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JOINT LIMIT THEOREMS FOR PERIODIC HURWITZ ZETA-FUNCTION. II

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Dedicated to Professors Zoltán Daróczy and Imre Kátai on their 75th birthday

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Abstract. In the paper, we prove a joint limit theorem for a collection of periodic Hurwitz zeta-functions with transcendental and rational parameters.

1. Introduction

Let $s = \sigma + it$ be a complex variable, α, $0 < \alpha \leqslant 1$, be a fixed parameter, and $\mathfrak{a} = \{a_m : m \in \mathbb{N} = \mathbb{N} \cup \{0\}$ be a periodic sequence of complex numbers with minimal period $k \in \mathbb{N}$. The periodic Hurwitz zeta-function $\zeta(s, \alpha; \mathfrak{a})$ is defined, for $\sigma > 1$, by the series

$$\zeta(s, \alpha, \mathfrak{a}) = \sum_{m=0}^{\infty} \frac{a_m}{(m+\alpha)^s},$$

and continues analytically to the whole complex plane, except, maybe, for a simple pole at the point $s = 1$ with residue

$$a \stackrel{def}{=} \frac{1}{k} \sum_{m=0}^{k-1} a_m.$$

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If $a = 0$, then the function $\zeta(s, \alpha; \mathfrak{a})$ is entire. This easily follows from the equality

$$\zeta(s, \alpha, \mathfrak{a}) = \frac{1}{k^s} \sum_{m=0}^{k-1} a_m \zeta\Big(s, \frac{\alpha + m}{k}\Big), \quad \sigma > 1,$$

where $\zeta(s, \alpha)$ is the classical Hurwitz zeta-function.

In [5], two joint limit theorems on the weak convergence of probability measures on the complex plane for periodic Hurwitz zeta-functions were proved. For $j = 1, \dots, r$, let $\zeta(s, \alpha_j, \mathfrak{a}_j)$ be a periodic Hurwitz zeta-function with parameter α_j, $0 < \alpha_j \leqslant 1$, and periodic sequence of complex numbers $\mathfrak{a}_j = \{a_{mj} : m \in \mathbb{N}_0\}$ with minimal period $k_j \in \mathbb{N}$. For brevity, we use the notation $\underline{\sigma} = (\sigma_1, \dots, \sigma_r)$, $\underline{\sigma} + it = (\sigma_1 + it, \dots, \sigma_r + it)$, $\underline{\alpha} = (\alpha_1, \dots, \alpha_r)$, $\underline{\mathfrak{a}} = (\mathfrak{a}_1, \dots, \mathfrak{a}_r)$ and $\zeta(s, \underline{\alpha}; \mathfrak{a}) = \big(\zeta(s, \alpha_1, ; \mathfrak{a}_1), \dots \zeta(s, \alpha_r; \mathfrak{a}_r)\big)$. Denote by $\mathcal{B}(S)$ the class of Borel sets of the space S, and by meas$A$ the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. Then in [5], the weak convergence as $T \to \infty$ of the probability measure

$$\widehat{P}_T(A) \overset{def}{=} \frac{1}{T} \text{meas}\big\{t \in [0, T] : \underline{\zeta}(\underline{\sigma} + it, \underline{\alpha}; \underline{\mathfrak{a}}) \in A\big\}, \quad A \in \mathcal{B}(\mathbb{C}^r)),$$

was discussed. The cases of algebraically independent and rational parameters $\alpha_1, \dots, \alpha_r$ were considered. For statements of the mentional results, we need some notation and definitions.

Denote by $\gamma = \{s \in \mathbb{C} : |s| = 1\}$ the unit circle on the complex pane, and define

$$\Omega_1 = \prod_{m=0}^{\infty} \gamma_m \quad \text{and} \quad \Omega_2 = \prod_p \gamma_p,$$

where $\gamma_m = \gamma$ for all $m \in \mathbb{N}$, and $\gamma_p = \gamma$ for all primes p, respectively. The tori Ω_1 and Ω_2 are compact topological Abelian groups with respect to the product topology and the operation of pointwise multiplication. Moreover, let

$$\underline{\Omega}_1 = \prod_{j=1}^{r} \Omega_{1j},$$

where $\Omega_{1j} = \Omega_1$ for $j = 1, \dots, r$. Then $\underline{\Omega}_1$ is also a compact topological group. This gives two probability spaces $(\underline{\Omega}_1, \mathcal{B}(\underline{\Omega}_1), \underline{m}_{1H})$ and $(\Omega_2, \mathcal{B}(\Omega_2), m_{2H})$, where $\underline{m}_{1H}$ and $\underline{m}_{2H}$ are the probability Haar measures on $(\underline{\Omega}_1, \mathcal{B}(\underline{\Omega}_1))$ and $(\Omega_2, \mathcal{B}(\Omega_2))$, respectively. Denote by $\omega_{1j}(m)$ and $\omega_2(p)$ the projections of $\omega_{1j} \in \Omega_{1j}$ to γ_m, and of $\omega_2 \in \Omega_2$ to γ_p, respectively. Let $\underline{\omega} = (\omega_{11} \dots, \omega_{1r})$ be the elements of $\underline{\Omega}_1$. On the probability space $(\underline{\Omega}_1, \mathcal{B}(\underline{\Omega}_1), m_{1H})$ define the $\mathbb{C}^r$-valued random element $\zeta(\underline{\sigma}, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}})$ by the formula $(\underline{\sigma}, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}) = \big(\zeta(\sigma_1, \alpha_1, \omega_{1j}; \mathfrak{a}_1), \dots,$

$\zeta(\sigma_r, \alpha_r, \omega_{1r}; \mathfrak{a}_r)\big)$, where, for $\sigma_j > \frac{1}{2}$,

$$\zeta(\sigma_j, \alpha_j, \omega_{1j}, \mathfrak{a}_j) = \sum_{m=0}^{\infty} \frac{a_{mj}\omega_{1j}(m)}{(m+\alpha_j)^{\sigma_j}}, \quad j = 1, \dots, r.$$

Let $P_{1,\underline{\zeta}}$ be the distribution of the random element $\underline{\zeta}(\underline{\sigma}, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}})$. The first joint theorem of [5] is the following statement,

Theorem 1.1. *Suppose that* $\min\limits_{1 \leqslant j \leqslant r} \sigma_j > \frac{1}{2}$*, and that the numbers* $\alpha_1, \dots, \alpha_r$ *are algebraically independent over* $\mathbb{Q}$*. Then* $\widehat{P}_T$ *converges weakly to* $P_{1,\underline{\zeta}}$ *as* $T \to \infty$*.*

Now let $\alpha_j = \frac{a_j}{q_j}$, $0 < a_j < q_j$, $a_j, q_j \in \mathbb{N}$, $(a_j, q_j) = 1$, $j = 1, \dots, r$. On the probability space $(\Omega_2, \mathcal{B}(\Omega_2), m_{2H})$, define the $\mathbb{C}^r$-valued random element $\underline{\zeta}(\sigma, \underline{\alpha}, \omega_2; \underline{\mathfrak{a}})$ by the formula $\underline{\zeta}(\underline{\sigma}, \underline{\alpha}, \omega_2; \underline{\mathfrak{a}}) = \big(\zeta(\sigma_1, \alpha_1, \omega_2; \mathfrak{a}_1), \dots, \zeta(\sigma_r, \alpha_r, \omega_2; \mathfrak{a}_r)\big)$, where, for $\sigma_j > \frac{1}{2}$,

$$\zeta(\sigma_j, \alpha_j, \omega_j; \mathfrak{a}_j) = \omega_2(q_j) q_j^{\sigma_j} \sum_{m=1}^{\infty} \frac{a_{(m-a_j)/q_j, j}\,\omega_2(m)}{m^{\sigma_j}}, \quad j = 1, \dots, r,$$

Let $P_{2,\underline{\zeta}}$ be the distribution of the random element $\underline{\zeta}(\underline{\sigma}, \underline{\alpha}, \omega_2; \underline{\mathfrak{a}})$. The second joint theorem of [5] is of the following form.

Theorem 1.2. *For* $j = 1, \dots, r$*, suppose that* $\alpha_j = \frac{a_j}{q_j}$, $0 < \alpha_j < q_j$, $a_j, q_j \in \mathbb{N}$, $(\alpha_j, q_j) = 1$*, and that* $\sigma_j > \frac{1}{2}$*. Then* $\widehat{P}_T$ *converges weakly to* $P_{2,\underline{\zeta}}$ *as* $T \to \infty$*.*

The aim of this note is to consider the weak convergence of the probability measure

$$P_T(A) = \frac{1}{T}\text{meas}\big\{t \in [0,T] : \underline{\underline{\zeta}}(\underline{\underline{\sigma}} + it, \underline{\underline{\alpha}}, \underline{\underline{\mathfrak{a}}}) \in A\big\}, \quad A \in \mathcal{B}(\mathbb{C}^{r+r_1}),$$

where $\underline{\underline{\sigma}} = (\sigma_1, \dots, \sigma_r, \widehat{\sigma}_1, \dots, \widehat{\sigma}_{r_1})$, $\underline{\underline{\alpha}} = (\alpha_1, \dots, \alpha_r, \widehat{\alpha}_1, \dots, \widehat{\alpha}_{r_1})$, $\underline{\underline{\mathfrak{a}}} = (\mathfrak{a}_1, \dots, \mathfrak{a}_r, \widehat{\mathfrak{a}}_1, \dots, \widehat{\mathfrak{a}}_{r_1})$, and $\underline{\underline{\zeta}}(s, \underline{\underline{\alpha}}; \underline{\underline{\mathfrak{a}}}) = (\zeta(s, \alpha_1; \mathfrak{a}_1), \dots, \zeta(a, \alpha_r; \mathfrak{a}_r), \zeta(s, \widehat{\alpha}_1; \widehat{\mathfrak{a}}_1), \dots, \zeta(s, \widehat{\alpha}_{r_1}; \widehat{\mathfrak{a}}_{r_1})$. Here the parameters $\alpha_1, \dots, \alpha_r$ are algebraically independent over $\mathbb{Q}$, while the parameters $\widehat{\alpha}_1, \dots, \widehat{\alpha}_{r_1}$ are rational. For $j = 1, \dots, r_1$, $\widehat{\mathfrak{a}}_j = \{\widehat{a}_{mj} : m \in \mathbb{N}\}$ is a periodic sequence of complex numbers with minimal period $\widehat{k}_j \in \mathbb{N}$.

Define $\Omega = \underline{\Omega}_1 \times \Omega_2$. Then again Ω is a topological compact group, and we have a new probability space $(\Omega, \mathcal{B}(\Omega), m_H)$, where m_H is the probability Haar measure on $(\Omega, \mathcal{B}(\Omega))$. Denote by $\underline{\omega} = (\omega_{11}, \dots, \omega_{1r}, \omega_2)$ the elements of Ω, and on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$, define the $\mathbb{C}^{r+r_1}$-valued random element $\underline{\underline{\zeta}}(\underline{\underline{\sigma}}, \underline{\underline{\alpha}}, \underline{\underline{\omega}}; \underline{\underline{\mathfrak{a}}})$ by the formula

$$\underline{\underline{\zeta}}(\underline{\underline{\sigma}}, \underline{\underline{\alpha}}, \underline{\underline{\omega}}; \underline{\underline{\mathfrak{a}}}) = \big(\zeta(\sigma_1, \alpha_1, \omega_{11}; \mathfrak{a}_1). \dots, \zeta(\sigma_r, \alpha_r, \omega_{1r}; \mathfrak{a}_r),$$
$$\zeta(\widehat{\sigma}_1, \widehat{\alpha}_1, \widehat{\omega}_2; \widehat{\mathfrak{a}}_1), \dots, \zeta(\widehat{\sigma}_{r_1}, \widehat{\alpha}_{r_1}, \omega_2; \widehat{\mathfrak{a}}_{r_1})\big),$$

where, for $\sigma_j > \frac{1}{2}$,

$$\zeta(\sigma_j, \alpha_j, \omega_{1j}; \mathfrak{a}_j) = \sum_{m=0}^{\infty} \frac{a_{mj}\omega_{1j}(m)}{(m+\alpha_j)^{\sigma_j}}, \quad j = 1, \dots, r,$$

and, for $\widehat{\sigma}_j > \frac{1}{2}$,

$$\zeta(\widehat{\sigma}_j, \widehat{\alpha}_j, \omega_2; \widehat{\mathfrak{a}}_j) = \omega_2(q_j) q_j^{\widehat{\sigma}_j} \sum_{\substack{m=1 \\ m \equiv a_j (\bmod q_j)}}^{\infty} \frac{\widehat{a}_{(m-a_j)/q_j, j}\,\omega_2(m)}{m^{\widehat{\sigma}_j}}, \quad j = 1, \dots, r_1.$$

Let $P_{\underline{\underline{\zeta}}}$ be the distribution of the random element $\underline{\underline{\zeta}}(\underline{\underline{\sigma}}, \underline{\underline{\alpha}}, \underline{\underline{\omega}}; \underline{\underline{\mathfrak{a}}})$. Now we state the main result of the paper.

Theorem 1.3. *Suppose that min* $(\min_{1 \leqslant j \leqslant r} \sigma_j, \min_{1 \leqslant j \leqslant r_1} \widehat{\sigma}_j) > \frac{1}{2}$, *the numbers* $\alpha_1, \dots, \alpha_r$ *are algebraically independent over* $\mathbb{Q}$, *and that, for* $j = 1, \dots, r_1$, $\widehat{\alpha}_j = \frac{a_j}{q_j}$, $0 < a_j < q_j$, $a_j, q_j \in \mathbb{N}$, $(a_j, q_j) = 1$. *Then* P_T *converges weakly to* $P_{\underline{\underline{\zeta}}}$ *as* $T \to \infty$.

2. A limit theorem on Ω

Denote by $\mathcal{P}$ the set of all prime numbers.

Lemma 2.1. *Suppose that the numbers* $\alpha_1, \dots, \alpha_r$ *are algebraically independent over* $\mathbb{Q}$. *Then*

$$Q_T(A) \stackrel{def}{=} \frac{1}{T} \text{meas}\big\{t \in [0, T] : \big(((m+\alpha_1)^{-it} : m \in \mathbb{N}_0), \dots,$$
$$((m+\alpha_r)^{-it} : m \in \mathbb{N}_0), (p^{-it} : p \in \mathcal{P})\big) \in A\big\}, \quad A \in \mathcal{B}(\Omega),$$

converges weakly to the Haar measure m_H *as* $T \to \infty$.

Proof of the lemma is given in [3, Theorem 3]. ∎

3. Limit theorems for absolutely convergent series

Let $\sigma_1 > \frac{1}{2}$ be a fixed number, and

$$u_n(m,\alpha_j) = \exp\left\{ -\left(\frac{m+\alpha_j}{n+\alpha_j}\right)^{\sigma_1}\right\}, \quad m \in \mathbb{N}_0,\ n \in \mathbb{N},\ j = 1,\dots,r,$$

$$v_n(m) = \exp\left\{ -\left(\frac{m}{n}\right)^{\sigma_1}\right\}, \quad m, n \in \mathbb{N}.$$

For $j = 1,\dots,r$, the sequence $\mathfrak{a}_j$ is bounded. Therefore, a standard application of the Mellin formula and contour integration imply the absolute convergence for $\sigma > \frac{1}{2}$ of the series

$$\zeta_n(s,\alpha_j;\mathfrak{a}_j) = \sum_{m=0}^{\infty} \frac{a_{mj} u_n(m,\alpha_j)}{(m+\alpha_j)^s}, \quad j = 1,\dots,r,$$

and

$$\zeta_n(s,\alpha_j,\omega_{1j};\mathfrak{a}_j) = \sum_{m=0}^{\infty} \frac{a_{mj}\omega_{1j}(m) u_n(m,\alpha_j)}{(m+\alpha_j)^s}, \quad j = 1,\dots,r.$$

For $j = 1,\dots,r_1$, define $f(s,\widehat{\alpha}_j) = q_j^s$ and

$$f_n(s,\widehat{\alpha}_j;\widehat{\mathfrak{a}}_j) = \sum_{\substack{m=1\\ m\equiv a_j (\bmod q_j)}}^{\infty} \frac{\widehat{a}_{(m-a_j)/q_j,j} v_n(m)}{m^s}.$$

Then we have that

$$\zeta(s,\widehat{\alpha}_j;\widehat{\mathfrak{a}}_j) = f(s,\widehat{\alpha}_j) f(s,\widehat{\alpha}_j;\widehat{\mathfrak{a}}_j), \quad j = 1,\dots,r.$$

Also, for $j = 1,\dots,r_1$, define $f(\widehat{\sigma}_j,\widehat{\alpha}_j,\omega_2) = \omega_2(q_j) q_j^{\widehat{\sigma}_j}$ and

$$f_n(\widehat{\sigma}_j,\widehat{\alpha}_j,\omega_2;\widehat{\mathfrak{a}}_j) = \sum_{\substack{m=1\\ m\equiv a_j (\bmod q_j)}}^{\infty} \frac{\widehat{a}_{(m-a_j)/q_j,j}\omega_2(m) v_n(m)}{m^{\sigma_j}}.$$

Then, similarly as above, we have that the series for $f_n(s,\widehat{\alpha}_j;\widehat{\mathfrak{a}}_j)$ and $f_n(\widehat{\sigma}_j,\widehat{\alpha}_j,\omega_2;\widehat{\mathfrak{a}}_j)$ converge absolutely for $\sigma > \frac{1}{2}$.

Let, for brevity,

$$\underline{F}_n(\underline{\underline{\sigma}},\underline{\underline{\alpha}};\underline{\underline{\mathfrak{a}}}) = \big(\zeta_n(\sigma_1,\alpha_1;\mathfrak{a}_1),\dots,\zeta_n(\sigma_r,\alpha_r;\mathfrak{a}_r), f(\widehat{\sigma}_1,\widehat{\alpha}_1), f_n(\widehat{\sigma}_1,\widehat{\alpha}_1;\widehat{\mathfrak{a}}_1),\dots,$$
$$\dots, f(\widehat{\sigma}_{r_1},\widehat{\alpha}_{r_1}), f_n(\widehat{\sigma}_{r_1},\widehat{\alpha}_{r_1};\widehat{\mathfrak{a}}_{r_1})\big)$$

and

$$\underline{F}_n(\underline{\underline{\sigma}}, \underline{\underline{\alpha}}, \underline{\underline{\omega}}; \underline{\underline{\mathfrak{a}}}) = \big(\zeta_n(\sigma_1, \alpha_1, \omega_{11}; \mathfrak{a}_1), \dots, \zeta_n(\sigma_r, \alpha_r, \omega_{1r}; \mathfrak{a}_r), f(\widehat{\sigma}_1, \widehat{\alpha}_1, \omega_2), \\ f_n(\widehat{\sigma}_1, \widehat{\alpha}_1, \omega_2; \widehat{\mathfrak{a}}_1), \dots, f(\widehat{\sigma}_{r_1}, \widehat{\alpha}_{r_1}, \omega_2), f_n(\widehat{\sigma}_{r_1}, \widehat{\alpha}_{r_1}, \omega_2; \widehat{\mathfrak{a}}_{r_1})\big).$$

Lemma 3.1. *Suppose that the numbers $\alpha_1, \dots, \alpha_r$ are algebraically independent over $\mathbb{Q}$, and that $\min\big(\min\limits_{1\leqslant j\leqslant r} \sigma_j, \min\limits_{1\leqslant j\leqslant r_1} \widehat{\sigma}_j\big) > \frac{1}{2}$. Then the probability measures*

$$P_{T,n}(A) \overset{def}{=} \frac{1}{T}\text{meas}\big\{t \in [0,T] : \underline{F}_n(\underline{\underline{\sigma}} + it, \underline{\underline{\alpha}}; \underline{\underline{\mathfrak{a}}}) \in A\big\}, \quad A \in \mathcal{B}(\mathbb{C}^{r+2r_1}),$$

and, for a fixed $\underline{\underline{\omega}}_0 \in \Omega$,

$$\tilde{P}_{T,n}(A) \overset{def}{=} \frac{1}{T}\text{meas}\big\{t \in [0,T] : \underline{F}_n(\underline{\underline{\sigma}} + it, \underline{\underline{\alpha}}, \underline{\underline{\omega_2}}; \underline{\underline{\mathfrak{a}}}) \in A\big\}, \quad A \in \mathcal{B}(\mathbb{C}^{r+2r_1}),$$

both converge weakly to the same probability measure P_n on $(\mathbb{C}^{r+2r_1}, \mathcal{B}(\mathbb{C}^{r+2r_1}))$ as $T \to \infty$.

Proof. The series defining $\zeta_n(s, \alpha_j; \mathfrak{a}_j)$, $j = 1, \dots, r$, and $f_n(s, \widehat{\alpha}_j; \widehat{\mathfrak{a}}_j)$, $j = 1, \dots, r_1$, converge absolutely for $\sigma > \frac{1}{2}$. Therefore, the function $h_n : \Omega \to \mathbb{C}^{r+2r_1}$ given by the formula $h_n(\underline{\underline{\omega}}) = \underline{F}_n(\underline{\underline{\sigma}}, \underline{\underline{\alpha}}, \underline{\underline{\omega}}; \underline{\underline{\mathfrak{a}}})$ is continuous. Moreover,

$$h_n\big((p^{-it} : p \in P), ((m+\alpha_1)^{-it} : m \in \mathbb{N}_0), \dots, ((m+\alpha_r)^{-it} : m \in \mathbb{N}_0)\big) = \\ = F_n(\underline{\underline{\sigma}} + it, \underline{\underline{\alpha}}; \underline{\underline{\mathfrak{a}}}).$$

Thus, we have that $P_{T,n} = Q_T h_n^{-1}$, where Q_T is the measure of Lemma 2.1. This, the continuity of h_n, Lemma 2.1 and Theorem 5.1 of [1] show that $P_{T,n}$ converges weakly to $P_n = m_H h_n^{-1}$ as $T \to \infty$.

Similar arguments give that the measure $\tilde{P}_{T,n}$ converges weakly to $m_H \tilde{h}_n^{-1}$ as $T \to \infty$, where the function $\tilde{h}_n : \Omega \to \mathbb{C}^{r+2r_1}$ is related to h_n by the equality $\tilde{h}_n(\underline{\underline{\omega}}) = h_n(\underline{\underline{\omega}}\,\underline{\underline{\omega}}_0)$. The invariance of the Haar measure m_H with respect to translates by points from Ω leads to the equality $m_H \tilde{h}_n^{-1} = m_H h_n^{-1}$. The lemma is proved. ■

4. Approximation in the mean

Let, for $j = 1, \dots, r_1$ and $\sigma > 1$,

$$f(s, \widehat{\alpha}_j; \widehat{\mathfrak{a}}_j) = \sum_{\substack{m=1 \\ m \equiv a_j (\bmod q_j)}} \frac{\widehat{a}_{(m-a_j)/q_j, j}}{m^s}$$

and

$$f(s, \widehat{\alpha}_j, \omega_2; \widehat{\mathfrak{a}}_j) = \sum_{\substack{m=1 \\ m \equiv a_j (\bmod q_j)}} \frac{\widehat{a}_{(m-a_j)/q_j, j}\omega_2(m)}{m^s}.$$

Define

$$\begin{aligned}\underline{F}(\underline{\underline{\sigma}}, \underline{\underline{\alpha}}; \underline{\underline{\mathfrak{a}}}) = &\big(\zeta(\sigma_1, \alpha_1; \mathfrak{a}_1), \dots, \zeta(\sigma_r, \alpha_r; \mathfrak{a}_r), f(\widehat{\sigma}_1, \widehat{\alpha}_1), f(\widehat{\sigma}_1, \widehat{\alpha}_1; \widehat{\mathfrak{a}}_1), \dots, \\ &f(\widehat{\sigma}_{r_1}, \widehat{\alpha}_{r_1}), f(\widehat{\sigma}_{r_1}, \widehat{\alpha}_{r_1}; \widehat{\mathfrak{a}}_{r_1})\big),\end{aligned}$$

and

$$\begin{aligned}\underline{F}(\underline{\underline{\sigma}}, \underline{\underline{\alpha}}, \underline{\underline{\omega}}; \underline{\underline{\mathfrak{a}}}) = &\big(\zeta(\sigma_1, \alpha_1, \omega_{11}; \mathfrak{a}_1), \dots, \zeta(\sigma_r, \alpha_r, \omega_{1r}; \mathfrak{a}_r), f(\widehat{\sigma}_1, \widehat{\alpha}_1, \omega_2), \\ &f(\widehat{\sigma}_1, \widehat{\alpha}_1, \omega_2; \widehat{\mathfrak{a}}_1), \dots, f(\widehat{\sigma}_{r_1}, \widehat{\alpha}_{r_1}, \omega_2), f(\widehat{\sigma}_{r_1}, \widehat{\alpha}_{r_1}, \omega_2; \widehat{\mathfrak{a}}_{r_1})\big).\end{aligned}$$

In this section, we approximate $\underline{F}(\underline{\underline{\sigma}}, \underline{\underline{\alpha}}; \underline{\underline{\mathfrak{a}}})$ by $\underline{F}_n(\underline{\underline{\sigma}}, \underline{\underline{\alpha}}; \underline{\underline{\mathfrak{a}}})$, and $\underline{F}(\underline{\underline{\sigma}}, \underline{\underline{\alpha}}, \underline{\underline{\omega}}; \underline{\underline{\mathfrak{a}}})$ by $\underline{F}_n(\underline{\underline{\sigma}}, \underline{\underline{\alpha}}, \underline{\underline{\omega}}; \underline{\underline{\mathfrak{a}}})$ in the mean. Denote by $\varrho = \varrho_{r+2r_1}$ the Euclidean metric on $\mathbb{C}^{r+2r_1}$.

Lemma 4.1. *Suppose that* $\min\big(\min\limits_{1\leqslant j\leqslant r} \sigma_j, \min\limits_{1\leqslant j\leqslant r_1} \widehat{\sigma}_j\big) > \frac{1}{2}$. *Then*

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int\limits_0^T \varrho\big(\underline{F}(\underline{\underline{\sigma}} + it, \underline{\underline{\alpha}}; \underline{\underline{\mathfrak{a}}}), F_n(\underline{\underline{\sigma}} + it, \underline{\underline{\alpha}}; \underline{\underline{\mathfrak{a}}})\big) dt = 0.$$

Proof. The lemma follows from one-dimensional results obtained in [4], Lemma 6 and equality (13), and from the definition of ϱ. ■

Lemma 4.2. *Suppose that the numbers* $\alpha_1, \dots, \alpha_r$ *are algebraically independent over* $\mathbb{Q}$, *and that* $\min\big(\min\limits_{1\leqslant j\leqslant r} \sigma_j, \min\limits_{1\leqslant j\leqslant r_1} \widehat{\sigma}_j\big) > \frac{1}{2}$. *Then, for almost all* $\underline{\underline{\omega}} \in \Omega$,

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int\limits_0^T \varrho\big(\underline{F}(\underline{\underline{\sigma}} + it, \underline{\underline{\alpha}}\, \underline{\underline{\omega}}; \underline{\underline{\mathfrak{a}}}), F_n(\underline{\underline{\sigma}} + it, \underline{\underline{\alpha}}\, \underline{\underline{\omega}}; \underline{\underline{\mathfrak{a}}})\big) dt = 0.$$

Proof. The algebraic independence of the numbers $\alpha_1, \dots, \alpha_r$ implies their transcendence. Therefore, the lemma is a consequence of similar one-dimensional equalities given in [4], Lemma 7 and equality (14), and of the fact that the Haar measure m_H is the product of the Haar measures on $(\Omega_{1j}, \mathcal{B}(\Omega_{1j}))$, $j = 1, \dots, r$, and $(\Omega_2, \mathcal{B}(\Omega_2))$. ∎

5. Proof of Theorem 1.3

We start with the following statement.

Lemma 5.1. *Suppose that the numbers $\alpha_1, \dots, \alpha_r$ are algebraically independent over $\mathbb{Q}$, and that $\min\big(\min\limits_{1\leqslant j\leqslant r}\sigma_j, \min\limits_{1\leqslant j\leqslant r_1}\widehat{\sigma}_j\big) > \frac{1}{2}$. Then the probability measures*

$$P_{1,T}(A) \overset{def}{=} \frac{1}{T}\big\{t \in [0,T] : \underline{F}(\underline{\underline{\sigma}}+it, \underline{\underline{\alpha}}; \underline{\underline{\mathfrak{a}}}) \in A\big\}, \quad A \in \mathcal{B}(\mathbb{C}^{r+2r_1}),$$

and

$$\tilde{P}_{1,T}(A) \overset{def}{=} \frac{1}{T}\big\{t \in [0,T] : \underline{F}(\underline{\underline{\sigma}}+it, \underline{\underline{\alpha}}\,\underline{\underline{\omega}}; \underline{\underline{\mathfrak{a}}}) \in A\big\}, \quad A \in \mathcal{B}(\mathbb{C}^{r+2r_1}),$$

converge weakly to the same probability measure P_1 on $(\mathbb{C}^{r+2r_1}, \mathcal{B}(\mathbb{C}^{r+2r_1}))$ as $T \to \infty$.

Proof. For the proof of the lemma, it suffices to pass from the measures $P_{T,n}$ and $\tilde{P}_{T,n}$ to the measures $P_{1,T}$ and $\tilde{P}_{1,T}$, respectively. Let θ be a random variable defined on a certain probability space $(\widehat{\Omega}, \mathcal{B}(\widehat{\Omega}), \mathbb{P})$ and uniformly distributed on $[0,1]$. Define

$$\begin{aligned}\underline{X}_{T,n}(\underline{\underline{\sigma}}) =&\big(X_{T,n,1}(\sigma_1), \dots, X_{T,n,r}(\sigma_r), \widehat{X}_{T,1}(\widehat{\sigma}_1), \widehat{X}_{T,n,1}(\widehat{\sigma}_1), \dots,\\ &\dots, \widehat{X}_{T,r_1}(\widehat{\sigma}_{r_1}), \widehat{X}_{T,n,r_1}(\widehat{\sigma}_{r_1})\big) = \underline{F}_n(\underline{\underline{\sigma}}+i\theta T, \underline{\underline{\alpha}}; \underline{\underline{\mathfrak{a}}}).\end{aligned}$$

Then, denoting by $\xrightarrow{D}$ the convergence in distribution, we have, in view of Lemma 3.1, that, for $\min(\min\limits_{1\leqslant j\leqslant r}\sigma_j, \min\limits_{1\leqslant j\leqslant r_1}\widehat{\sigma}_j) > \frac{1}{2}$,

$$X_{T,n}(\underline{\underline{\sigma}}) \xrightarrow[T\to\infty]{D} \underline{X}_n(\underline{\underline{\sigma}}), \tag{5.1}$$

where

$$\underline{X}_n(\underline{\underline{\sigma}}) = \big(X_{n,1}(\sigma_1), \dots, X_{n,r}(\sigma_r), \widehat{X}_1(\widehat{\sigma}_1), \widehat{X}_{n,1}(\widehat{\sigma}_1), \dots, \widehat{X}_{r_1}(\widehat{\sigma}_{r_1})\widehat{X}_{n,r_1}(\widehat{\sigma}_{r_1})\big)$$

is the $\mathbb{C}^{r+2r_1}$-valued random element with the distribution P_n, and P_n is the limit measure in Lemma 3.1.

It is not difficult to see that the family of probability measures $\{P_n : n \in \in \mathbb{N}\}$ is tight. Indeed, the series for $\zeta_n(s, \alpha_j; \mathfrak{a})$ and $f_n(s, \widehat{\alpha}_j; \widehat{\mathfrak{a}}_j)$ are convergent absolutely for $\sigma > \frac{1}{2}$. Therefore, we have that, for $\sigma_j > \frac{1}{2}$ and $\widehat{\sigma}_j > \frac{1}{2}$,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T |\zeta_n(\sigma_j + it, \alpha_j; \mathfrak{a}_j)|^2 dt = \sum_{m=0}^{\infty} \frac{|a_{mj}|^2 u_n^2(m, \alpha_j)}{(m+\alpha_j)^{2\sigma_j}} \leqslant$$
$$\leqslant \sum_{m=0}^{\infty} \frac{|a_{mj}|^2}{(m+\alpha_j)^{2\sigma_j}}$$

and

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T |f_n(\widehat{\sigma}_j + it, \widehat{\alpha}_j; \widehat{\mathfrak{a}}_j)|^2 dt = \sum_{\substack{m=1 \\ m\equiv a_j (\bmod q_j)}}^{\infty} \frac{|\widehat{a}_{(m-a_j)/q_j, j}|^2 v_n^2(m)}{m^{2\widehat{\sigma}_j}} \leqslant$$
$$\leqslant \sum_{\substack{m=1 \\ m\equiv a_j (\bmod q_j)}}^{\infty} \frac{|\widehat{a}_{(m-a_j)/q_j, j}|^2}{m^{2\widehat{\sigma}_j}}$$

for $n \in \mathbb{N}$, $j = 1, \ldots, r$, and $j = 1, \ldots, r_1$, respectively. Now, denoting

$$R_j = R_j(\sigma_j) = \left(\sum_{m=0}^{\infty} \frac{|a_{mj}|^2}{(m+\alpha_j)^{2\sigma_j}} \right)^{1/2}$$

and

$$\widehat{R}_j = \widehat{R}_j(\widehat{\sigma}_j) = \left(\sum_{\substack{m=1 \\ m\equiv a_j (\bmod q_j)}} \frac{|\widehat{a}_{(m-a_j)/q_j, j}|^2}{m^{2\widehat{\sigma}_j}} \right)^{1/2},$$

we obtain that

$$\limsup_{T\to\infty} \frac{1}{T} \int_0^T |\zeta_n(\sigma_j + it, \alpha_j; \mathfrak{a}_j)| dt \leqslant R_j(\sigma_j), \quad j = 1, \ldots, r, \tag{5.2}$$

and

$$\limsup_{T\to\infty} \frac{1}{T} \int_0^T |f_n(\widehat{\sigma}_j + it, \widehat{\alpha}_j; \widehat{\mathfrak{a}}_j)| dt \leqslant \widehat{R}_j(\widehat{\sigma}_j), \quad j = 1, \ldots, r_1. \tag{5.3}$$

Let ε be an arbitrary positive number, and $M_j = R_j(3r)^{-1}\varepsilon^{-1}$, $j = 1, \ldots, r$, $\widehat{M}_{1j} = \widehat{q}_j(3r_1)^{-1}\varepsilon^{-1}$, $\widehat{M}_{2j} = \widehat{R}_j(3r)^{-1}\varepsilon^{-1}$, $j = 1, \ldots, r_1$. Then we deduce from (5.2) and (5.3) that

$$\begin{aligned}
&\limsup_{T\to\infty} \mathbb{P}\big((\exists j : |X_{T,n,j}(\sigma_j)| > M_j) \bigwedge (\exists j : |\widehat{X}_{T,j}(\widehat{\sigma}_j)| > \\
&> \widehat{M}_{1j} \wedge (\exists j : |\widehat{X}_{T,n,j}(\widehat{\sigma}_j)| > \widehat{M}_{2j})\big) \leqslant \\
&\leqslant \sum_{j=1}^{r} \limsup_{T\to\infty} \mathbb{P}(|X_{T,n,j}(\sigma_j)| > M_j) + \\
&+ \sum_{j-1}^{r_1} \limsup_{T\to\infty} \mathbb{P}(|\widehat{X}_{T,j}(\sigma_j)| > \widehat{M}_{1j}) + \sum_{j=1}^{r_1} \limsup_{T\to\infty} \mathbb{P}(|\widehat{X}_{T,n,j}(\widehat{\sigma}_j)| > \widehat{M}_{2j}) \\
&\sum_{j=1}^{r} \sup_{n\in\mathbb{N}} \limsup_{T\to\infty} \frac{1}{M_j} \int_0^T |\zeta_n(\sigma_j + it, \alpha_j; \mathfrak{a}_j)| dt \\
&+ \sum_{j=1}^{r_1} \limsup_{T\to\infty} \frac{1}{\widehat{M}_{1j}} \int_0^T |f(\widehat{\sigma}_j + it, \widehat{\alpha}_j)| dt \\
&+ \sum_{j=1}^{r_1} \sup_{n\in\mathbb{N}} \limsup_{T\to\infty} \frac{1}{\widehat{M}_{2j}} \int_0^T |f_n(\widehat{\sigma}_j + it, \widehat{\alpha}_j; \widehat{\mathfrak{a}}_j)| dt \leqslant \varepsilon.
\end{aligned} \tag{5.4}$$

This and (5.1) show that, for all $n \in \mathbb{N}$,

$$\mathbb{P}\big((\exists j : |X_{n,j}(\sigma_j)| > M_j) \wedge (\exists j : |\widehat{X}_j(\widehat{\sigma}_j)| > \widehat{M}_{1j}) \wedge (\exists j : |\widehat{X}_{n,j}(\widehat{\sigma}_j)| > \widehat{M}_{2j}\big) \leqslant \varepsilon.$$

Let

$$M = \bigg(\sum_{j=1}^{r} M_j^2 + \sum_{j=1}^{r_1} \widehat{M}_{1j}^2 + \sum_{j=1}^{r_1} \widehat{M}_{2j}^2 \bigg)^{1/2}.$$

Define the set $K_\varepsilon = \{\underline{\underline{z}} \in \mathbb{C}^{2+2r} : \varrho(\underline{\underline{z}}, 0) \leqslant M\}$. Then K_ε is a compact subset of $\mathbb{C}^{r+2r_1}$, and, by (4) $\mathbb{P}(\underline{X}_n(\underline{\underline{\sigma}}) \in K_\varepsilon) \geqslant 1 - \varepsilon$ for all $n \in \mathbb{N}$, or equivalently, $P_n(K_\varepsilon) \geqslant 1 - \varepsilon$ for all $n \in \mathbb{N}$. This means that the family $\{P_n : n \in \mathbb{N}\}$ is tight. Hence, by the Prokhorov theorem, Theorem 6.1 of [1], it is relatively compact. Therefore, there exists a subsequence $\{P_{n_k}\} \subset \{P_n\}$ such that P_{n_k} converges weakly to a certain probability measure P_1 on $(\mathbb{C}^{r+2r_1}, \mathcal{B}(\mathbb{C}^{r+2r_1}))$ as $k \to \infty$, that is

$$\underline{X}_{n_k}(\underline{\underline{\sigma}}) \xrightarrow[k\to\infty]{D} P_1. \tag{5.5}$$

Define the $\mathbb{C}^{r+2r_1}$-valued random element $\underline{X}_T(\underline{\underline{\sigma}}) = \underline{F}(\underline{\underline{\sigma}} + i\theta T, \underline{\underline{\alpha}}; \underline{\underline{\mathfrak{a}}})$. Then, using Lemma 4.1, we find that, for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \limsup_{T\to\infty} \mathbb{P}(\varrho(\underline{X}_T(\underline{\underline{\sigma}}), \underline{X}_{T,n}(\underline{\underline{\sigma}})) \geqslant \varepsilon) = 0.$$

This, (5.1), (5.5) and Theorem 4.2 of [1] give the relation

$$\underline{X}_T(\underline{\underline{\sigma}}) \xrightarrow[T\to\infty]{D} P_1, \tag{5.6}$$

and we have that $P_{1,T}$ converges weakly to P as $T \to \infty$. Moreover, (5.6) shows that the measure P_1 is independent of the sequence $\{P_{n_k}\}$. Hence,

$$\underline{X}_n(\underline{\underline{\sigma}}) \xrightarrow[n\to\infty]{D} P_1.$$

Similar arguments applied for the $\mathbb{C}^{r+2r_1}$-valued random elements $\tilde{X}_{T,n}(\underline{\underline{\sigma}}) = \underline{F}_n(\underline{\underline{\sigma}} + i\theta T, \underline{\underline{\alpha}}, \underline{\underline{\omega}}; \underline{\underline{\mathfrak{a}}})$ and $\underline{\tilde{X}}_T(\underline{\underline{\sigma}}) = \underline{F}(\underline{\underline{\sigma}} + i\theta T, \underline{\underline{\alpha}}, \underline{\underline{\omega}}; \underline{\underline{\mathfrak{a}}})$ together with Lemma 4.2 and (6) show that the measure $\tilde{P}_{1,T}$ also converges weakly to P_1 as $T \to \infty$.

Proof of Theorem 1.3. First we identify the limit measure P_1 in Lemma 5.1. For this, we apply the ergodicity of the one-parameter group $\{\varphi_t : t \in \mathbb{R}\}$, where

$$\begin{aligned} \varphi_t(\underline{\underline{\omega}}, \underline{\underline{\alpha}}) \quad = \quad & \big((m+\alpha_1)^{-it} : m \in \mathbb{N}_0), \dots, (m+\alpha_r)^{-it} : m \in \mathbb{N}_0), \\ & (p - it : p \in \mathcal{P})\big)\, \underline{\underline{\omega}}, \quad \underline{\underline{\omega}} \in \Omega, \end{aligned}$$

of measurable measure preserving transformations on Ω [3], Lemma 7.

We fix a continuity set A of the measure P_1 in Lemma 5.1. Then, by Theorem 2.1 of [1] and Lemma 5.1, we have that

$$\lim_{T\to\infty} \frac{1}{T} \text{meas}\{t \in [0,T] : \underline{F}(\underline{\underline{\sigma}} + it, \underline{\underline{\alpha}}, \underline{\underline{\omega}}; \underline{\underline{\mathfrak{a}}}) \in A\} = P_1(A). \tag{5.7}$$

Let the random variable ξ be defined on $(\Omega, \mathcal{B}(\Omega), m_H)$ by the formula

$$\xi(\underline{\underline{\omega}}) = \begin{cases} 1 & \text{if } \underline{F}(\underline{\underline{\sigma}}, \underline{\underline{\alpha}}, \underline{\underline{\omega}}; \underline{\underline{\mathfrak{a}}}) \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Then the expectation

$$\mathbb{E}\xi = m_H(\underline{\underline{\omega}} \in \Omega : \underline{F}(\underline{\underline{\sigma}}, \underline{\underline{\alpha}}, \underline{\underline{\omega}}; \underline{\underline{\mathfrak{a}}}) \in A) = P_{\underline{F}}(A), \tag{5.8}$$

where $P_{\underline{F}}$ is the distribution of $\underline{F}$. The ergodicity of the group $\{\varphi_t : t \in \mathbb{R}\}$ implies that of thew random process $\xi(\varphi_t(\underline{\underline{\omega}}, \underline{\underline{\alpha}}))$. Therefore, by the Birkhoff-Khintchine theorem. see, for example, [2], we obtain that, for almost all $\underline{\underline{\omega}} \in \Omega$,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \xi(\varphi_t(\underline{\underline{\omega}}; \underline{\underline{\mathfrak{a}}}))dt = \mathbb{E}\xi. \tag{5.9}$$

However, the definitions of ξ and φ_t show that

$$\frac{1}{T} \int_0^T \xi(\varphi_t(\underline{\underline{\omega}}; \underline{\underline{\mathfrak{a}}}))dt = \frac{1}{T}\text{meas}\{t \in [0, T] : \underline{F}(\underline{\underline{\sigma}} + it, \underline{\underline{\alpha}}, \underline{\underline{\omega}}; \underline{\underline{\mathfrak{a}}}) \in A\}.$$

This together with (5.8) and (5.9) leads, for almost all $\underline{\underline{\omega}} \in \Omega$, to

$$\lim_{T\to\infty} \frac{1}{T}\text{meas}\{t \in [0, T] : \underline{F}(\underline{\underline{\sigma}} + it, \underline{\underline{\alpha}}, \underline{\underline{\omega}}; \underline{\underline{\mathfrak{a}}}) \in A\} = P_{\underline{F}}(A).$$

Hence, find that $P_1(A) = P_{\underline{F}}(\overline{A})$ for all continuity sets A of P_1. Hence, P_1 coincides with $P_{\underline{F}}$.

It remains to pass from $P_{1,T}$ to P_T. Define the function $h : \mathbb{C}^{r+2r_1} \to \mathbb{C}^{r+r_1}$ by the formula

$$h(z_1, \ldots, z_r, z_{11}, z_{12}, \ldots, z_{r1}, z_{r2}) = (z_1, \ldots, z_r, z_{11}, z_{12}, \ldots, z_{r_1}, z_{r2}).$$

Then h is a continuous function, and $P_T = P_{1,T}h^{-1}$. This, the weak convergence of $P_{1,T}$ to $P_{\underline{F}}$ and Theorem 5.1 of [1] show that the measure P_T converges weakly to $P_{\underline{F}}h^{-1}$ as $T \to \infty$. Moreover, for $A \in \mathcal{B}(\mathbb{C}^{r+re_1})$,

$$\begin{aligned}
P_{\underline{F}}h^{-1}(A) &= m_H h^{-1}(\underline{\underline{\omega}} \in \Omega : \underline{F}(\underline{\underline{\sigma}}, \underline{\underline{\alpha}}, \underline{\underline{\omega}}; \underline{\underline{\mathfrak{a}}}) \in A) = \\
&= m_H(\underline{\underline{\omega}} \in \Omega : \underline{F}(\underline{\underline{\sigma}}, \underline{\underline{\alpha}}, \underline{\underline{\omega}}; \underline{\underline{\mathfrak{a}}}) \in h^{-1}A) = \\
&= m_H(\underline{\underline{\omega}} \in \Omega : h(\underline{F}(\underline{\underline{\sigma}}, \underline{\underline{\alpha}}, \underline{\underline{\omega}}; \underline{\underline{\mathfrak{a}}}) \in A) = \\
&= m_H\big(\underline{\underline{\omega}} \in \Omega : (\zeta(\sigma_1, \alpha_1, \omega_1; \mathfrak{a}_1), \ldots, \zeta(\sigma_r, \alpha_r, \omega_{1r}; \mathfrak{a}_r), f(\widehat{\sigma}_1, \widehat{\alpha}_1, \omega_2) \\
&\qquad f(\widehat{\sigma}_1, \widehat{\alpha}_1, \omega_2; \widehat{\mathfrak{a}}_1), \ldots, f(\widehat{\sigma}_{r_1}, \widehat{\alpha}_{r_1}, \omega_2) f(\widehat{\sigma}_{r_1}, \widehat{\alpha}_{r_1}, \omega_2; \widehat{\mathfrak{a}}_{r_1})) \in A\big) = \\
&= m_H\big(\underline{\underline{\omega}} \in \Omega : (\zeta(\sigma_1, \alpha_1, \omega_{11}; \mathfrak{a}_1), \ldots, \zeta(\sigma_r, \alpha_r, \omega_{11}; \mathfrak{a}_r), \\
&\qquad \zeta(\widehat{\sigma}_1, \widehat{\alpha}_1, \omega_2; \widehat{\mathfrak{a}}_1), \ldots, \zeta(\widehat{\sigma}_{r_1}, \widehat{\alpha}_{r_1}, \omega_2; \widehat{\mathfrak{a}}_{r_1})) \in A\big) = \\
&= m_H(\underline{\underline{\omega}} \in \Omega : \underline{\underline{\zeta}}(\underline{\underline{\sigma}}, \underline{\underline{\alpha}}, \underline{\underline{\omega}}; \underline{\underline{\mathfrak{a}}}) \in A) = P_{\underline{\underline{\zeta}}}(A).
\end{aligned}$$

Thus, the measure P_T converges weakly to $P_{\underline{\underline{\zeta}}}$ as $T \to \infty$. The theorem is proved. ■

References

[1] **Billingsley, P.,** *Convergence of Probability Measures*, Wiley, New York, 1968.

[2] **Cramér H., M.R. Leadbetter,** *Stationary and Related Stochastic Processes*, Wiley, Nework, 1967.

[3] **Laurinčikas, A.,** Joint universality of zeta-functions with periodic coefficients, *Izv.RAN, Ser.Matem.*, **74**(3) (2010), 79–102 (in Russian) = *Izv.:Math.*, **74**(3) (2010), 515–539.

[4] **Rimkevičienė, A.,** Limit theorems for the periodic Hurwitz zeta-function, *Šiauliai Math. Seminar*, **5**(13) (2010), 55–69.

[5] **Rimkevičienė, A.,** Joint limit theorems for periodic Hurwitz zeta-functions, *Šiauliai Math. Seminar*, **6**(14) (2011), 53–68.

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