QUASI-SUM REPRESENTATION FOR AN m-PLACE FUNCTION

A. A. J. Marley (Victoria, Canada)
Che Tat Ng (Waterloo, Canada)

Dedicated to the 75th birthday of Professors Zoltán Daróczy and Imre Kátai

Communicated by Antal Járai

(Received March 30, 2013; accepted May 23, 2013)

Abstract. Aczél, in his book, reported on the associativity equation

$$F(F(x,y),z) = F(x,F(y,z))$$

on a real interval. Conditions leading to the quasi-sum representation

$$F(x,y) = f(f^{-1}(x) + f^{-1}(y))$$

were obtained. We expand on this and obtain results for an m-place function.

1. Introduction

The associativity equation $F(F(x,y),z) = F(x,F(y,z))$ is featured in the book of Aczél [1]. In the case of semigroups the main result is the following

Key words and phrases: Additive representation, quasi-sum, associativity equation.
2010 Mathematics Subject Classification: 39B22, 39B12.
We thank David Curry for stimulating our interest in this problem. This research has been supported by Natural Science and Engineering Research Council Discovery Grant 8124-98 to the University of Victoria for Marley.

Theorem 1.1 (see [1], §6.2.2). *Let I be an interval. If for all x, y in I, $F(x,y)$ always lies in I and F is reducible on both sides, (i.e. F is injective in both variables), then*

$$F(x,y) = f(f^{-1}(x) + f^{-1}(y)) \tag{1.1}$$

with continuous and strictly monotonic f is the general continuous solution of the associativity equation

$$F(F(x,y),z) = F(x,F(y,z)). \tag{1.2}$$

Craigen and Páles ([2]) offered a new proof.

We shall extend the result to m-place functions. The following uniqueness theorem will be invoked.

Theorem 1.2 (see [3]). *Let X and Y be real intervals and $T : X \times Y \to \mathbb{R}$ be a function continuous in each of its two variables. Consider the functional equation*

$$\begin{aligned} &\phi(x) + \psi(y) = \eta(T(x,y)), \quad x \in X, y \in Y, \\ &\phi : X \to \mathbb{R},\ \psi : Y \to \mathbb{R},\ \eta : T(X \times Y) \to \mathbb{R}. \end{aligned} \tag{1.3}$$

If (ϕ_0, ψ_0, η_0) is a solution with non-constant continuous ϕ_0 and non-constant ψ_0, then

$$\phi = \alpha\phi_0 + \beta_1,\ \psi = \alpha\psi_0 + \beta_2,\ \eta = \alpha\eta_0 + \beta_1 + \beta_2$$

where α, β_1, $\beta_2 \in \mathbb{R}$ are constants, give the general solutions (ϕ, ψ, η) with continuous ϕ.

2. A result for m-place functions

Theorem 2.1. *Let I be an interval. Let $F : I^m \to I$ be a function having the following properties:*

(i) *$F(x_1, x_2, ..., x_m)$ is continuous in the variables (x_1, x_2), injective in x_1 and in x_2.*

(ii) *$F(x_1, x_2, ..., x_m)$ is symmetric in the variables x_2, x_3,..., x_m.*

(iii)

$$\begin{aligned} &F(F(x_1, x_2, ..., x_m), x_{m+1}, ..., x_{2m-1}) = \\ &= F(x_1, F(x_2, ..., x_m, x_{m+1}), x_{m+2}, ..., x_{2m-1}). \end{aligned} \tag{2.1}$$

Then F admits the representation

$$f\left(F(x_1, x_2, ..., x_m)\right) = \sum_{i=1}^{m} f(x_i) \tag{2.2}$$

for some strictly increasing continuous bijective function $f : I \to J$ (for some interval J).

Proof. *Step 1.* We claim that, for arbitrarily fixed $x_3, ..., x_m$, the function

$$H(x_1, x_2) := F(x_1, x_2, ..., x_m)$$

satisfies the associativity equation

$$H(H(u, v), w) = H(u, H(v, w)).$$

Proof of the claim: If $m = 2$, there is really no $x_3, ..., x_m$ to fix, assumption (ii) is redundant and $H = F$. The associativity equation and the equation (2.1) coincide.

We now work with $m > 2$.

$$\begin{aligned} H(H(u,v),w) &= F(F(u,v,x_3,...,x_m),w,x_3,...,x_m) = \\ &= F(u,F(v,x_3,...,x_m,w),x_3,...,x_m) = \qquad \text{by (2.1)} \\ &= F(u,F(v,w,x_3,...,x_m),x_3,...,x_m) = \qquad \text{by assumption (ii)} \\ &= H(u,H(v,w)). \end{aligned}$$

Step 2. By Theorem 1.1 there exists a function f such that H has the representation

$$H(u, v) = f^{-1}(f(u) + f(v)).$$

As a consequence, H is symmetric.

If $m = 2$, we have arrived at the representation (2.2) as $F = H$ and we are done.

We now continue with the case $m > 2$.

To make clear that f may depend on the fixed $x_3, ..., x_m$ we denote it as $f_{x_3,...,x_m}$ and write the above representation as

$$F(x_1, x_2, ..., x_m) = f^{-1}_{x_3,...,x_m}(f_{x_3,...,x_m}(x_1) + f_{x_3,...,x_m}(x_2)). \tag{2.3}$$

The symmetry of H translates into

$$F(x_1, x_2, ..., x_m) \text{ is symmetric in } x_1 \text{ and } x_2.$$

Step 3. The symmetry of $F(x_1, x_2, ..., x_m)$ in x_1 and x_2 and the symmetry assumption (ii) imply that

(2.4) F is a symmetric function of all of its variables.

Step 4. Consider the function $G : I^{2m-1} \to I$ induced by

$$G(x_1, x_2, ..., x_m, x_{m+1}, ..., x_{2m-1}) := F(F(x_1, x_2, ..., x_m), x_{m+1}, ..., x_{2m-1}).$$

The symmetry of F, (2.4), and the identity (2.1), along with combinatorial manipulation, imply that

(2.5) G is a symmetric function of all of its variables.

Step 5. Let $c \in I$ be arbitrarily fixed. Then (2.5) yields in particular

$$F(F(x_1, x_2, ..., x_m), c, c, ..., c) = F(F(x_1, x_2, c, c, ..., c), c, x_3, ..., x_m). \tag{2.6}$$

Note that we do not require the existence of a common c such that (2.6) holds. In fact, one can fix $x_{m+1} = c_2, x_{m+2} = c_3, .., x_{2m-1} = c_m$ arbitrarily and the remainder of the proof can be adjusted by just adding extra subscripts to c.

Step 6. The strict monotonicity of F in its first argument allows us to infer from (2.6) that

$$F(x_1, x_2, ..., x_m) = \phi(F(x_1, x_2, c, c, ..., c), x_3, ..., x_m) \tag{2.7}$$

for some function ϕ.

In fact, let ψ denote the inverse function of $y \mapsto F(y, c, c, ..., c)$ and we may take $\phi = \psi \circ F$.

Step 7. Consider (2.3) which gives rise to

$$F(x_1, x_2, ..., x_m) = f^{-1}_{x_3,...,x_m}(f_{x_3,...,x_m}(x_1) + f_{x_3,...,x_m}(x_2))$$

and

$$F(x_1, x_2, c, ..., c) = f^{-1}_{c,c,...,c}(f_{c,c,...,c}(x_1) + f_{c,c,...,c}(x_2)).$$

Putting them into (2.7) we get

$$f^{-1}_{x_3,...,x_m}(f_{x_3,...,x_m}(x_1) + f_{x_3,...,x_m}(x_2)) =$$
$$= \phi(f^{-1}_{c,c,...,c}(f_{c,c,...,c}(x_1) + f_{c,c,...,c}(x_2)), x_3, ..., x_m).$$

It can be rewritten:

$$\begin{aligned}&f_{x_3,\ldots,x_m}(x_1)+f_{x_3,\ldots,x_m}(x_2)=\\&=f_{x_3,\ldots,x_m}(\phi(f^{-1}_{c,c,\ldots,c}(f_{c,c,\ldots,c}(x_1)+f_{c,c,\ldots,c}(x_2)),x_3,\ldots,x_m))=\\&=h_{x_3,\ldots,x_m}(f_{c,c,\ldots,c}(x_1)+f_{c,c,\ldots,c}(x_2))\end{aligned}$$

where $h_{x_3,\ldots,x_m}$ is defined by

$$h_{x_3,\ldots,x_m}(t)=f_{x_3,\ldots,x_m}(\phi(f^{-1}_{c,c,\ldots,c}(t),x_3,\ldots,x_m)).$$

The resulting equation

$$\begin{aligned}&f_{x_3,\ldots,x_m}(x_1)+f_{x_3,\ldots,x_m}(x_2)=\\&=h_{x_3,\ldots,x_m}(f_{c,c,\ldots,c}(x_1)+f_{c,c,\ldots,c}(x_2))\end{aligned}$$

is in a form to which Theorem 1.2 can be readily applied. It leads to the existence of constants $\alpha(x_3,\ldots,x_m)>0$ and $\beta(x_3,\ldots,x_m)$ such that

$$f_{x_3,\ldots,x_m}(x)=\alpha(x_3,\ldots,x_m)f_{c,c,\ldots,c}(x)+\beta(x_3,\ldots,x_m). \tag{2.8}$$

That is, $f_{x_3,\ldots,x_m}$ and $f_{c,c,\ldots,c}$ are affine functions of each other. Rewrite (2.3) first as

$$f_{x_3,\ldots,x_m}(F(x_1,x_2,\ldots,x_m))=f_{x_3,\ldots,x_m}(x_1)+f_{x_3,\ldots,x_m}(x_2)$$

and then replace $f_{x_3,\ldots,x_m}$ using (2.8) we get

$$\begin{aligned}&\alpha(x_3,\ldots,x_m)f_{c,c,\ldots,c}(F(x_1,x_2,\ldots,x_m))+\beta(x_3,\ldots,x_m)=\\&=\alpha(x_3,\ldots,x_m)f_{c,c,\ldots,c}(x_1)+\beta(x_3,\ldots,x_m)+\\&\quad+\alpha(x_3,\ldots,x_m)f_{c,c,\ldots,c}(x_2)+\beta(x_3,\ldots,x_m).\end{aligned}$$

Eliminating a common term and dividing through by $\alpha(x_3,\ldots,x_m)$ we get

$$\begin{aligned}&f_{c,c,\ldots,c}(F(x_1,x_2,\ldots,x_m))=\\&=f_{c,c,\ldots,c}(x_1)+f_{c,c,\ldots,c}(x_2)+\gamma(x_3,\ldots,x_m)\end{aligned} \tag{2.9}$$

where $\gamma:=\beta/\alpha$.

Step 8. The symmetry of F on the left side of (2.9) yields the symmetry of γ and the symmetry of $f_{c,c,\ldots,c}(x_1)+f_{c,c,\ldots,c}(x_2)+\gamma(x_3,\ldots,x_m)$ in all variables. The latter further implies that

$$\gamma(x_3,\ldots,x_m)=f_{c,c,\ldots,c}(x_3)+f_{c,c,\ldots,c}(x_4)+\cdots+f_{c,c,\ldots,c}(x_m)+k \tag{2.10}$$

for some constant k. This is done by implicit simple induction as indicated below.

(a) The symmetry of $f_{c,c,\dots,c}(x_1) + f_{c,c,\dots,c}(x_2) + \gamma(x_3, ..., x_m)$ gives in particular

$$f_{c,c,\dots,c}(x_1) + f_{c,c,\dots,c}(x_2) + \gamma(x_3, x_4, ..., x_m) = \\ = f_{c,c,\dots,c}(x_1) + f_{c,c,\dots,c}(x_3) + \gamma(x_2, x_4, ..., x_m).$$

Simplifying and rearranging terms we get

$$\gamma(x_2, x_4, ..., x_m) - f_{c,c,\dots,c}(x_2) = \gamma(x_3, x_4, ..., x_m) - f_{c,c,\dots,c}(x_3).$$

Because the left side does not depend on x_3 and the right side does not depend on x_2 we get

$$\gamma(x_3, x_4, ..., x_m) - f_{c,c,\dots,c}(x_3) =: \gamma_4(x_4, ..., x_m).$$

Hence

$$\gamma(x_3, x_4, ..., x_m) = f_{c,c,\dots,c}(x_3) + \gamma_4(x_4, ..., x_m) \tag{2.11}$$

where, as mentioned earlier, γ is symmetric.

(b) Repeating the scheme of step (a) we get from the symmetry of γ and (2.11) the symmetry of γ_4 and that

$$\gamma_4(x_4, ..., x_m) = f_{c,c,\dots,c}(x_4) + \gamma_5(x_5, ..., x_m)$$

for some function γ_5.

(c) Repeating the above inductively till we reach

$$\gamma_m(x_m) = f_{c,c,\dots,c}(x_m) + k.$$

This proves (2.10) following a sequence of substitutions.

Step 9. Putting (2.10) into (2.9) we obtain

$$f_{c,c,\dots,c}(F(x_1, x_2, ..., x_m)) \\ = k + \sum_{i=1}^{m} f_{c,c,\dots,c}(x_i). \tag{2.12}$$

The constant k can be absorbed into the function $f_{c,c,\dots,c}$. In fact, letting

$$f(x) := f_{c,c,\dots,c}(x) + \frac{k}{m-1}$$

(2.12) becomes

$$f(F(x_1, x_2, ..., x_m)) = \sum_{i=1}^{m} f(x_i).$$

This proves the representation (2.2) in the case $m > 2$. ■

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A.A.J. Marley
Department of Psychology
University of Victoria
Victoria, B.C. V8W 3P5
Canada
ajmarley@uvic.ca

C.T. Ng
Department of Pure Mathematics
University of Waterloo
Waterloo, Ontario N2L 3G1
Canada
ctng@uwaterloo.ca