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ON HYBRID UNIVERSALITY OF CERTAIN COMPOSITE FUNCTIONS INVOLVING DIRICHLET L-FUNCTIONS

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Dedicated to Professors Zoltán Daróczy and Imre Kátai on their 75*th anniversary*

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Abstract. It is proved that a certain composite function involving several Dirichlet L-functions satisfies a hybrid universality property, that is, a combination of usual universality and certain Diophantine inequalities. A joint version for several composite functions is also obtained.

1. Introduction and statement of results

It is well known that the distribution of the values of the Riemann zeta-function $\zeta(s)$, $s = \sigma + it$, is very mysterious and fascinating. More than one hundred years ago, H. Bohr obtained in [3] that the function $\zeta(s)$, in the region $1 < \sigma < 1 + \delta$ with arbitrary $\delta > 0$, takes every non-zero value infinitely many times. A little later, jointly with R. Courant, he proved in [4] that, for every σ, $\frac{1}{2} < \sigma \leq 1$, the set

$$\{\zeta(\sigma + it) : \ t \in \mathbb{R}\} \tag{1}$$

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is dense in $\mathbb{C}$. The extension of the Bohr-Courant result to the space $\mathbb{C}^k$, $k \in \mathbb{N}$, was done by S. M. Voronin [23] in 1970s. He obtained that, for every σ, $\frac{1}{2} < \sigma < 1$, the set

$$\left\{ \left(\zeta(\sigma + it), \zeta'(\sigma + it), \dots, \zeta^{(k-1)}(\sigma + it) \right) : \, t \in \mathbb{R} \right\}$$

is dense in $\mathbb{C}^k$. Voronin did not stop there. He developed his method further and found an infinite-dimensional version of the Bohr-Courant theorem on the denseness of the set (1), by proving in [24] a very interesting theorem on the approximation of analytic functions by shifts $\zeta(s+i\tau)$, $\tau \in \mathbb{R}$. Now this theorem is called the Voronin universality theorem. We will state an improved version of the Voronin theorem. Let $D = \{s \in \mathbb{C} : \, \frac{1}{2} < \sigma < 1\}$. Denote by $\mathcal{K}$ the class of compact subsets of the strip D with connected complements, and let $H_0(K)$, $K \in \mathcal{K}$, stand for the class of continuous non-vanishing functions on K which are analytic in the interior of K. Also let $\operatorname{meas}\{A\}$ be the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. Then the modern version of the Voronin theorem has the following form (see, for example, [8, 22]):

Theorem 1. *Let $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T \to \infty} \frac{1}{T} \operatorname{meas} \left\{ \tau \in [0, T] : \, \sup_{s \in K} |\zeta(s + i\tau) - f(s)| < \varepsilon \right\} > 0.$$

Theorem 1 asserts that the set of shifts $\zeta(s + i\tau)$ approximating a given analytic function is infinite and has a positive lower density.

Theorem 1 attracted attention of many mathematicians; it was generalized to other zeta and L-functions, and the problem of its effectivization was investigated. See [8, 9, 13, 14, 15, 22] for references and results.

Voronin also succeeded in generalizing Theorem 1 to a collection of Dirichlet L-functions $L(s, \chi)$. In [25], in a non-explicit form, he proved a joint universality theorem on a simultaneous approximation to a collection of analytic functions by shifts $L(s + i\tau, \chi_1), \dots, L(s + i\tau, \chi_r)$. We state a modern version of this theorem (see [11]):

Theorem 2. *Let $\chi_1, \dots, \chi_r$ be pairwise non-equivalent Dirichlet characters. For $j = 1, \dots, r$, let $K_j \in \mathcal{K}$ and $f_j(s) \in H_0(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T \to \infty} \frac{1}{T} \operatorname{meas} \left\{ \tau \in [0, T] : \, \sup_{1 \le j \le r} \sup_{s \in K_j} |L(s + i\tau, \chi_j) - f_j(s)| < \varepsilon \right\} > 0.$$

B. Bagchi in [1] and [2] obtained, independently of [25], a similar joint universality theorem for distinct Dirichlet characters $\chi_1, \dots, \chi_r$ modulo q.

S. M. Gonek in his thesis [6] proposed a new version of the joint universality for Dirichlet L-functions. Denote by $\|u\|$ the distance from u to the nearest integer. The following theorem of Gonek connects the Kronecker theorem in the theory of Diophantine approximation with Voronin's theorem on joint universality.

Theorem 3. *Let $q \in \mathbb{N}$, and let K be a simply connected compact subset of the strip D. To each prime $p \mid q$, we attach a real number θ_p with $0 \le \theta_p < 1$, and to each Dirichlet character $\chi(\operatorname{mod} q)$, we attach a function $f_\chi(s)$ which is continuous on K and analytic in the interior of K. Then, for every $\varepsilon > 0$, there exists a number $\tau \in \mathbb{R}$ such that*

$$\max_{s\in K}\left|L(s+i\tau,\chi) - \mathrm{e}^{f_\chi(s)}\right| < \varepsilon, \quad \textit{for all} \quad \chi(\operatorname{mod} q),$$

and

$$\left\|\frac{-\tau\log p}{2\pi} - \theta_p\right\| < \varepsilon, \quad \textit{for all} \quad p \mid q.$$

This theorem was improved by J. Kaczorowski and M. Kulas [7]. They proved the following hybrid joint universality theorem.

Theorem 4. *Let $\chi_1, \dots, \chi_r$ be pairwise non-equivalent Dirichlet characters and $K \in \mathcal{K}$. For $j = 1, \dots, r$, let $f_j(s) \in H_0(K)$. Then, for every sequence $\{\theta_p\}_{p\le z}$ of real numbers indexed by primes up to z and for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty}\frac{1}{T}\operatorname{meas}\left\{\tau\in[0,T]: \max_{1\le j\le r}\max_{s\in K}|L(s+i\tau,\chi_j) - f_j(s)| < \varepsilon,\right.$$
$$\left.\max_{p\le z}\left\|\frac{\tau\log p}{2\pi} - \theta_p\right\| < \varepsilon\right\} > 0.$$

Ł. Pańkowski in [21] replaced the sequence $\{\frac{\log p}{2\pi}\}_{p\le z}$ in Theorem 4 by an arbitrary sequence $\{\alpha_j\}_{1\le j\le m}$ of real numbers linearly independent over the field of rational numbers $\mathbb{Q}$.

It is to be noted that, applying hybrid universality theorems, we can show new universality results and also new zero-distribution results on various zeta and multiple zeta-functions (see Nakamura and Pańkowski [18], [19], [20]).

In [11], universality theorems for $F(L(s,\chi_1), \dots, L(s,\chi_r))$ for some classes of operators F were obtained. Denote by $H(D)$ the space of analytic functions on D equipped with the topology of uniform convergence on compacta, and let

$$S = \{g \in H(D):\ g^{-1}(s) \in H(D) \text{ or } g(s) \equiv 0\}.$$

Moreover, let $H(K)$, $K \in \mathcal{K}$, be the class of continuous functions on K which are analytic in the interior of K. Then the following theorem is one of the results from [11].

Theorem 5. *Suppose that $\chi_1, \ldots, \chi_n$ are pairwise non-equivalent Dirichlet characters, and that $F : H^r(D) \to H(D)$ is a continuous operator such that, for every open set $G \in H(D)$, the set $(F^{-1}G) \cap S^r$ is non-empty. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \mathrm{meas}\Big\{ \tau \in [0,T] \ : \\ : \sup_{s\in K} |F(L(s+i\tau,\chi_1), \ldots, L(s+i\tau,\chi_r)) - f(s)| < \varepsilon \Big\} > 0.$$

This type of universality theorem for composite functions was first treated in [10] in the case of the Riemann zeta-function; see also [12] and [5].

For the proof of Theorem 5, a probabilistic method based on limit theorems for weakly convergent probabilistic measures in the space of analytic functions is applied. However, this method does not work for the proof of hybrid joint universality theorems. Therefore, to obtain some hybrid version of Theorem 5, we have to search for a different way. In the present paper we propose the following generalization of the Pańkowski theorem [21].

Let $\beta_1 > 0, \ldots, \beta_r > 0$, and $\beta = \min_{1\le j\le r} \beta_j$. We say that the operator $F : H^r(D) \to H(D)$ belongs to the Lipschitz class $Lip(\beta_1, \ldots, \beta_r)$ if the following hypotheses are satisfied:

1° For each polynomial $p = p(s)$, and any $K \in \mathcal{K}$, there exists an element $(g_1, \ldots, g_r) \in F^{-1}\{p\} \subset H^r(D)$ such that $g_j(s) \neq 0$ on K, $j = 1, \ldots, r$;

2° For any $K \in \mathcal{K}$, there exist a positive constant c, and a set $\widehat{K} \in \mathcal{K}$ such that

$$\sup_{s\in K} |F(g_{11}(s), \ldots, g_{1r}(s)) - F(g_{21}(s), \ldots, g_{2r}(s))| \le c \sup_{1\le j\le r} \sup_{s\in \widehat{K}} |g_{1j}(s) - g_{2j}(s)|^{\beta_j}$$

for all $(g_{h1}, \ldots g_{hr}) \in H^r(D)$, $h = 1, 2$.

Theorem 6. *Suppose that $\chi_1, \ldots, \chi_r$ are pairwise non-equivalent Dirichlet characters, and that $F \in Lip(\beta_1, \ldots, \beta_r)$. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Moreover, let $\{\alpha_j : j = 1, \ldots, m\}$ be any sequence of real numbers linearly independent over $\mathbb{Q}$ and $\{\theta_j : j = 1, \ldots, m\}$ be any sequence of real numbers. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \mathrm{meas}\Big\{ \tau \in [0,T] : \sup_{s\in K} |F(L(s+i\tau,\chi_1), \ldots, L(s+i\tau,\chi_r)) - f(s)| < \varepsilon, \\ \max_{1\le j\le m} \|\tau\alpha_j - \theta_j\| < \Big(\frac{\varepsilon}{2}\Big)^{\frac{1}{\beta}} \Big\} > 0.$$

Theorem 6 admits a generalization to the "joint" case. Let $n \in \mathbb{N}\setminus\{1\}$, and $F_l : H^r(D) \to H(D)$, $1 \le l \le n$. We say that the operator $F_{r,n} = (F_1, \dots, F_n) :$ $: H^r(D) \to H^n(D)$ is in the class $Lip_n(\beta_1, \dots, \beta_r)$, $\beta_1 > 0, \dots, \beta_r > 0$, if:

1° For all polynomials $p_1(s), \dots, p_n(s)$, and any $K \in \mathcal{K}$, there exists an element $(g_1, \dots, g_r) \in F_{r,n}^{-1}\{p_1, \dots, p_n\} \subset H^r(D)$ such that $g_j(s) \neq 0$ on K, $j = 1, \dots, r$;

2° For any $K_1, \dots, K_n \in \mathcal{K}$, there exist a positive constant c, and a set $\widehat{K} \in \mathcal{K}$ such that

$$\sup_{1\le l\le n} \sup_{s\in K_l} |F_l(g_{11}(s), \dots, g_{1r}(s)) - F_l(g_{21}(s), \dots, g_{2r}(s))| \le$$
$$\le c \sup_{1\le j\le r} \sup_{s\in \widehat{K}} |g_{1j}(s) - g_{2j}(s)|^{\beta_j}$$

for all $(g_{h1}, \dots g_{hr}) \in H^r(D)$, $h = 1, 2$.

Theorem 7. *Suppose that $\chi_1, \dots, \chi_r$ are pairwise non-equivalent Dirichlet characters, and that $F_{r,n} = (F_1, \dots, F_n) \in Lip_n(\beta_1, \dots, \beta_r)$. For $l = 1, \dots, n$, let $K_l \in \mathcal{K}$ and $f_l(s) \in H(K_l)$. Moreover, let α_j and θ_j be the same as in Theorem 6. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\mathrm{meas}\Bigg\{\tau \in [0, T] : \sup_{1\le l\le n} \sup_{s\in K_l} |F_l(L(s + i\tau, \chi_1), \dots, L(s + i\tau, \chi_r))-$$
$$-f_l(s)| < \varepsilon, \quad \max_{1\le j\le m} \|\tau\alpha_j - \theta_j\| < \left(\frac{\varepsilon}{2}\right)^{\frac{1}{\beta}} \Bigg\} > 0.$$

2. Proof of Theorems 6 and 7

We deduce Theorem 6 directly from Theorem 1.1 of [21] and the definition of the class $Lip(\beta_1, \dots, \beta_r)$.

Proof of Theorem 6. We may assume that ε is sufficiently small. By the Mergelyan theorem on the approximation of analytic functions by polynomials ([16] [17], see also [26]), we see that there exists a polynomial $p = p(s)$ such that

$$\sup_{s\in K} |f(s) - p(s)| < \frac{\varepsilon}{2}. \tag{2}$$

Let $\widehat{K} \in \mathcal{K}$ be the set corresponding to K in hypothesis 2° of the class $Lip(\beta_1, \dots, \beta_r)$. By hypothesis 1° of the class $Lip(\beta_1, \dots, \beta_r)$, there exists an element $(g_1,\dots,g_r)\in F^{-1}\{p\}\subset H^r(D)$ such that $g_j(s)\neq 0$ on $\widehat{K}$, $j=1,\dots,r$. Let $A(g_1,\dots,g_r)$ be the set of all $\tau\in\mathbb{R}$ satisfying

$$\sup_{1\le j\le r}\sup_{s\in\widehat{K}}|L(s+i\tau,\chi_j)-g_j(s)|<c_1^{-\frac{1}{\beta}}\left(\frac{\varepsilon}{2}\right)^{\frac{1}{\beta}}$$

and

$$\max_{1\le j\le m}\|\tau\alpha_j-\theta_j\|<c_1^{-\frac{1}{\beta}}\left(\frac{\varepsilon}{2}\right)^{\frac{1}{\beta}},$$

where $c_1=\max(1,c)$. Theorem 1.1 of Pańkowski [21] assures the existence of such a τ; in fact, his theorem asserts

$$\liminf_{T\to\infty}\frac{1}{T}\operatorname{meas}\{A(g_1,\dots,g_r)\}>0. \tag{3}$$

By hypothesis 2° of the class $Lip(\beta_1,\dots,\beta_r)$, for τ satisfying the above inequalities, we have

$$\begin{aligned}&\sup_{s\in K}|F(L(s+i\tau,\chi_1),\dots,L(s+i\tau,\chi_r))-p(s)|\\ &\le c\sup_{1\le j\le r}\sup_{s\in\widehat{K}}|L(s+i\tau,\chi_j)-g_j(s)|^{\beta_j}\le c\,c_1^{-\frac{\beta}{\beta}}\left(\frac{\varepsilon}{2}\right)^{\frac{\beta}{\beta}}\le\frac{\varepsilon}{2}.\end{aligned}\tag{4}$$

From (3) and (4) it follows that

$$\begin{aligned}&\liminf_{T\to\infty}\frac{1}{T}\operatorname{meas}\Big\{\tau\in[0,T]\ :\\ &:\ \sup_{s\in K}|F(L(s+i\tau,\chi_1),\dots,L(s+i\tau,\chi_r))-p(s)|<\frac{\varepsilon}{2},\\ &\max_{1\le j\le m}\|\tau\alpha_j-\theta_j\|<\left(\frac{\varepsilon}{2}\right)^{\frac{1}{\beta}}\Big\}>0.\end{aligned}\tag{5}$$

However, taking into account (2), we find that

$$\begin{aligned}&\Big\{\tau\in[0,T]:\ \sup_{s\in K}|F(L(s+i\tau,\chi_1),\dots,L(s+i\tau,\chi_r))-p(s)|<\frac{\varepsilon}{2},\\ &\qquad\max_{1\le j\le m}\|\tau\alpha_j-\theta_j\|<\left(\frac{\varepsilon}{2}\right)^{\frac{1}{\beta}}\Big\}\subset\\ &\subset\Big\{\tau\in[0,T]:\ \sup_{s\in K}|F(L(s+i\tau,\chi_1),\dots,L(s+i\tau,\chi_r))-f(s)|<\varepsilon,\\ &\qquad\max_{1\le j\le m}\|\tau\alpha_j-\theta_j\|<\left(\frac{\varepsilon}{2}\right)^{\frac{1}{\beta}}\Big\}.\end{aligned}$$

Combining this with (5) we obtain the theorem. ∎

Remark. Clearly, if $\beta \leq 1$, the inequality

$$\max_{1\leq j\leq m} \|\tau\alpha_j - \theta_j\| < \left(\frac{\varepsilon}{2}\right)^{\frac{1}{\beta}}$$

in Theorem 6 for small ε can be replaced by

$$\max_{1\leq j\leq m} \|\tau\alpha_j - \theta_j\| < \varepsilon.$$

Proof of Theorem 7. The argument is similar to the proof of Theorem 6. By the Mergelyan theorem, there exist polynomials $p_1 = p_1(s), \ldots, p_n = p_n(s)$ such that

$$\sup_{1\leq l\leq n} \sup_{s\in K_l} |f_l(s) - p_l(s)| < \frac{\varepsilon}{2}. \tag{6}$$

Let $\widehat{K} \in \mathcal{K}$ be the set corresponding to the sets $K_1, \ldots, K_n$ in hypothesis 2° of the class $Lip_n(\beta_1, \ldots, \beta_r)$. Then, in view of hypothesis 1° of the class $Lip_n(\beta_1, \ldots, \beta_r)$, there exists an element $(g_1, \ldots, g_r) \in F^{-1}_{n,r}\{p_1, \ldots, p_n\} \subset$ $\subset H^r(D)$ such that $g_j(s) \neq 0$ on $\widehat{K}$, $j = 1, \ldots, r$.

Let $A(g_1, \ldots, g_r)$ be the same as in the proof of Theorem 6. For $\tau \in$ $\in A(g_1, \ldots, g_r)$, from hypothesis 2° of the class $Lip_n(\beta_1, \ldots, \beta_r)$ we have

$$\begin{aligned}&\sup_{1\leq l\leq n} \sup_{s\in K_l} |F_l(L(s+i\tau,\chi_1), \ldots, L(s+i\tau,\chi_r)) - p_l(s)| \leq \\ &\leq c \sup_{1\leq j\leq r} \sup_{s\in\widehat{K}} |L(s+i\tau,\chi_j) - g_j(s)|^{\beta_j} < \frac{\varepsilon}{2}.\end{aligned} \tag{7}$$

Combining (3) and (7) we have

$$\begin{aligned}\liminf_{T\to\infty} \frac{1}{T}\text{meas}\Bigg\{\tau \in [0,T] : \sup_{1\leq l\leq n} \sup_{s\in K_l} |F_l(L(s+i\tau,\chi_1), \ldots, L(s+i\tau,\chi_r)) -\\ -p_l(s)| < \frac{\varepsilon}{2}, \max_{1\leq j\leq m} \|\tau\alpha_j - \theta_j\| < \left(\frac{\varepsilon}{2}\right)^{\frac{1}{\beta}}\Bigg\} > 0.\end{aligned}$$

From this and (6) the theorem follows. ■

3. Examples

(I) *An example for Theorem 6*: For $(g_1, \ldots, g_r) \in H^r(D)$, let

$$F(g_1, \ldots, g_r) = c_1 g_1^{(k_1)} + \cdots + c_r g_r^{(k_r)},$$

where $c_1,\dots,c_r\in\mathbb{C}\setminus\{0\}$, $k_1,\dots,k_r\in\mathbb{N}$ and $g^{(k)}$ denotes the k-th derivative of $g\in H(D)$. We prove $F\in Lip(1,\dots,1)$.

We take an arbitrary polynomial

$$p(s)=a_ns^n+\cdots+a_1s+a_0$$

and an arbitrary $K\in\mathcal{K}$. Then, taking

$$g_1(s)=\frac{a_ns^{n+k_1}}{c_1(n+1)(n+2)\cdots(n+k_1)}+\cdots+\frac{a_1s^{1+k_1}}{c_1(k_1+1)!}+\frac{a_0s^{k_1}}{c_1k_1!}+C,$$

where $C\in\mathbb{C}$ is chosen to be $|C|$ large enough so that $g_1(s)\neq 0$ for $s\in K$, and $g_2(s)=\cdots=g_r(s)\equiv 1$, we have that $(g_1,\dots,g_r)\in H^r(D)$,

$$F(g_1,\dots,g_r)=p(s),$$

and $g_j(s)\neq 0$ on K, $j=1,\dots,r$. Therefore, hypothesis 1° of the class $Lip(\beta_1,\dots,\beta_r)$ is satisfied.

Now for $K\in\mathcal{K}$, let $K\subset G\subset\widehat{K}$ where G is an open set and $\widehat{K}\in\mathcal{K}$, and L be a simple closed contour lying in $\widehat{K}\setminus G$ and enclosing the set K. Then the Cauchy integral formula, for all $(g_{k1},\dots,g_{kr})\in H^r(D)$, $k=1,2$, and $s\in K$, yields

$$\begin{aligned}
&|F(g_{11}(s),\dots,g_{1r}(s))-F(g_{21}(s),\dots,g_{2r}(s))|=\\
&=\left|\sum_{j=1}^r c_j\frac{k_j!}{2\pi i}\int\limits_L\frac{g_{1j}(z)-g_{2j}(z)}{(z-s)^{k_j+1}}\mathrm{d}z\right|\leq\\
&\leq\sum_{j=1}^r|c_j|C_j\sup_{s\in L}|g_{1j}(s)-g_{2j}(s)|\leq\sum_{j=1}^r|c_j|C_j\sup_{s\in\widehat{K}}|g_{1j}(s)-g_{2j}(s)|\leq\\
&\leq c\sup_{1\leq j\leq r}\sup_{s\in\widehat{K}}|g_{1j}(s)-g_{2j}(s)|,
\end{aligned}$$

where $c=r\max_{1\leq j\leq r}(|c_j|C_j)$, and C_j is a positive constant, $j=1,\dots,r$. Thus, hypothesis 2° of the class $Lip(\beta_1,\dots,\beta_r)$ is also satisfied with $\beta_1=\cdots=\beta_r=1$. Therefore, $F\in Lip(1,\dots,1)$, and for the function

$$F(L(s,\chi_1),\dots,L(s,\chi_r))=c_1L(s,\chi_1)^{(k_1)}+\cdots+c_rL(s,\chi_r)^{(k_r)}$$

the assertion of Theorem 6 is true.

(II) *An example for Theorem 7*: Suppose that $n\leq r$, and

$$F_l(g_1,\dots,g_r)=c_lg_l^{(k_l)},$$

where $c_l \in \mathbb{C} \setminus \{0\}$, $k_l \in \mathbb{N}$, $l = 1, \dots, n$. Define

$$F_{n,r} = (F_1, \dots, F_n) : \ H^r(D) \to H^n(D),$$

and we check the hypotheses of the class $Lip_n(1, \dots, 1)$ for $F_{n,r}$.

Let $K \in \mathcal{K}$, and let $p_1 = p_1(s)$ be an arbitrary polynomial. Analogously to the first example, we solve the equation

$$F_1(g_{11}, \dots, g_{1r}) = p_1$$

and find $g_{11}(s) \in H(D)$ such that $g_{11}(s) \neq 0$ on K. For $j = 2, \dots, r$, the function $g_{1j}(s) \in H(D)$ can be arbitrary, but $g_{1j}(s) \neq 0$ on K. Similarly, from the equation

$$F_2(g_{21}, \dots, g_{2r}) = p_2$$

with arbitrary polynomial $p_2 = p_2(s)$, we find $g_{22}(s) \in H(D)$ and $g_{22}(s) \neq 0$ on K. Moreover, we take $g_{21}(s) = g_{11}(s)$, and, for $j = 3, \dots, r$, the function $g_{2j}(s) \in H(D)$ is arbitrary such that $g_{2j}(s) \neq 0$ on K. Proceeding with this process, we obtain $(g_{11}, g_{22}, \dots, g_{rr}) \in F_{n,r}^{-1}\{p_1, \dots, p_n\} \subset H^r(D)$ such that $g_{jj}(s) \neq 0$ on K, $j = 1, \dots, n$, where $g_{jj}(s) = g_{nj}(s)$ for $j = n+1, \dots, r$. Thus hypothesis 1° of the class $Lip_n(\beta_1, \dots, \beta_r)$ is satisfied.

For $K_l \in \mathcal{K}$, $1 \leq l \leq n$, let $K_l \subset G_l \subset \widetilde{K}_l$ where G_l is an open set and $\widetilde{K}_l \in \mathcal{K}$, and let L_l be a simple closed contour lying in $\widetilde{K}_l \setminus G_l$ and enclosing the set K_l. Then, for any fixed l and all $(g_{h1}, \dots, g_{hr}) \in H^r(D)$, $h = 1, 2$, $s \in K_l$, in view of the Cauchy integral formula,

$$\begin{aligned} &|F_l\left(g_{11}(s), \dots, g_{1r}(s)\right) - F_l\left(g_{21}(s), \dots, g_{2r}(s)\right)| = \\ &= \left| c_l \frac{k_l!}{2\pi i} \int\limits_{L_l} \frac{g_{1l}(z) - g_{2l}(z)}{(z-s)^{k_l+1}} \mathrm{d}z \right| \leq \\ &\leq |c_l| C_l \sup_{s \in L_l} |g_{1l}(s) - g_{2l}(s)| \leq |c_l| C_l \sup_{s \in \widetilde{K}_l} |g_{1l}(s) - g_{2l}(s)| \leq \\ &\leq c \sup_{1 \leq j \leq r} \sup_{s \in \widetilde{K}_l} |g_{1j}(s) - g_{2j}(s)|, \end{aligned} \tag{8}$$

where $c = \max\limits_{1 \leq l \leq n} (|c_l| C_l)$, and C_l is a positive constant, $l = 1, \dots, n$. Let $\widehat{K} \in \mathcal{K}$ such that

$$\bigcup_{l=1}^{n} \widetilde{K}_l \subset \widehat{K}.$$

Then we obtain from (8) that

$$\sup_{1\le l\le n}\sup_{s\in K_l}|F_l\left(g_{11}(s),\dots,g_{1r}(s)\right)-F_l\left(g_{21}(s),\dots,g_{2r}(s)\right)|\le$$
$$\le c\sup_{1\le j\le r}\sup_{s\in\widehat{K}}|g_{1j}(s)-g_{2j}(s)|.$$

Thus we conclude that $F_{n,r}\in Lip_n(1,\dots,1)$. Therefore, for $F_{n,r}$, a hybrid joint universality theorem (Theorem 7) can be applied. For example, if χ_1,χ_2 and χ_3 are pairwise non-equivalent Dirichlet characters, $K_1,K_2,K_3\in\mathcal{K}$ and $f_1(s)\in H(K_1)$, $f_2(s)\in H(K_2)$ and $f_3(s)\in H(K_3)$, then, for every $\varepsilon>0$,

$$\liminf_{T\to\infty}\frac{1}{T}\mathrm{meas}\Bigg\{\tau\in[0,T]:\sup_{s\in K_1}|L^{(5)}(s+i\tau,\chi_1)-f_1(s)|<\varepsilon,$$
$$\sup_{s\in K_2}|L^{(8)}(s+i\tau,\chi_2)-f_2(s)|<\varepsilon,\sup_{s\in K_3},|L^{(10)}(s+i\tau,\chi_3)-f_3(s)|<\varepsilon,$$
$$\max_{1\le j\le m}\|\tau\alpha_j-\theta_j\|<\varepsilon\Bigg\}>0.$$

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