Annales Univ. Sci. Budapest., Sect. Comp. 41 (2013) 97–102

A $(0,2)$-TYPE PÁL INTERPOLATION PROBLEM

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Dedicated to Professors Zoltán Daróczy and Imre Kátai on the occasion of their 75th birthday

Communicated by Ferenc Schipp

(Received May 31, 2013; accepted July 28, 2013)

Abstract. We consider the following Pál interpolation problem: On two sets of nodes (one consists of the zeros of a polynomial p_n of degree n, while the elements of the other one are the zeros of p_n') different interpolation conditions are prescribed simultaneously. Weighted (0,1,…,r-2,r)-interpolation conditions ($r \geq 2$) are given on one of the sets of the nodes with Hermite conditions on the other set. We study the regularity of the problem and the explicit representation of the fundamental polynomials of interpolation. We obtain regular examples on the zeros of the classical orthogonal polynomials.

1. Introduction

In 1975 L. Pál introduced a modification of Hermite-Fejér interpolation in which the function values and the first derivatives were prescribed on two sets of nodes $\{x_i\}$ and $\{x_i^*\}$, respectively. In [8] he investigated the case when $p_n(x_i) = 0$ for $i = 1, \dots, n$ and $p_n'(x_i^*) = 0$ for $i = 1, \dots, n-1$, where p_n is a polynomial of degree n with distinct zeros. In 1983 L. Szili [9] studied the inverse problem when the roles of p_n and p_n' are interchanged, namely, the

Key words and phrases: Lacunary nterpolation, Pál interpolation, weighted $(0,1,\dots,r-2,r)$-interpolation, regularity

2010 Mathematics Subject Classification: 41A05

first derivatives are interpolated at the zeros of p_n and the function values are interpolated at the zeros of p_n'. Several authors studied these two problems, the Pál-type (0;1)- and (1;0)-interpolation using different choices of nodes.

We consider the Pál interpolation with different kind of interpolation conditions. Instead of the Lagrange condition we prescribe Hermite conditions, consecutive derivatives up to a higher order. The lacunary condition we substitute with the weighted (0,1,...,r-2,r)-interpolation ($r \geq 2$), introduced by J. Balázs [1] as a generalization of P. Turán's problem [10]. We study the regularity of the problem and the explicit representation of the fundamental polynomials of interpolation. We give some examples on the zeros of the classical orthogonal polynomials.

2. The interpolation problem

For given n and $m \in \mathbb{N}$, on a finite or infinite interval I let $\{x_i\}_{i=1}^n$ and $\{\bar{x}_i\}_{i=1}^m$ be disjoint sets of points, the nodes, and let $w \in C^r(I)$ be a given function, the weight function. Furthermore, let $y_i^{(l)}$ $(l = 0, 1, \ldots, r-2, r;$ $i = 1, \ldots, n)$ and $\bar{y}_i^{(j)}$ $(j = 0, \ldots, j_i - 1;\ i = 1, \ldots, m)$ be arbitrary given real numbers, $M = j_1 + \cdots + j_m$ and $N = rn + M$.

Find a minimal degree polynomial R_N of degree less than N satisfying the weighted $(0, 1, \ldots, r-2, r)$-interpolation conditions on $\{x_i\}_{i=1}^n$

$$R_n^{(l)}(x_i) = y_i^{(l)} \qquad (l = 0, 1, \ldots, r-2), \qquad (wR_n)^{(r)}(x_i) = y_i^{(r)}, \tag{2.1}$$

with Hermite interpolation conditions on $\{\bar{x}_i\}_{i=1}^m$

$$R_N^{(j)}(\bar{x}_i) = \bar{y}_i^{(j)} \qquad (j = 0, \ldots, j_i - 1). \tag{2.2}$$

For $\ell \geq 1$, the (2.1) – (2.2) interpolation problem is called *Pál-type weighted* $(0, \ldots, r-2, r; 0, \ldots, \ell-1)$*-interpolation on the zeros of* p_n , if $\{\bar{x}_i\}_{i=1}^{n-1}$ are the zeros of p_n' and the derivatives up to the $\ell - 1$-th order are prescribed at each $\bar{x}_i$ $(i = 1, \ldots, n-1)$ in (2.2). The *inverse Pál-type* problem is the *weighted* $(0, \ldots, \ell-1; 0, \ldots, r-2, r)$*-interpolation on the zeros of* p_n, when the roles of p_n and p_n' are interchanged, having Hermite conditions at the zeros of p_n, while the weighted $(0, 1, \ldots, r-2, r)$-interpolation conditions are prescribed at the zeros of p_n'.

An interpolation problem is called regular on a given set of nodes if there exists a unique interpolant for any choice of the data in the conditions. The (2.1)–(2.2) interpolation problem is not regular in general, as it is shown for $r = 2$ and $r = 3$ in [3] and [2] respectively. In [5], [6] and [7] the authors considered the problem on special set of nodes with special additional condition. In Theorem (2.1) we gave sufficient conditions on the nodes and the weight function, for the problem to be regular under different choices of additional interpolation conditions.

In what follows, let p_n and q be polynomials of degree n and M, respectively, associated with the interpolation conditions (2.1) – (2.2), that is

$$\begin{aligned} p_n(x_i) &= 0 \qquad (i = 1, \ldots, n), \\ q^{(j)}(\bar{x}_i) &= 0 \qquad (j = 0, \ldots, j_i - 1;\ i = 1, \ldots, m). \end{aligned} \tag{2.3}$$

Theorem 2.1 (Lénárd [4]). *If on the nodes* $\{x_i\}_{i=1}^n$ *and* $\{\bar{x}_i\}_{i=1}^m$ *the weight function* w *satisfies*

$$\left(wqp_n^{r-1}\right)^{(r)}(x_i) = 0 \qquad \text{and} \qquad w(x_i) \neq 0 \qquad (i = 1, \ldots, n), \tag{2.4}$$

then the interpolation problem (2.1) – (2.2) *is regular under each of the additional conditions* (i)-(v) *in Table* 1 *if and only if the corresponding condition in the third column of the table is fulfiled.*

For the explicit forms of the fundamental polynomials of interpolation associated with the weighted $(0, 1, \ldots, r-2, r)$-interpolation conditions see Theorem 3.1 in [4].

3. Weighted $(0,2)$-type Pál interpolation on the zeros of the classical orthogonal polynomials

The classical orthogonal polynomials fulfil the differential equation

$$(\omega y)'' + f \cdot (\omega y) = 0$$

with some weight function ω and function f. Therefore, if the nodes are the zeros of the classical orthogonal polynomial p_n and ω is the function associated with p_n in Table 2, then

$$(\omega p_n)''(x_i) = 0 \qquad (i = 1, \ldots, n).$$

It follows

$$\left((\omega p_n)^{r-1}\right)^{(r)}(x_i) = 0 \quad \text{iff} \quad (\omega p_n)''(x_i) = 0 \qquad (i = 1, \ldots, n). \tag{3.1}$$

Applying (3.1), the condition (2.4) of Theorem 2.1 is fulfiled with $q = (p_n')^\ell$ and $w = \omega^{r+2\ell-1}$ for $m = n - 1$ and $j_i = \ell$ $(i = 1, \ldots, n-1)$. Hence we obtain

Table 1.

	Additional Interpolatory Condition(s)	Condition for Regularity
(i)	$R_N^{(j)}(x_0) = y_0^{(j)}$	$\left(qp_n^{r-1}\right)^{(j)}(x_0) \neq 0$
(ii)	$(wR_N)^{(r)}(x_0) = y_0^{(r)}$	$\left(wqp_n^{r-1}\right)^{(r)}(x_0) \neq 0$
(iii)	$R_N^{(r-2)}(x_0) = y_0^{(r-2)}$, $(wR_N)^{(r)}(x_0) = y_0^{(r)}$	$\left(qp_n^{r-1}\right)^{(r-2)}(x_0)\left(wqp_n^{r-1}\int_{x_0}^x p_n(t)dt\right)^{(r)}(x_0) - \left(wqp_n^{r-1}\right)^{(r)}(x_0)\left(qp_n^{r-1}\int_{x_0}^x p_n(t)dt\right)^{(r-2)}(x_0) \neq 0$
(iv)	$R_N^{(j)}(x_0) = y_0^{(j)}$, $R_N^{(j)}(x_{n+1}) = y_{n+1}^{(j)}$	$(qp_n^{r-1})^{(j)}(x_{n+1})\left(qp_n^{r-1}\int_{x_0}^x p_n(t)dt\right)^{(j)}(x_0) - (qp_n^{r-1})^{(j)}(x_0)\left(qp_n^{r-1}\int_{x_0}^x p_n(t)dt\right)^{(j)}(x_{n+1}) \neq 0$
(v)	$(wR_N)^{(r)}(x_0) = y_0^{(r)}$, $(wR_N)^{(r)}(x_{n+1}) = y_{n+1}^{(r)}$	$\left(wqp_n^{r-1}\right)^{(r)}(x_{n+1})\left(wqp_n^{r-1}\int_{x_0}^x p_n(t)dt\right)^{(r)}(x_0) - \left(wqp_n^{r-1}\right)^{(r)}(x_0)\left(wqp_n^{r-1}\int_{x_0}^x p_n(t)dt\right)^{(r)}(x_{n+1}) \neq 0$

Theorem 3.1. *Let the nodes* $\{x_i\}_{i=1}^n$ *be the zeros of a classical orthogonal polynomial* p_n *and let* ω *be the function associated with* p_n *in Table 2. Then the Pál-type weighted* $(0,1,\ldots,r-2,r\,;0,1,\ldots,\ell-1)$*-interpolation problem with respect to the weight function* $w=\omega^{r+2\ell-1}$ *is regular under each of the additional conditions* (i)-(v) *in Table* 1 *if and only if the corresponding condition in the third column of the table is fulfiled.*

Corollary 3.1. *The Pál-type weighted* $(0,1,\ldots,r-2,r\,;0,1,\ldots,\ell-1)$*-interpolation with respect to the* $w=e^{-(r+2\ell-1)x^2/2}$ *weight function is regular on the zeros of the Hermite polynomial* H_n *with the additional condition* (i) *in Table* 1 *with* $x_0=0$ *for odd* n *if and only if* $j=\ell$*, but it is not regular for even* n*.*

The special case $r=2$ and $\ell=1$, the Pál-type weighted $(0,2;0)$-interpolation on the zeros of the Hermite polynomials was discussed by P. Mathur and S. Datta in [6].

In a similar way, choosing the zeros of the Laguerre polynomial $L_n^{(\alpha)}$ and

Table 2.

	p_n	I	ω
Hermite	H_n	$(-\infty,\infty)$	$e^{-x^2/2}$
Laguerre	$L_n^{(\alpha)}$ $(\alpha>-1)$	$[0,\infty)$	$e^{-x/2}x^{(\alpha+1)/2}$
Jacobi	$P_n^{(\alpha,\beta)}$ $(\alpha,\beta>-1)$	$[-1,1]$	$(1-x)^{(\alpha+1)/2}(1+x)^{(\beta+1)/2}$

$xL_n^{(\alpha-1)}$ to be the nodes $\{x_i\}_{i=1}^n$ and $\{\bar{x}_i\}_{i=0}^n$ in Theorem 2.1, the condition (2.4) is satisfied with the weight function $w(x)=\left(e^{-3x/2}x^{(\alpha-1)/2}\right)^{r-1}$.

Theorem 3.2. *Let the nodes $\{x_i\}_{i=1}^n$ and $\{\bar{x}_i\}_{i=0}^n$ be the zeros of the polynomials $L_n^{(1)}$ and $xL_n^{(0)}$, respectively. If the derivatives up to the $\ell=r-2$-th order are prescribed in* (2.2)*, then the* (2.1)–(2.2) *interpolation problem with respect to the weight function $w(x)=e^{-3(r-1)x/2}$ is regular with the additional condition* (ii) *in Table 1 at $x_0=0$.*

Applying Theorem 3.2 and using the fact

$$\left(L_n^{(-1)}\right)'(x)=\left(xL_{n-1}^{(1)}\right)'(x)=nL_{n-1}^{(0)}(x),$$

we obtain some regular cases:

Theorem 3.3. *On the zeros of $L_n^{(-1)}$ the Pál-type weighted* $(0,1,\dots,r-2,r;0,1,\dots,r-2)$*-interpolation is regular with respect to the weight function $w(x)==e^{-3(r-1)x/2}$.*

For further results on Pál-type weighted $(0,2;0)$-interpolation on the zeros of Jacobi polynomials we refer to [3]. For Pál-type weighted $(0,1,3;0,1,\dots,\dots,\ell-1)$-interpolation on the zeros of the Hermite polynomials we refer to [2].

As an application of Theorem 2.1, we obtain similar results for the inverse Pál-type interpolation problem, as well.

Theorem 3.4. *Let the nodes $\{x_i\}_{i=1}^n$ be the zeros of a classical orthogonal polynomial p_n and let ω be the function associated with p_n in Table 2. Then the Pál-type weighted* $(0,1,\dots,\ell-1;0,1,\dots,r-2,r)$*-interpolation problem with respect to the weight function $w=\omega^{r-1}$ is regular under each of the additional conditions* (i)-(v) *in Table* 1 *if and only if the corresponding condition in the third column of the table is fulfiled.*

P. Mathur and S. Datta in [7] studied the special case $\ell=1$ and $r=2$ on the zeros of the Hermite polynomials, K. K. Mathur and A. K. Srivastava [5] discussed the problem on the same nodes for $r=3$. In [3] we studied the problem on the zeros of the classical orthogonal polynomials for $\ell = 1$ and $r = 2$ and we obtained regular case on the zeros of the integrated Legendre polynomial with respect to the weight function $w(x) = 1$.

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