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QUASI-SUM REPRESENTATION FOR AN m-PLACE FUNCTION

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Dedicated to the 75th birthday
of Professors Zoltán Daróczy and Imre Kátai

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Abstract. Aczél, in his book, reported on the associativity equation

$$F(F(x,y),z) = F(x,F(y,z))$$

on a real interval. Conditions leading to the quasi-sum representation

$$F(x,y) = f(f^{-1}(x) + f^{-1}(y))$$

were obtained. We expand on this and obtain results for an m-place function.

1. Introduction

The associativity equation $F(F(x,y),z) = F(x,F(y,z))$ is featured in the book of Aczél [1]. In the case of semigroups the main result is the following

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Theorem 1.1 (see [1], §6.2.2). *Let I be an interval. If for all x, y in I, $F(x,y)$ always lies in I and F is reducible on both sides, (i.e. F is injective in both variables), then*

$$F(x,y) = f(f^{-1}(x) + f^{-1}(y)) \tag{1.1}$$

with continuous and strictly monotonic f is the general continuous solution of the associativity equation

$$F(F(x,y),z) = F(x, F(y,z)). \tag{1.2}$$

Craigen and Páles ([2]) offered a new proof.

We shall extend the result to m-place functions. The following uniqueness theorem will be invoked.

Theorem 1.2 (see [3]). *Let X and Y be real intervals and $T : X \times Y \to \mathbb{R}$ be a function continuous in each of its two variables. Consider the functional equation*

$$\begin{aligned} &\phi(x) + \psi(y) = \eta(T(x,y)), \quad x \in X, y \in Y, \\ &\phi : X \to \mathbb{R},\ \psi : Y \to \mathbb{R},\ \eta : T(X \times Y) \to \mathbb{R}. \end{aligned} \tag{1.3}$$

If (ϕ_0, ψ_0, η_0) is a solution with non-constant continuous ϕ_0 and non-constant ψ_0, then

$$\phi = \alpha\phi_0 + \beta_1,\ \psi = \alpha\psi_0 + \beta_2,\ \eta = \alpha\eta_0 + \beta_1 + \beta_2$$

where α, β_1, $\beta_2 \in \mathbb{R}$ are constants, give the general solutions (ϕ, ψ, η) with continuous ϕ.

2. A result for m-place functions

Theorem 2.1. *Let I be an interval. Let $F : I^m \to I$ be a function having the following properties:*

(i) *$F(x_1, x_2, ..., x_m)$ is continuous in the variables (x_1, x_2), injective in x_1 and in x_2.*

(ii) *$F(x_1, x_2, ..., x_m)$ is symmetric in the variables x_2, x_3,..., x_m.*

(iii)

$$\begin{aligned} &F(F(x_1, x_2, ..., x_m), x_{m+1}, ..., x_{2m-1}) = \\ &= F(x_1, F(x_2, ..., x_m, x_{m+1}), x_{m+2}, ..., x_{2m-1}). \end{aligned} \tag{2.1}$$

Then F admits the representation

$$f\left(F(x_1,x_2,...,x_m)\right)=\sum_{i=1}^{m}f(x_i) \tag{2.2}$$

for some strictly increasing continuous bijective function $f: I \to J$ (for some interval J).

Proof. *Step 1.* We claim that, for arbitrarily fixed $x_3,...,x_m$, the function

$$H(x_1,x_2):=F(x_1,x_2,...,x_m)$$

satisfies the associativity equation

$$H(H(u,v),w)=H(u,H(v,w)).$$

Proof of the claim: If $m=2$, there is really no $x_3,...,x_m$ to fix, assumption (ii) is redundant and $H=F$. The associativity equation and the equation (2.1) coincide.

We now work with $m>2$.

$$\begin{aligned}H(H(u,v),w)&=F(F(u,v,x_3,...,x_m),w,x_3,...,x_m)= \\ &=F(u,F(v,x_3,...,x_m,w),x_3,...,x_m)= &&\text{by (2.1)}\\ &=F(u,F(v,w,x_3,...,x_m),x_3,...,x_m)= &&\text{by assumption (ii)}\\ &=H(u,H(v,w)).\end{aligned}$$

Step 2. By Theorem 1.1 there exists a function f such that H has the representation

$$H(u,v)=f^{-1}(f(u)+f(v)).$$

As a consequence, H is symmetric.

If $m=2$, we have arrived at the representation (2.2) as $F=H$ and we are done.

We now continue with the case $m>2$.

To make clear that f may depend on the fixed $x_3,...,x_m$ we denote it as $f_{x_3,...,x_m}$ and write the above representation as

$$F(x_1,x_2,...,x_m)=f^{-1}_{x_3,...,x_m}(f_{x_3,...,x_m}(x_1)+f_{x_3,...,x_m}(x_2)). \tag{2.3}$$

The symmetry of H translates into

$$F(x_1,x_2,...,x_m)\text{ is symmetric in }x_1\text{ and }x_2.$$

Step 3. The symmetry of $F(x_1, x_2, ..., x_m)$ in x_1 and x_2 and the symmetry assumption (ii) imply that

(2.4) F is a symmetric function of all of its variables.

Step 4. Consider the function $G : I^{2m-1} \to I$ induced by

$$G(x_1, x_2, ..., x_m, x_{m+1}, ..., x_{2m-1}) := F(F(x_1, x_2, ..., x_m), x_{m+1}, ..., x_{2m-1}).$$

The symmetry of F, (2.4), and the identity (2.1), along with combinatorial manipulation, imply that

(2.5) G is a symmetric function of all of its variables.

Step 5. Let $c \in I$ be arbitrarily fixed. Then (2.5) yields in particular

$$F(F(x_1, x_2, ..., x_m), c, c, ..., c) = F(F(x_1, x_2, c, c, ..., c), c, x_3, ..., x_m). \tag{2.6}$$

Note that we do not require the existence of a common c such that (2.6) holds. In fact, one can fix $x_{m+1} = c_2, x_{m+2} = c_3, .., x_{2m-1} = c_m$ arbitrarily and the remainder of the proof can be adjusted by just adding extra subscripts to c.

Step 6. The strict monotonicity of F in its first argument allows us to infer from (2.6) that

$$F(x_1, x_2, ..., x_m) = \phi(F(x_1, x_2, c, c, ..., c), x_3, ..., x_m) \tag{2.7}$$

for some function ϕ.

In fact, let ψ denote the inverse function of $y \mapsto F(y, c, c, ..., c)$ and we may take $\phi = \psi \circ F$.

Step 7. Consider (2.3) which gives rise to

$$F(x_1, x_2, ..., x_m) = f^{-1}_{x_3,...,x_m}(f_{x_3,...,x_m}(x_1) + f_{x_3,...,x_m}(x_2))$$

and

$$F(x_1, x_2, c, ..., c) = f^{-1}_{c,c,...,c}(f_{c,c,...,c}(x_1) + f_{c,c,...,c}(x_2)).$$

Putting them into (2.7) we get

$$\begin{aligned} &f^{-1}_{x_3,...,x_m}(f_{x_3,...,x_m}(x_1) + f_{x_3,...,x_m}(x_2)) = \\ &= \phi(f^{-1}_{c,c,...,c}(f_{c,c,...,c}(x_1) + f_{c,c,...,c}(x_2)), x_3, ..., x_m). \end{aligned}$$

It can be rewritten:

$$\begin{aligned}&f_{x_3,\dots,x_m}(x_1)+f_{x_3,\dots,x_m}(x_2)=\\&=f_{x_3,\dots,x_m}(\phi(f^{-1}_{c,c,\dots,c}(f_{c,c,\dots,c}(x_1)+f_{c,c,\dots,c}(x_2)),x_3,...,x_m))=\\&=h_{x_3,\dots,x_m}(f_{c,c,\dots,c}(x_1)+f_{c,c,\dots,c}(x_2))\end{aligned}$$

where $h_{x_3,\dots,x_m}$ is defined by

$$h_{x_3,\dots,x_m}(t)=f_{x_3,\dots,x_m}(\phi(f^{-1}_{c,c,\dots,c}(t),x_3,...,x_m)).$$

The resulting equation

$$\begin{aligned}&f_{x_3,\dots,x_m}(x_1)+f_{x_3,\dots,x_m}(x_2)=\\&=h_{x_3,\dots,x_m}(f_{c,c,\dots,c}(x_1)+f_{c,c,\dots,c}(x_2))\end{aligned}$$

is in a form to which Theorem 1.2 can be readily applied. It leads to the existence of constants $\alpha(x_3,...,x_m)>0$ and $\beta(x_3,...,x_m)$ such that

$$f_{x_3,\dots,x_m}(x)=\alpha(x_3,...,x_m)f_{c,c,\dots,c}(x)+\beta(x_3,...,x_m). \tag{2.8}$$

That is, $f_{x_3,\dots,x_m}$ and $f_{c,c,\dots,c}$ are affine functions of each other. Rewrite (2.3) first as

$$f_{x_3,\dots,x_m}(F(x_1,x_2,...,x_m))=f_{x_3,\dots,x_m}(x_1)+f_{x_3,\dots,x_m}(x_2)$$

and then replace $f_{x_3,\dots,x_m}$ using (2.8) we get

$$\begin{aligned}&\alpha(x_3,...,x_m)f_{c,c,\dots,c}(F(x_1,x_2,...,x_m))+\beta(x_3,...,x_m)=\\&=\alpha(x_3,...,x_m)f_{c,c,\dots,c}(x_1)+\beta(x_3,...,x_m)+\\&\quad+\alpha(x_3,...,x_m)f_{c,c,\dots,c}(x_2)+\beta(x_3,...,x_m).\end{aligned}$$

Eliminating a common term and dividing through by $\alpha(x_3,...,x_m)$ we get

$$\begin{aligned}&f_{c,c,\dots,c}(F(x_1,x_2,...,x_m))=\\&=f_{c,c,\dots,c}(x_1)+f_{c,c,\dots,c}(x_2)+\gamma(x_3,...,x_m)\end{aligned} \tag{2.9}$$

where $\gamma:=\beta/\alpha$.

Step 8. The symmetry of F on the left side of (2.9) yields the symmetry of γ and the symmetry of $f_{c,c,\dots,c}(x_1)+f_{c,c,\dots,c}(x_2)+\gamma(x_3,...,x_m)$ in all variables. The latter further implies that

$$\gamma(x_3,...,x_m)=f_{c,c,\dots,c}(x_3)+f_{c,c,\dots,c}(x_4)+\cdots+f_{c,c,\dots,c}(x_m)+k \tag{2.10}$$

for some constant k. This is done by implicit simple induction as indicated below.

(a) The symmetry of $f_{c,c,\ldots,c}(x_1) + f_{c,c,\ldots,c}(x_2) + \gamma(x_3, ..., x_m)$ gives in particular

$$f_{c,c,\ldots,c}(x_1) + f_{c,c,\ldots,c}(x_2) + \gamma(x_3, x_4, ..., x_m) = \\ = f_{c,c,\ldots,c}(x_1) + f_{c,c,\ldots,c}(x_3) + \gamma(x_2, x_4, ..., x_m).$$

Simplifying and rearranging terms we get

$$\gamma(x_2, x_4, ..., x_m) - f_{c,c,\ldots,c}(x_2) = \gamma(x_3, x_4, ..., x_m) - f_{c,c,\ldots,c}(x_3).$$

Because the left side does not depend on x_3 and the right side does not depend on x_2 we get

$$\gamma(x_3, x_4, ..., x_m) - f_{c,c,\ldots,c}(x_3) =: \gamma_4(x_4, ..., x_m).$$

Hence

$$\gamma(x_3, x_4, ..., x_m) = f_{c,c,\ldots,c}(x_3) + \gamma_4(x_4, ..., x_m) \tag{2.11}$$

where, as mentioned earlier, γ is symmetric.

(b) Repeating the scheme of step (a) we get from the symmetry of γ and (2.11) the symmetry of γ_4 and that

$$\gamma_4(x_4, ..., x_m) = f_{c,c,\ldots,c}(x_4) + \gamma_5(x_5, ..., x_m)$$

for some function γ_5.

(c) Repeating the above inductively till we reach

$$\gamma_m(x_m) = f_{c,c,\ldots,c}(x_m) + k.$$

This proves (2.10) following a sequence of substitutions.

Step 9. Putting (2.10) into (2.9) we obtain

$$f_{c,c,\ldots,c}(F(x_1, x_2, ..., x_m)) \\ = k + \sum_{i=1}^{m} f_{c,c,\ldots,c}(x_i). \tag{2.12}$$

The constant k can be absorbed into the function $f_{c,c,\ldots,c}$. In fact, letting

$$f(x) := f_{c,c,\ldots,c}(x) + \frac{k}{m-1}$$

(2.12) becomes

$$f(F(x_1, x_2, ..., x_m)) = \sum_{i=1}^{m} f(x_i).$$

This proves the representation (2.2) in the case $m > 2$. ■

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