mathbb{R}$ *such that*

$$\begin{cases} Z_f = (\alpha, \infty), Z_\psi = (\alpha + T, \infty), \\ u(y,z) = T + \max\{y, z\}, \\ f(x) = c \text{ for } x \le \alpha, \psi(x) = c^2 \text{ for } x \le \alpha + T \end{cases},$$

or

(6) $f(x)$, $\psi(x) > 0$ *for every* $x \in \mathbb{R}$ *and functions* $F = \ln f$, $\Psi = \ln \psi$ *fulfill the equation* (1.4) *for* x, y, $z \in \mathbb{R}$,

or

(7) $f(x) < 0$, $\psi(x) > 0$ *for every* $x \in \mathbb{R}$ *and functions* $F = \ln(-f)$, $\Psi = \ln \psi$ *fulfill the equation* (1.4) *for* x, y, $z \in \mathbb{R}$.

Remark. The constants α, c, T in (2) $-$ (5) are exactly these occurring in Lemma 2.1.

Proof. Observe that putting $x - u(0,0)$ instead of x and $y = z = 0$ in (1.3), we get

$$\psi(x) = \psi((x - u(0,0)) + u(0,0)) = f(x - u(0,0))^2 \tag{2.16}$$

for all $x \in \mathbb{R}$.

If $Z_f = \mathbb{R}$, then from (2.16) follows that ψ is constantly equal to 0, too.

If $Z_f = \emptyset$, then (2.16) implies that $Z_\psi = \emptyset$, and $\psi(x) > 0$ for every $x \in \mathbb{R}$. Therefore, taking $x = 0$ and arbitrary $y, z \in \mathbb{R}$ in (1.3), we have

$$f(y)f(z) = \psi(u(y,z)) > 0,$$

which means that f is of constant sign on $\mathbb{R}$. Put $\Psi = \ln \psi$ and $F = \ln f$ if $f > 0$ or $F = \ln(-f)$ if $f < 0$. Then from (1.3) it follows that F, Ψ fulfill the equation $F(x + y) + F(x + z) = \Psi(x + u(y,z))$ for x, y, $z \in \mathbb{R}$.

If $\emptyset \ne Z_f \ne \mathbb{R}$, then we apply Lemma 1.

Conversely, it is a matter of simple calculation to show that if a triple (f, u, ψ) satisfies any of the conditions (1) - (7) then it actually is a solution of (1.3). ■

3. Case $X = [0, \infty)$

We assume that

(**B**) $f : [0,\infty) \longrightarrow \mathbb{R}$, $u : [0,\infty)^2 \longrightarrow [0,\infty)$, $\psi : u([0,\infty)^2) \longrightarrow \mathbb{R}$, u is continuous.

Lemma 3.1. *Assume* (**B**). *If a triple* (f, u, ψ) *solves the equation* (1.3) *for all* $x, y, z \geq 0$, *and if* $\emptyset \neq Z_f \neq [0,\infty)$, *then there exist* α, c, $T \in \mathbb{R}$ *such that some of the following conditions is satisfied*

$$\begin{cases} Z_f = [0,\alpha], Z_\psi = [0, \alpha + T], \\ u(y,z) = T + \min\{y,z\}, \\ f(x) = c \text{ for } x \notin Z_f, \psi(x) = c^2 \text{ for } x \notin Z_\psi, \end{cases}$$

$$\begin{cases} Z_f = [0,\alpha), Z_\psi = [0, \alpha + T), \\ u(y,z) = T + \min\{y,z\}, \\ f(x) = c \text{ for } x \notin Z_f, \psi(x) = c^2 \text{ for } x \notin Z_\psi, \end{cases}$$

$$\begin{cases} Z_f = [\alpha,\infty), Z_\psi = [\alpha + T, \infty), \\ u(y,z) = T + \max\{y,z\}, \\ f(x) = c \text{ for } x \notin Z_f, \psi(x) = c^2 \text{ for } x \notin Z_\psi, \end{cases}$$

$$\begin{cases} Z_f = (\alpha,\infty), Z_\psi = (\alpha + T, \infty), \\ u(y,z) = T + \max\{y,z\}, \\ f(x) = c \text{ for } x \notin Z_f, \psi(x) = c^2 \text{ for } x \notin Z_\psi, \end{cases}$$

$$\begin{cases} Z_f = \{0\},\ Z_\psi = \{T\}, \\ f \text{ is of constant sign in } (0,\infty), \psi \text{ is positive in } (T,\infty) = u([0,\infty)^2) \setminus \{T\}, \\ (F,\ \Psi) \text{ satisfy } (1.4) \text{ for } x,y,z > 0 \text{ with } F = \ln|f|_{(0,\infty)}|, \Psi = \ln \psi|_{(T,\infty)}. \end{cases}$$

Remark. The constants α, c, T in the above lemma are exactly these occurring in Lemma 2.1.

Proof. Define u_1 and u_2 by (2.1). Continuity of u implies continuity of u_1, u_2, hence $u_1([0,y] \times [0,+\infty))$, $u_2([0,+\infty) \times [0,z])$ are intervals (perhaps degenerated to a point) for arbitrary $y, z \geq 0$. Moreover, $T = u(0,0) = u_1(0,0) = = u_2(0,0)$ belongs to each of these intervals. Therefore, we have

$$u_1([0,y] \times [0,\infty)) = |l_1(y), r_1(y)|,\ u_2([0,\infty) \times [0,z]) = |l_2(z), r_2(z)| \tag{3.1}$$

for all y, $z \geq 0$ and

$$l_1(y) \leq T \leq r_1(y),\ \ l_2(z) \leq T \leq r_2(z) \tag{3.2}$$

for all y, $z \geq 0$.

If $0 \le y < t$ then $u_1([0,y] \times [0,\infty)) \subseteq u_1([0,t] \times [0,\infty))$ and $u_2([0,\infty) \times \times [0,y]) \subseteq u_2([0,\infty) \times [0,t])$, so functions r_1, r_2 are weakly increasing and functions l_1, l_2 are weakly decreasing.

Putting $y = z = 0$ in (1.3), we obtain

$$f(x)^2 = \psi(x+T) \qquad (x \ge 0). \tag{3.3}$$

Fix arbitrary $w_0 \in Z_f$. For every $x, y, z \ge 0$ such that $x + y = w_0$ equation (1.3) implies that $x + u(y,z) \in Z_\psi$. Thus $w_0 - y + u(y,z) \in Z_\psi$ for arbitrary $y \in [0, w_0],\ z \in [0,\infty)$, so $w_0 + u_1([0,w_0] \times [0,\infty)) \subseteq Z_\psi$. In other words,

$$w_0 + |l_1(w_0), r_1(w_0)| \subseteq Z_\psi \tag{3.4}$$

and similarly one can show that

$$w_0 + |l_2(w_0), r_2(w_0)| \subseteq Z_\psi. \tag{3.5}$$

From the above inclusions and (3.3) as well as (3.2) it follows that

$$\Big(w_0 + (|l_1(w_0) - T, r_1(w_0) - T| \cup |l_2(w_0) - T, r_2(w_0) - T|)\Big) \cap [0,\infty) \subseteq Z_f. \tag{3.6}$$

If for every $w \in Z_f$ we have $l_1(w) = r_1(w) = l_2(w) = r_2(w) = T$, then we also have $u(y,z) = y + T$ for $y \in [0,w], z \ge 0$ and $u(y,z) = z + T$ for $z \in [0,w], y \ge 0$. This leads to a contradiction if $Z_f \ne \{0\}$.

If $Z_f = \{0\}$, then $u(0,z) = u(y,0) = T$ for every $y,\ z \ge 0$. From (3.3) we get $Z_\psi \cap [T,\infty) = \{T\}$. Moreover, if $w_0 \in Z_\psi$ then there are $x_0, y_0, z_0 \ge 0$ such that $w_0 = x_0 + u(y_0, z_0)$. Then (3.10) implies that $x_0 + y_0 \in Z_f = \{0\}$ or $x_0 + z_0 \in Z_f = \{0\}$. Suppose e. g. $x_0 = y_0 = 0$ (the other case is analogous). We get $w_0 = 0 + u(0, z_0) = T$. Therefore $Z_\psi = \{T\}$ and $x + u(y,z) = T \iff (x = y = 0 \le z$ or $x = z = 0 \le y)$. Thus $(0,\infty) + u((0,\infty)^2) = (T,\infty)$. Observe that (3.3) implies that ψ is positive in (T,∞). Furthermore, f is of constant sign in $(0,\infty)$ since if we had $f(y)f(z) < 0$ for some $y, z \ge 0$ then we would have $\psi(u(y,z)) = f(y)f(z) < 0$, which contradicts nonnegativity of ψ. Let us define $\Psi = \ln \psi|_{(T,\infty)}$ and $F = \ln \mid f|_{(0,\infty)} \mid$. It is easy to check that F, Ψ, u fulfill (1.4) for $x, y, z > 0$.

Let us consider now the case where there exist $k \in \{1,2\}$ and $w \in Z_f$ such that $l_k(w) < T$ or $r_k(w) > T$. Without loss of generality we can assume that $k = 1$.

Suppose that for some $w_1,\ w_2 \in Z_f$ we have $l_1(w_1) < T$ and $r_1(w_2) > T$. From monotonicity of l_1 and r_1 we get that if $w_1 \le w_2$ then $l_1(w_2) < T$, and if $w_1 \geq w_2$ then $r_1(w_1) > T$. Hence we can admit that there exists a $w_0 \in Z_f$ such that $l_1(w_0) < T < r_1(w_0)$. Let $\varepsilon = 0,5 \min\{T - l_1(w_0), r_1(w_0) - T\}$. The inclusion (3.6) implies that $[w_0 - \varepsilon, w_0 + \varepsilon] \cap [0, \infty) \subseteq Z_f$. In particular $w_0 + \varepsilon \in Z_f$. Since $r_1(w_0 + \varepsilon) \geq r_1(w_0)$, we obtain by an analogous argument that $[w_0 + \varepsilon, w_0 + 2\varepsilon] \subseteq Z_f$. We can proceed with this operation and obtain $[w_0, \infty) \subseteq Z_f$. Similarly, if $w_0 - \varepsilon > 0$, then $l_1(w_0 - \varepsilon) \leq l_1(w_0)$, hence $[w_0 - 2\varepsilon, w_0 - \varepsilon] \cap [0, \infty) \subseteq Z_f$. Continuing this process, we get $[0, w_0] \subseteq Z_f$. Thus $[0, \infty) = [0, w_0] \cup [w_0, \infty) \subseteq Z_f$, which is excluded by the assumption.

Therefore, we can assume without loss of generality that there is a $w \in Z_f$ such that $T < r_1(w)$, and for every $w \in Z_f$ we have $l_1(w) = T$. Let $w \in Z_f$ be such that $r_1(w) > T$. From (3.6) follows that $[w, w + \varepsilon] \subseteq Z_f$ for $\varepsilon = 0,5(r_1(w) - T)$. In particular $w + \varepsilon \in Z_f$ and $r_1(w + \varepsilon) \geq r_1(w)$, so $[w + \varepsilon, w + 2\varepsilon] \subseteq Z_f$. Repeating this procedure, we get $[w, \infty) \subseteq Z_f$.

Now let define $\alpha = \inf\{w \in Z_f \colon T < r_1(w)\}$. The above argumentation shows that $(\alpha, \infty) \subseteq Z_f$. We will prove that there is no element of Z_f smaller than α. Indeed, suppose that there is a $w \in Z_f$, $w < \alpha$. Then of course $r_1(w) = T = l_1(w)$, so $u(y, z) = y + T$ for every $y \in [0, w]$, $z \geq 0$. Therefore we have

$$f(x + y)f(x + z) = \psi(x + y + T) \tag{3.7}$$

for all x, $z \geq 0$, $y \in [0, w]$. Putting in (3.7) $y = 0$, $x_0 \notin Z_f$ and arbitrary $z \geq 0$ we get $f(x_0 + z) = \frac{\psi(x_0+T)}{f(x_0)}$, which means that the function f is constant on the set $[x_0, \infty)$. Since $(\alpha, \infty) \subseteq Z_f$, we have $f|_{[x_0,\infty)} = 0$, which is a contradiction with the choice of x_0. It proves that either $Z_f = (\alpha, \infty)$ or $Z_f = [\alpha, \infty)$. In the further part of the proof we consider the case $Z_f = [\alpha, \infty)$.

Fix a $w_0 \in Z_\psi$ and let x_0, y_0, $z_0 \geq 0$ be such that $x_0 + u(y_0, z_0) = w_0$. From equation (1.3) it follows that $x_0 + y_0 \in Z_f$ or $x_0 + z_0 \in Z_f$. Assume for example that $x_0 + y_0 \in Z_f$, so $x_0 + y_0 \geq \alpha$. Obviously for $x \geq x_0$, $y \geq y_0$ we have $x + y \geq \alpha$, so $x + y \in Z_f$ as well. Therefore, equation (1.3) implies that for arbitrary $x \geq x_0$, $y \geq y_0$, $z \geq 0$ we have $x + u(y, z) \in Z_\psi$, so $I := [x_0, \infty) + u([y_0, \infty) \times [0, \infty)) \subseteq Z_\psi$. Moreover, I is an interval unbounded from right side, which contains the point $x_0 + u(y_0, z_0) = w_0$. Thus we proved that if $w_0 \in Z_\psi$, then $[w_0, \infty) \subseteq Z_\psi$. Define $\beta = \inf Z_\psi$. $Z_\psi = (\beta, \infty)$ or $Z_\psi = [\beta, \infty)$. However, from (3.3) we have $Z_\Psi \cap [T, \infty) = [\alpha + T, \infty)$, so finally $Z_\psi = [\alpha + T, +\infty)$.

7. Equation (3.10) implies that for arbitrary $x, y, z \geq 0$ we have

$$x + u(y, z) \notin Z_\psi \iff (x + y, x + z \notin Z_f).$$

Taking advantage of the form of sets Z_f, Z_ψ we can write

$$x + u(y, z) < \alpha + T \iff (x + y, x + z < \alpha)$$

and

$$u(y,z) < \alpha + T - x \iff (y, z < \alpha - x),$$

which proves that $u^{-1}((-\infty, s)) = [0, s-T)^2$ for $s \in \mathbb{R}$. Making use of properties of counterimages, we get subsequently

$$u^{-1}([s,+\infty)) = \Big([0,+\infty) \times [s-T,+\infty)\Big) \cup \Big([s-T,+\infty) \times [0,+\infty)\Big),$$

$$u^{-1}([s,t)) = \Big([0,t-T) \times [s-T,t-T)\Big) \cup \Big([s-T,t-T) \times [0,t-T)\Big)$$

and eventually

$$u^{-1}(\{s\}) = \Big(\{s-T\} \times [0,s-T]\Big) \cup \Big([0,s-T] \times \{s-T\}\Big).$$

From the last relation follows that

$$u(y,z) = s \iff \Big(s-T = y \geq z\Big) \text{ or } \Big(s-T = z \geq y\Big),$$

so $u(y,z) = \max\{y,z\} + T$ for every $y, z \geq 0$.

8. From 5., 6., 7. follows

$$f(x+y)f(x+z) = \psi(x+y+T) \qquad (x,y,z \geq 0,\ y \geq z). \tag{3.8}$$

Putting in (3.8) $y = z \geq 0 = x$, we obtain $f(y)^2 = \psi(y+T)$, so we have

$$f(x+y)f(x+z) = f(x+y)^2 \qquad (x,y,z \geq 0, y \geq z), \tag{3.9}$$

which implies $f(x+z) = f(x+y)$ for $\alpha > x+y \geq x+z \geq 0$. Therefore, $f(x) = c,\ \psi(y) = c^2$ for $x \in [0,\alpha),\ y \in [T, \alpha+T)$. ■

Theorem 3.1. *Assume* (**B**). *A triple* (f,u,ψ) *is a solution of the equation*

$$f(x+y)f(x+z) = \psi(x+u(y,z)) \qquad (x,y,z \geq 0) \tag{3.10}$$

if and only if some of the following conditions is satisfied

$$(1) \begin{cases} Z_f = [0,\alpha], Z_\psi = [0,\alpha+T] \ \textit{ for some } \alpha \geq 0, T \in \mathbb{R}, \\ u(y,z) = T + \min\{y,z\}, \\ f(x) = c \textit{ for } x \notin Z_f, \psi(x) = c^2 \textit{ for } x \notin Z_\psi \ \textit{ for some } c \in \mathbb{R} \end{cases},$$

$$(2) \begin{cases} Z_f = [0,\alpha), Z_\psi = [0,\alpha+T) \ \textit{ for some } \alpha \geq 0, T \in \mathbb{R}, \\ u(y,z) = T + \min\{y,z\}, \\ f(x) = c \textit{ for } x \notin Z_f, \psi(x) = c^2 \textit{ for } x \notin Z_\psi \ \textit{ for some } c \in \mathbb{R} \end{cases},$$

$$(3)\quad \begin{cases} Z_f = [\alpha, +\infty), Z_\psi = [\alpha + T, +\infty) \quad \textit{for some } \alpha \geq 0, T \in \mathbb{R}, \\ u(y,z) = T + \max\{y, z\}, \\ f(x) = c \textit{ for } x \notin Z_f, \psi(x) = c^2 \textit{ for } x \notin Z_\psi \quad \textit{for some } c \in \mathbb{R} \end{cases},$$

$$(4)\quad \begin{cases} Z_f = (\alpha, +\infty), Z_\psi = (\alpha + T, +\infty) \quad \textit{for some } \alpha \geq 0, T \in \mathbb{R}, \\ u(y,z) = T + \max\{y, z\}, \\ f(x) = c \textit{ for } x \notin Z_f, \psi(x) = c^2 \textit{ for } x \notin Z_\psi \quad \textit{for some } c \in \mathbb{R} \end{cases},$$

$$(5)\quad \begin{cases} Z_f = \{0\},\ Z_\psi = \{T\} \quad \textit{for some } T \in \mathbb{R}, \\ f \textit{ is of constant sign in } (0, +\infty), \\ \psi \textit{ is positive in } (T, +\infty) = u([0, +\infty)^2) \setminus \{T\}, \\ F(x+y) + F(x+z) = \Psi(x + u(y,z)) \\ \textit{for } x, y, z > 0 \textit{ with } F = \ln |f|_{(0,+\infty)}|, \Psi = \ln \psi|_{(T,+\infty)} \end{cases},$$

$$(6)\quad \begin{cases} Z_f = Z_\psi = \emptyset,\ \psi \textit{ is positive in } [T, +\infty) \quad \textit{for some } T \in \mathbb{R}, \\ F(x+y) + F(x+z) = \Psi(x + u(y,z)) \\ \textit{for } x, y, z \geq 0 \textit{ with } F = \ln |f|, \Psi = \ln |\psi| \end{cases},$$

$$(7)\quad \begin{cases} Z_f = [0, +\infty),\ Z_\psi = u([0, +\infty)^2), \\ f \textit{ is constantly equal to 0 in } [0, +\infty), \\ \psi \textit{ is constantly equal to 0 in } u([0, +\infty)^2) \end{cases}.$$

Remark. The constants α, c, T in (1) $-$ (6) are exactly these occurring in Lemma 3.1.

Proof. From the previous lemma follows that it is enough to consider two cases: $Z_f = \emptyset$ and $Z_f = [0, +\infty)$.

If $Z_f = \emptyset$, then $Z_\psi = \emptyset$. Indeed, if there where $w_0 \in Z_\psi$, then there are $x_0, y_0, z_0 \geq 0$ such that $x_0 + u(y_0, z_0) = w_0$. Equation (3.10) implies that $x_0 + y_0 \in Z_f = \emptyset$ or $x_0 + z_0 \in Z_f = \emptyset$. Moreover, from (3.3) follows that ψ is positive on the set $[T, +\infty)$. We can define $F = \ln |f|$, $\Psi = \ln |\psi|$ and easily check that functions F, Ψ, u fulfil $F(x+y) + F(x+z) = \Psi(x + u(y,z))$ for $x, y, z \geq 0$.

If $Z_f = [0, +\infty)$, then of course for every $x, y, z \geq 0$ we have $x + u(y,z) \in Z_\psi$, so $Z_\psi = u([0, +\infty)^2)$ and both the function f and the function ψ are constantly equal to 0.

Conversely, it is a matter of simple calculation to show that if a triple (f, u, ψ) satisfies any of the conditions (1) - (7) then it actually is a solution of (3.10). ■

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