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ITERATIONS OF THE MEAN-TYPE MAPPINGS AND UNIQUENESS OF INVARIANT MEANS

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Abstract. It is known, that under some natural conditions, the iterates of any continuous mean-type mapping converge to a continuous invariant mean-type mapping, and, moreover, the invariant continuous mean-type mapping is unique. In this paper we show that the assumption of the continuity in the "moreover" part of this result is superfluous. We also show that every increasing and homogeneous mean is continuous, and we give some new conditions on convergence of the sequence of iterates of, not necessarily continuous, mean-type mapping to a unique invariant mean-type mapping. Some examples and applications are presented.

1. Introduction

Let $I \subset \mathbb{R}$ be an interval and $p \geq 2$ a fixed integer number. A function $M : I^p \to I$ is called a mean if $\min(x_1, ..., x_p) \leq M(x_1, ..., x_p) \leq \max(x_1, ..., x_p)$ for all $(x_1, ..., x_p) \in I^p$ ([1],[2]). The mean M is reflexive, i.e., $M(x, ..., x) = x$ for $x \in I$. In general the mean M need not be continuous.

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It is known that *if* $M_i : I^p \to I$ *for* $i = 1, ..., p$ *are continuous means and*

$$\max(M_1(\mathbf{x}), ..., M_p(\mathbf{x})) - \min(M_1(\mathbf{x}), ..., M_p(\mathbf{x})) < \max(\mathbf{x}) - \min(\mathbf{x}),$$

for all $\mathbf{x} = (x_1, ..., x_p) \in I^p$ *such that* $\min(\mathbf{x}) < \max(\mathbf{x})$, *then the sequence of iterates of the mean-type mapping* $\mathbf{M} = (M_1, ..., M_p)$ *converges to a mean-type mapping* $\mathbf{K} = (K, ..., K)$, *where* $K : I^p \to I$ *is a continuous and* $\mathbf{M}$*-invariant mean, i.e.* $K \circ \mathbf{M} = K$; *moreover, a continuous* $\mathbf{M}$*-invariant mean is unique* ([8], Theorem 1, also [6]). At this background it was an open and frequently asked question if, under these assumptions, there can exists another (necessarily discontinuous) **M**-invariant mean. In this paper we show that the answer is "no". In section 2 we observe that every mean $M : I^p \to I$ is continuous on the diagonal of I^p (Theorem 1). It turns out that this fact allows to prove the uniqueness of **M**-invariant mean K without the assumption of the continuity of K (Theorem 3). We also show that, if I is an open interval and $0 \notin I$, then every increasing and homogeneous mean is continuous (Theorem 2). Moreover we give new conditions on convergence of the sequence of iterates of a mean-type mapping **M** to a unique **M**-invariant mean-type mapping (Theorem 4). This is the first result on convergence of iterates of a mean-type mapping where the discontinuous means are admitted. The relevant examples and some applications are also presented.

2. Means and their continuity

Let $I \subset \mathbb{R}$ be an interval and $p \in \mathbb{N}$, $p \geq 2$, fixed. A function $M : I^p \to \mathbb{R}$ is said to be a *p-variable mean in* I (briefly, a *mean in* I) if, for all $x_1, ..., x_p \in I$,

$$\min(x_1, ..., x_p) \leq M(x_1, .., x_p) \leq \max(x_1, ..., x_p).$$

A mean M in I is called *strict* if these inequalities are sharp whenever

$\min(x_1, ..., x_p) < \max(x_1, ..., x_p)$;

symmetric if $M\left(x_{\sigma(1)}, .., x_{\sigma(p)}\right) = M(x_1, .., x_p)$ for all permutations σ of the set $\{1, ..., p\}$ and for all $x_1, ..., x_p \in I$;

homogeneous, if for all permissible $x_1, ..., x_p$ and $t > 0$,

$$M(tx_1, ..., tx_p) = tM(x_1, .., x_p).$$

Remark 2.1. A function $M : I^p \to \mathbb{R}$ is a mean iff $M(J^p) = J$ for every subinterval $J \subset I$.

By $\Delta(I^p)$ we denote the diagonal of the cube I^p; thus

$$\Delta(I^p) := \{(x_1, ..., x_p) \in I^p : x_1 = \cdots = x_p\}.$$

Remark 2.2. If $M : I^p \to \mathbb{R}$ is a mean then M maps I^p onto I and, moreover, M is reflexive, that is, for all $(x_1, ..., x_p) \in \Delta(I^p)$ such that $x_1 = ... = x_p = x$ we have $M(x, ..., x) = x$.

To see that there are a lot of discontinuous (even homogeneous) means, consider the following

Remark 2.3. ([8]) A function $M : (0, \infty)^2 \to (0, \infty)$ is a homogeneous mean iff

$$M(x, y) = yf\left(\frac{x}{y}\right), \qquad x, y > 0,$$

where $f : (0, \infty) \to (0, \infty)$ is such that $\min(x, 1) \leq f(x) \leq \max(x, 1)$ for all $x > 0$; M is symmetric iff $f(x) = xf\left(\frac{1}{x}\right)$ for all $x > 0$; M is strict iff $\min(x, 1) < f(x) < \max(x, 1)$ for all $x > 0$, $x \neq 1$, or, equivalently iff $0 < \frac{f(x)-1}{x-1} < 1$ for all $x \in (0, \infty)$, $x \neq 1$. Thus, for a discontinuous function $f : (0, \infty) \to (0, \infty)$ such that $\min(x, 1) < f(x) < \max(x, 1)$ and $f(x) = xf\left(\frac{1}{x}\right)$ for all $x > 0$, we get a discontinuous homogeneous, symmetric and strict mean M. Note, however, that f is continuous at the point $x = 1$, which implies the continuity of M on the diagonal $\{(x, x) : x > 0\}$.

We prove the following property of an arbitrary mean.

Theorem 2.4. *If M is a p-variable mean in an interval I, then it is continuous at every point of the diagonal $\Delta(I^p)$.*

Proof. From the definition of the mean, for arbitrary $(x_0, ..., x_0) \in \Delta(I^p)$, we have

$$\min(x_1, ..., x_p) \leq M(x_1, ..., x_p) \leq \max(x_1, ..., x_p),$$

whence, by the reflexivity of M,

$$\lim_{x_1 \to x_0, \ldots, x_p \to x_0} M(x_1, ..., x_p) = x_0 = M(x_0, \ldots, x_0). \qquad \blacksquare$$

3. Increasing and homogeneous means are continuous

Remark 3.1. (cf. [7], [8]) If a function $M : I^p \to \mathbb{R}$ is reflexive, i.e., $M(x, ..., x) = x$ for $x \in I$, and (strictly) increasing with respect to each variable, then M is a (strict) mean.

The mean described by this remark is referred to as *increasing* one.

Theorem 3.2. *Suppose that $I \subset \mathbb{R}$ is an open interval and $0 \notin I$. If $M : I^p \to I$ is a homogeneous and increasing mean, then it is continuous.*

Proof. We may assume that I is open. Take arbitrary $(x_1, ..., x_p) \in I^p$. Since M is increasing, the one sided limits $M(x_1-, ..., x_p-)$, $M(x_1+, ..., x_p+)$ exist and

$$M(x_1-, ..., x_p-) \leq \liminf_{u_1 \to x_1, ..., u_p \to x_p} M(u_1, ..., u_p) \leq M(x_1, ..., x_p) \leq$$
$$\leq \limsup_{u_1 \to x_1, ..., u_p \to x_p} M(u_1, ..., u_p) \leq M(x_1+, ..., x_p+).$$

From the homogeneity of M, for t close to 1, we have

$$M(x_1, ..., x_p) = \frac{1}{t} M(tx_1, ..., tx_p).$$

Hence

$$M(x_1, ..., x_p) = \lim_{t \to 1-} \frac{1}{t} M(tx_1, ..., tx_p) = M(x_1-, ..., x_p-),$$

$$M(x_1, ..., x_p) = \lim_{t \to 1+} \frac{1}{t} M(tx_1, ..., tx_p) = M(x_1+, ..., x_p+),$$

which implies that M is continuous at the point $x = (x_1, ..., x_p)$.

With reference to Remark 3, this result implies the following

Remark 3.3. Let $f : (0, \infty) \to (0, \infty)$ be a function such that $\min(x, 1) \leq$ $\leq f(x) \leq \max(x, 1)$ for all $x > 0$. If f is increasing and discontinuous at least at one point, then the mean $M : (0, \infty)^2 \to (0, \infty)$ given by $M(x, y) = yf\left(\frac{x}{y}\right)$ for $x, y > 0$, is not increasing.

In fact, M is increasing mean if, and only if, f is increasing and the function $(0, \infty) \ni x \longmapsto \frac{f(x)}{x}$ is decreasing.

4. Mean-type mappings, iterations and invariant means

A mapping $\mathbf{M} : I^p \to I^p$ is referred to as a *p-variable mean-type mapping in I* (briefly, *mean-type mapping in I*), if there are some means $M_i : I^p \to I$, $i = 1, ..., p$, such that $\mathbf{M} = (M_1, ..., M_p)$. We say that the mean-type mapping $\mathbf{M}$ is *strict (homogeneous, symmetric)* if each of its coordinate means $M_1, ..., M_p$ is strict (homogeneous, symmetric).

If $\mathbf{M} : I^p \to I^p$ is a mean-type mapping then, the sequence $(\mathbf{M}^n)_{n=0}^{\infty}$ of the iterates of $\mathbf{M}$ is defined by

$$\mathbf{M}^0 := \mathrm{Id}\,|_{I^p}\ ; \qquad \mathbf{M}^{n+1} := \mathbf{M} \circ\ \mathbf{M}^n \quad \text{for } \ n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}.$$

Remark 4.1. Suppose that $\mathbf{M} : I^p \to I^p$, $\mathbf{M} = (M_1, ..., M_p)$, is a mean-type mapping. Then, for every $n \in \mathbb{N}_0$,

$$\mathbf{M}^n = (M_{n,1}, ..., M_{n,p})$$

where, for all $i = 1, ..., p$; $(x_1, ..., x_p) \in I^p$,

$$M_{i,0}\,(x_1, ..., x_p) = x_i,$$

and, since $\mathbf{M}^{n+1} = \mathbf{M} \circ \mathbf{M}^n = \mathbf{M}^n \circ \mathbf{M}$, we have for all $n \in \mathbb{N}_0$, $i = 1, ..., p$, $(x_1, ..., x_p) \in I^p$,

$$M_{i,n+1}\,(x_1, ..., x_p) = M_i\,(M_{1,n}\,(x_1, ..., x_p)\,, ..., M_{p,n}\,(x_1, ..., x_p)) \tag{1}$$

$$M_{i,n+1}\,(x_1, ..., x_p) = M_{i,n}\,(M_1\,(x_1, ..., x_p)\,, ..., M_p\,(x_1, ..., x_p))\,. \tag{2}$$

Moreover, for each $i = 1, .., p$ and for every $n \in \mathbb{N}_0$,the function $M_{i,n}$ is a mean in I; in particular, each iterate of a mean-type mapping is a mean-type mapping.

The "moreover" part of this remark follows from the definition of the mean.

Given a mean-type mapping $\mathbf{M} : I^p \to I^p$ and a mean $K : I^p \to I$ we say that K is *invariant with respect to the mean-type mapping* $\mathbf{M}$, briefly, $\mathbf{M}$-*invariant*, if

$$K \circ \mathbf{M} = K;$$

and a mean-type mapping $\mathbf{K} : I^p \to I^p$ is $\mathbf{M}$-*invariant,* if $\mathbf{K} = \mathbf{K} \circ \mathbf{M}$.

The invariant mean $\mathbf{K}$ is also called the Gauss composition of the coordinate means of means of $\mathbf{M}$ ([3], cf. also [4], [5]).

Remark 4.2. A mean-type mapping $\mathbf{K} : I^p \to I^p$, $\mathbf{K} = (K_1, ..., K_p)$, is $\mathbf{M}$-*invariant* iff, for each $i = 1, .., p$, the mean $K_i : I^p \to I$ is $\mathbf{M}$-*invariant.*

Now we prove the main result of this section.

Theorem 4.3. *Let an interval $I \subset \mathbb{R}$ and $p \in \mathbb{N}$, $p \geq 2$, be fixed. Suppose that $\mathbf{M} : I^p \to I^p$, $\mathbf{M} = (M_1, ..., M_p)$, is a mean-type mapping such that the sequence of iterates $(\mathbf{M}^n)_{n=0}^{\infty}$ converges pointwise in I^p, and put*

$$\mathbf{K} := \lim_{n\to\infty} \mathbf{M}^n, \qquad \mathbf{K} = (K_1, ..., K_p).$$

Then $\mathbf{K}$ is an $\mathbf{M}$-invariant mean-type mapping. If moreover $K_1 = \cdots = K_p$, then $K := K_1$ is a unique $\mathbf{M}$-invariant mean.

Proof. Suppose that the sequence of iterates $(\mathbf{M}^n)_{n=0}^{\infty}$ converges pointwise to a mapping $\mathbf{K} = (K_1, ..., K_p)$ in I^p, that is

$$K_i(x) = \lim_{n\to\infty} M_{i,n}(\mathbf{x}), \qquad \mathbf{x} = (x_1, ..., x_p) \in I^p;\ i = 1, ..., p,$$

where, by Remark 6, $(M_{i,n})_{n=0}^{\infty}$, $i = 1, ..., p$, are the sequences of means. Clearly, $K_1, ..., K_p$ are means. From (2) we have

$$M_{i,n+1}(\mathbf{x}) = M_{i,n}(M_1(\mathbf{x}), ..., M_p(\mathbf{x})), \qquad \mathbf{x} \in I^p;\ i = 1, ..., p;\ n \in \mathbb{N}.$$

Hence, letting $n \to \infty$, we get

$$K_i(\mathbf{x}) = K_i(M_1(\mathbf{x}), ..., M_p(\mathbf{x})), \qquad \mathbf{x} \in I^p;\ i = 1, ..., p, \tag{3}$$

so the means $K_1, ..., K_p$, as well as the mean-type mapping $\mathbf{K}$, are $\mathbf{M}$-invariant. Assuming that $K_1 = ... = K_p$ and putting $K = K_1$, we hence get $K = K \circ \mathbf{M}$. To prove the uniqueness of K assume that $K^* : I^p \to I$ is an $\mathbf{M}$-invariant mean, i.e. that $K^* = K^* \circ \mathbf{M}$. Hence, by induction, $K^* = K^* \circ \mathbf{M}^n$ for all $n \in \mathbb{N}$, that is

$$K^*(\mathbf{x}) = K^*(M_{1,n}(\mathbf{x}), ..., M_{p,n}(\mathbf{x})), \qquad \mathbf{x} \in I^p;\ n \in \mathbb{N}. \tag{4}$$

Since

$$K(x) = \lim_{n\to\infty} M_{i,n}(\mathbf{x}), \qquad \mathbf{x} = (x_1, ..., x_p) \in I^p;\ i = 1, ..., p,$$

we have

$$\lim_{n\to\infty} \mathbf{M}^n(\mathbf{x}) = \lim_{n\to\infty} (M_{1,n}(\mathbf{x}), ..., M_{p,n}(\mathbf{x})) = (K(\mathbf{x}), ..., K(\mathbf{x})) \in \Delta(I^p).$$

By Theorem 1 the mean K^* is continuous on the diagonal $\Delta(I^p)$. Hence, letting $n \to \infty$ in (4), and making use of the reflexivity of K^*, we obtain

$$K^*(\mathbf{x}) = K^*(K(\mathbf{x}), ..., K(\mathbf{x})) = K(\mathbf{x}), \qquad \mathbf{x} \in I^p. \qquad \blacksquare$$

Theorem 3 gives the following improvement of the main results of [6] and [8].

Corollary 4.4. *Let $I \subset \mathbb{R}$ be an interval and $p \in \mathbb{N}$, $p \geq 2$, fixed. Suppose that $\mathbf{M} : I^p \to I^p$, $\mathbf{M} = (M_1, ..., M_p)$, is a continuous mean-type mapping of I^p such that, for all $(x_1, ..., x_p) \in I^p \backslash \Delta(I^p)$,*

$$\max(M_1(x_1, ..., x_p), ..., M_p(x_1, ..., x_p)) - \min(M_1(x_1, ..., x_p), ..., M_p(x_1, ..., x_p))$$
$$< \max(x_1, ..., x_p) - \min(x_1, ..., x_p).$$

Then

(i) *for every* $n \in \mathbb{N}$, *the* n*-th iterate* $\mathbf{M}^n = (M_{n,1}, ..., M_{n,p})$, *is a mean-type mapping of* I^p;

(ii) *there is a continuous mean* $K : I^p \to I$ *such that the sequence of iterates* $(\mathbf{M}^n)_{n=0}^{\infty}$ *converges uniformly on compact subsets of* I^p, *to the mean-type mapping* $\mathbf{K} : I^p \to I^p$, $\mathbf{K} = (K_1, ..., K_p)$ *such that*

$$K_1 = \cdots = K_p = K;$$

(iii) $\mathbf{K} : I^p \to I^p$ *is* $\mathbf{M}$*-invariant (equivalently,* K *is* $\mathbf{M}$*-invariant);*

(iv) *the* $\mathbf{M}$*-invariant mean (and* $\mathbf{M}$*-invariant mean-type mapping) is unique;*

(v) *if* $\mathbf{M}$ *is strict then so is* K *(and* $\mathbf{K}$*);*

(vi) *if* $\mathbf{M}$ *is (strictly) increasing with respect to each variable then so is* K*;*

(vii) *if* $\mathbf{M}$ *is homogeneous, then every iterate of* $\mathbf{M}$ *and* K *are homogeneous.*

Indeed, except for (iv), all the remaining results coincide with Theorem 1 in [8]. The counterpart of (iv) in [8] reads as follows: "a continuous $\mathbf{M}$-invariant mean (mean-type mapping) is unique". In view of Theorem 2 the assumed here continuity of the $\mathbf{M}$-invariant mean is redundant.

The assumption of the continuity of mean-type mapping in Corollary 1 (as well as in [8], Theorem 1) can be weakened. Namely, we shall prove the following

Theorem 4.5. *Let* $I \subset \mathbb{R}$ *be an interval and* $p \in \mathbb{N}$, $p \geq 2$, *fixed. Suppose that* $\mathbf{M} : I^p \to I^p$, $\mathbf{M} = (M_1, ..., M_p)$, *is a mean-type mapping such that*

$$\max(M_1(\mathbf{x}), ..., M_p(\mathbf{x})) - \min(M_1(\mathbf{x}), ..., M_p(\mathbf{x})) < \max(\mathbf{x}) - \min(\mathbf{x})$$

and

$$\limsup_{\mathbf{u} \to \mathbf{x}} [\max(M_1(\mathbf{u}), ..., M_p(\mathbf{u})) - \min(M_1(\mathbf{u}), ..., M_p(\mathbf{u}))] < \max \mathbf{x} - \min \mathbf{x}$$

for all $\mathbf{x} = (x_1, ..., x_p) \in I^p \backslash \Delta(I^p)$. *Then*

(i) *there is a mean* $K : I^p \to I$ *such that the sequence of iterates* $(\mathbf{M}^n)_{n=0}^{\infty}$ *converges pointwise to the mean-type mapping* $\mathbf{K} : I^p \to I^p$, $\mathbf{K} = (K_1, ..., K_p)$ *such that* $K_1 = ... = K_p = K$

(ii) $K : I^p \to I$ *is a unique* $\mathbf{M}$*-invariant mean;*

(iii) *if* $\mathbf{M}$ *is strict then so is* K*;*

(iv) *if* $I = (0, \infty)$ *and* $\mathbf{M}$ *is homogeneous, then every iterate of* $\mathbf{M}$ *and* K *are homogeneous.*

Proof. To avoid writing too long expression we assume that $p = 2$. It is easy to see that the same reasoning works in general case.

Let $M, N : I^2 \to I$ be means satisfying the assumptions. Consider the sequence $(M, N)^n$, $n \in \mathbb{N}$, of the iterates of the mean-type mapping $(M, N) :$ $: I^2 \to I^2$. Putting (cf. Remark 6)

$$M_0(x, y) := x, \quad N_0(x, y) := y, \quad (x, y) \in I^2,$$

$$(M_n, N_n) := (M, N)^n, \quad n \in \mathbb{N}_0,$$

we have

$$M_{n+1} = M \circ (M_n, N_n) = M_n \circ (M, N), \quad n \in \mathbb{N}_0;$$

$$N_{n+1} = N \circ (M_n, N_n) = N_n \circ (M, N), \quad n \in \mathbb{N}_0.$$

Define

$$\alpha_n := \min(M_n, N_n), \quad \beta_n := \max(M_n, N_n), \quad n \in \mathbb{N}_0.$$

Since

$$M_{n+1}(x, y) = M(M_n(x, y), N_n(x, y)),$$
$$N_{n+1}(x, y) = N(M_n(x, y), N_n(x, y))$$

by the definition of the mean, we have, for all $n \in \mathbb{N}_0$, $x, y \in I$,

$$\min((M_n(x, y), N_n(x, y))) \le M_{n+1}(x, y) \le \max(M_n(x, y), N_n(x, y)),$$

$$\min((M_n(x, y), N_n(x, y))) \le N_{n+1}(x, y) \le \max(M_n(x, y), N_n(x, y)),$$

whence

$$\alpha_n \le \alpha_{n+1} \le \beta_{n+1} \le \beta_n, \quad n \in \mathbb{N}_0.$$

It follows that

$$\alpha := \sup\{\alpha_n : n \in \mathbb{N}_0\} = \lim_{n\to\infty} \alpha_n, \quad \beta := \inf\{\beta_n : n \in \mathbb{N}_0\} = \lim_{n\to\infty} \beta_n$$

and $\alpha \le \beta$.We shall show that $\alpha = \beta$. Assume, for the contrary, that for some $x_0, y_0 \in I$,

$$\alpha = \alpha(x_0, y_0) < \beta(x_0, y_0) = \beta.$$

By the assumption we have

$$\limsup_{(x,y)\to(\alpha,\beta)} (\max(M(x, y), N(x, y)) - \min(M(x, y), N(x, y))) <$$
$$< \max(\alpha, \beta) - \min(\alpha, \beta) = \beta - \alpha,$$

so there exists a $\delta > 0$ such that for all $x \in (\alpha - \delta, \alpha + \delta)$, and $y \in (\beta - \delta, \beta + \delta)$, we have

$$\max(M(x,y), N(x,y)) - \min(M(x,y), N(x,y)) < \beta - \alpha. \tag{5}$$

Since $\alpha_n(x_0, y_0) \in (\alpha - \delta, \alpha]$ and $\beta_n(x_0, y_0) \in [\beta, \beta + \delta)$ for all n sufficiently large, we hence get

$$M_n(x_0, y_0),\ N_n(x_0, y_0) \in (\alpha - \delta, \beta + \delta)$$

for all sufficiently large n. Taking into account that

$$M_{n+1}(x,y) = M\left(M_n(x,y), N_n(x,y)\right), \quad N_{n+1}(x,y) = N\left(M_n(x,y), N_n(x,y)\right),$$

and applying (5) with $x = x_0$, $y := y_0$ we conclude that

$$\max(M_{n+1}(x_0, y_0), N_{n+1}(x_0, y_0)) - \min(M_{n+1}(x_0, y_0), N_{n+1}(x_0, y_0)) < \beta - \alpha,$$

i.e. that $\beta_{n+1} - \alpha_{n+1} < \beta - \alpha$ for all sufficiently large n. This contradiction completes the proof that $\alpha = \beta$.Thus, putting

$$K(x,y) := \alpha(x,y), \qquad x, y \in I,$$

we obtain

$$K(x,y) = \lim_{n\to\infty} M_n(x,y) = \lim_{n\to\infty} N_n(x,y) = \alpha(x,y), \qquad x, y \in I,$$

whence $\lim_{n\to\infty} (M,N)^n = (K,K)$ and, obviously, K is a mean in I. Moreover, for all $(x,y) \in I^2$, we have

$$\begin{aligned} K(x,y) = \alpha(x,y) = \lim_{n\to\infty} M_{n+1}(x,y) = \lim_{n\to\infty} M_n(M(x,y), N(x,y)) = \\ = \alpha(M(x,y), N(x,y)) = K(M(x,y), N(x,y)), \end{aligned}$$

which shows that K is invariant with respect to the mapping (M,N). The uniqueness of K follows from Theorem 3. This completes the proof of (i) and (ii). We omit obvious arguments for (iii) and (iv). ■

Let $I \subset \mathbb{R}$ be an open interval. If $M : I^p \to I$ is increasing with respect to each variable and $\mathbf{x} = (x_1, ..., x_p) \in I^p$, then

$$\begin{aligned} M(\mathbf{x}+) &:= \lim_{u_1 \to x_1+, \dots, u_p \to x_p+} M(u_1, ..., u_p), \\ M(\mathbf{x}-) &:= \lim_{u_1 \to x_1-, \dots, u_p \to x_p-} M(u_1, ..., u_p), \end{aligned}$$

exist. From Theorem 4 we immediately obtain the following

Corollary 4.6. *Let $I \subset \mathbb{R}$ be an interval and $p \in \mathbb{N}$, $p \geq 2$, fixed. Suppose that $\mathbf{M} : I^p \to I^p$, $\mathbf{M} = (M_1, ..., M_p)$, is a mean-type mapping with increasing means $M_1, ..., M_p$, and such that, for all $\mathbf{x} = (x_1, ..., x_p) \in I^p \backslash \Delta(I^p)$,*

$$\max(M_1(\mathbf{x}+), ..., M_p(x+)) - \min(M_1(\mathbf{x}-), ..., M_p(\mathbf{x}-)) < \\ < \max \mathbf{x} - \min \mathbf{x}. \tag{6}$$

Then

(i) *there is an increasing mean $K : I^p \to I$ such that the sequence of iterates $(\mathbf{M}^n)_{n=0}^{\infty}$ converges pointwise to the mean-type mapping $\mathbf{K} : I^p \to I^p$, $\mathbf{K} = (K_1, ..., K_p)$ such that $K_1 = ... = K_p = K$;*

(ii) *$K : I^p \to I$ is a unique $\mathbf{M}$-invariant mean;*

(iii) *if $\mathbf{M}$ is strict then so is K;*

(iv) *if $\mathbf{M}$ is homogeneous, then every iterate of $\mathbf{M}$ and K are homogeneous.*

To see that the assumption (6) is essential consider the following

Example 4.7. Let $I \subset \mathbb{R}$ be an interval. Assuming that $M, N : I^2 \to I$ are given by $M(x, y) := \max(x, y)$ and $N(x, y) := \min(x, y)$, we have

$$M \circ (M, N) = M, \quad N \circ (M, N) = N, \quad A \circ (M, N) = A,$$

where $A(x, y) = \frac{x+y}{2}$. So the means M, N and A are (M, N)-invariant.

The mean type mapping $(A, H) : (0, \infty)^2 \to (0, \infty)^2$ where $A(x, y) = \frac{x+y}{2}$, $H(x, y) = \frac{2xy}{x+y}$, satisfies the equality $G \circ (A, H) = G$ where $G(x, y) = \sqrt{xy}$, that is G is (A, H)-invariant. Corollary 1 implies $\lim_{n\to\infty} (A, H)^n = (G, G)$ (cf. [6]). In this simple example all the means are continuous.

Example 4.8. The functions $f, g : (0, \infty) \to (0, \infty)$ given by

$$f(x) := \begin{cases} \frac{7}{4}x & 0 < x < \frac{1}{2} \\ \frac{x+1}{2} & \frac{1}{2} \leq x \leq 2 \\ \frac{7}{4} & x > 0 \end{cases}, \qquad g(x) := \begin{cases} 1 - \frac{3}{4}x & 0 < x < \frac{1}{2} \\ \frac{x+1}{2} & \frac{1}{2} \leq x \leq 2 \\ x - \frac{3}{4} & x > 0 \end{cases},$$

are such that $\min(x, 1) \leq f(x) \leq \max(x, 1)$, $\min(x, 1) \leq g(x) \leq \max(x, 1)$ and $f(x) = xf\left(\frac{1}{x}\right)$ and $g(x) = xg\left(\frac{1}{x}\right)$ for all $x > 0$. By Remark 3, the functions $M, N : (0, \infty)^2 \to (0, \infty)$,

$$M(x, y) = xf\left(\frac{y}{x}\right), \qquad N(x, y) = xg\left(\frac{y}{x}\right), \qquad x, y > 0,$$

are homogeneous and symmetric means. Since f and g are not continuous, in view of Theorem 2, neither M nor N is increasing. Moreover we have

$$xf\left(\frac{y}{x}\right) + xg\left(\frac{y}{x}\right) = x + y, \qquad x, y > 0,$$

so the arithmetic mean $A(x,y) = \frac{x+y}{2}$ is (M,N)-invariant and. by Theorem 4,

$$\lim_{n\to\infty} (M,N)^n (x,y) = \left(\frac{x+y}{2}, \frac{x+y}{2}\right), \qquad x,y>0.$$

Thus a discontinuous mean-type mapping can have a continuous invariant mean. The next example shows that the invariant mean need not be continuous.

Example 4.9. The functions $f,g,h : (0,\infty) \to (0,\infty)$

$$f(t) = \begin{cases} \frac{t+2}{3} & \text{for} \quad 0<t\le 2 \\ \frac{7}{4} & \text{for} \quad t>2 \end{cases}, \qquad g(t) = \begin{cases} \frac{2t+1}{3} & \text{for} \quad 0<t\le 2 \\ \frac{7}{4} & \text{for} \quad t>2 \end{cases},$$

$$h(t) = \begin{cases} \frac{t+1}{2} & \text{for} \quad 0<t\le 2 \\ \frac{7}{4} & \text{for} \quad t>2 \end{cases}$$

are discontinuous and such that $\min(t,1) \le f(t) \le \max(t,1)$, $\min(t,1) \le$ $\le g(t) \le \max(t,1)$ and $\min(t,1) \le h(t) \le \max(t,1)$ for all $t>0$. Moreover, it is easy to verify that

$$g(t)\, h\left(\frac{f(t)}{g(t)}\right) = h(t), \qquad t>0. \tag{7}$$

By Remark 3 the functions $M,N,K : (0,\infty)^2 \to (0,\infty)$,

$$M(x,y) = xf\left(\frac{y}{x}\right), \quad N(x,y) = xg\left(\frac{y}{x}\right), \quad K(x,y) = xh\left(\frac{y}{x}\right), \qquad x,y>0,$$

are discontinuous homogeneous means. Moreover, from (7) we get

$$K(M(x,y), N(x,y)) = K(x,y), \qquad x,y>0,$$

that is K is (M,N)-invariant.

5. Examples of applications

We apply Corollary 1 to find the explicit form of the limit of the sequence of iterates of some mean-type mapping, that is, in general, a nontrivial problem.

Example 5.1. It easy to verify that the functions $M,N : (0,\infty)^2 \to (0,\infty)$,

$$M(x,y) = \left(\sqrt{x} + \log\frac{y+1}{x+1}\right)^2, \qquad N(x,y) = \left(\sqrt{y} + \log\frac{x+1}{y+1}\right)^2,$$

are continuous and strictly increasing means. Since the mean

$$K(x,y) = \left(\frac{\sqrt{x}+\sqrt{y}}{2}\right)^2, \qquad x,y>0,$$

is (M,N)-invariant, in view of Corollary 1, $\lim_{n\to\infty}(M,N)^n = (K,K)$. Moreover K is a unique (M,N)-invariant mean.

Hence, as an almost immediate consequence, we obtain

Corollary 5.2. *Assume that $F:(0,\infty)^2 \to \mathbb{R}$ is continuous on the diagonal $\{(x,x): x>0\}$. The function satisfies the functional equation*

$$F\left(\sqrt{x}+\log\frac{y+1}{x+1}, \sqrt{y}+\log\frac{x+1}{y+1}\right) = F(x,y), \qquad x,y>0,$$

if, and only if, there is a continuous function $\varphi:(0,\infty)\to\mathbb{R}$ such that

$$F(x,y) = \varphi\left(\sqrt{x}+\sqrt{y}\right), \qquad x,y>0, \qquad x,y>0$$

In the same way we can apply Corollary 1 in the following examples.

Example 5.3. The functions $M,N:\mathbb{R}^2\to\mathbb{R}$ defined by

$$M(x,y) = \tan\left(\arctan x - \frac{x}{|x|+1} + \frac{y}{|y|+1}\right),$$

$$N(x,y) = \tan\left(\arctan y - \frac{y}{|y|+1} + \frac{x}{|x|+1}\right),$$

are continuous, strictly increasing means, and the mean

$$K(x,y) = \tan\left(\frac{\arctan x + \arctan y}{2}\right) \quad \text{for} \quad x,y\in\mathbb{R},$$

is (M,N)-invariant.

Example 5.4. The functions $M,N:(1,\infty)^2\to(1,\infty)$,

$$M(x,y) = x\exp\frac{y-x}{xy}, \qquad N(x,y) = y\exp\frac{x-y}{xy},$$

are increasing means, and the mean $G(x,y)=\sqrt{xy}$ is (M,N)-invariant.

Example 5.5. The functions $M,N:(0,\infty)^2\to(0,\infty)$,

$$M(x,y) = \ln\left(e^x - \arctan x + \arctan y\right),$$
$$N(x,y) = \ln\left(e^y - \arctan y + \arctan x\right),$$

are continuous strictly increasing means, and the mean $K(x,y) = \ln\left(\frac{e^x+e^y}{2}\right)$ is (M,N)-invariant.

Example 5.6. The functions $M, N : (0,1)^2 \to (0,1)$,

$$M(x,y) = \frac{x}{1 + x \ln \frac{x}{y}}, \qquad N(x,y) = \frac{y}{1 + y \ln \frac{y}{x}},$$

are continuous strictly increasing means, and the harmonic mean $H(x,y) = \frac{2xy}{x+y}$ is (M,N)-invariant.

Example 5.7. The functions $M, N : (1,\infty)^2 \to (1,\infty)$,

$$M(x,y) = \frac{2x^2y^2}{2xy^2 - y^2 + x^2}, \qquad N(x,y) = \frac{2x^2y^2}{2x^2y - x^2 + y^2},$$

are continuous strictly increasing means, and the harmonic mean $H(x,y) = \frac{2xy}{x+y}$ is (M,N)-invariant.

Example 5.8. Let $p > q > 0$. The functions $M, N : (1,\infty)^2 \to (1,\infty)$,

$$M(x,y) = \left(x^p - \frac{p}{q}\left(x^q - y^q\right)\right)^{1/p}, \quad N(x,y) = \left(y^p - \frac{p}{q}\left(y^q - x^q\right)\right)^{1/p},$$

are continuous strictly increasing means, and the power mean

$$K(x,y) = \left(\frac{x^p + y^p}{2}\right)^{1/p}$$

is (M,N)-invariant.

The means in these examples belong to a family of generalized quasi-arithmetic weighted means (cf. [9]).

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