Annales Univ. Sci. Budapest., Sect. Comp. 41 (2013) 73–84

CHARACTERIZATION OF BIVARIATE DISTRIBUTIONS WITH CONDITIONALS OF THE SAME TYPE

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Dedicated to Professors Zoltán Daróczy and Imre Kátai on their 75th birthday

Communicated by Antal Járai
(Received May 31, 2013; accepted June 28, 2013)

Abstract. Here we deal with a special characterization problem of a conditionally specified absolutely continuous bivariate distribution by the help of a functional equation satisfied almost everywhere on its domain for the unknown density functions. We consider the case of conditionals of the same type (in other words, in the same location-scale families) with specified moments (we have linear regressions and conditional standard deviations).

1. Introduction

Let (X, Y) be an absolutely continuous bivariate random vector, whose joint, marginal and conditional density functions are denoted by $f_{(X,Y)}$, f_X,

Key words and phrases: Functional equations, density function solutions.
2010 Mathematics Subject Classification: 39B22.
This research has been supported by the Hungarian Scientific Research Fund (OTKA), Grant NK81402.

f_Y, $f_{X|Y}$, $f_{Y|X}$ respectively. Note that these functions are not uniquely determined, only almost everywhere. One can write $f_{(X,Y)}$ in two different ways and obtain the equation

$$f_{X|Y}(x,y)f_Y(y) = f_{Y|X}(x,y)f_X(x) \tag{1.1}$$

valid for almost every $(x,y) \in \mathbb{R}^2$. This can be considered as a functional equation for the marginal and conditional density functions. Assuming that the conditional densities admit some properties (important from the point of view of probability theory), the solutions can be given.

Probably Narumi was the first who studied some related questions in [7]. Later in [1] Arnold, Castillo and Sarabia showed how solutions of functional equations can be used in characterizing joint distributions by certain properties of conditional distributions.

In this paper we will suppose that the conditional density functions have the form

$$f_{X|Y}(x,y) = g_1\left(\frac{x-a(y)}{c(y)}\right)\frac{1}{c(y)}, \tag{1.2}$$

$$f_{Y|X}(x,y) = g_2\left(\frac{y-b(x)}{d(x)}\right)\frac{1}{d(x)} \tag{1.3}$$

for given functions a, b and for given positive functions c, d, where g_1, g_2 are unknown functions (necessarily densities). Equation (1.2) means that the conditional distributions of the coordinate X with respect to the condition $Y=y$ are of the same type, i.e., they belong to the same location-scale family for all y.

Then we can deduce from (1.1) a functional equation

$$g_1\left(\frac{x-a(y)}{c(y)}\right)\frac{1}{c(y)}f_Y(y) = g_2\left(\frac{y-b(x)}{d(x)}\right)\frac{1}{d(x)}f_X(x), \tag{1.4}$$

where g_1, g_2, f_Y, f_X are unknown densities.

2. Types of distributions and location-scale families

Two distribution functions F and H are of the same type if for some $a \in \mathbb{R}$, $c > 0$

$$F(x) = H\left(\frac{x-a}{c}\right) \tag{2.1}$$

for all $x \in \mathbb{R}$, see, e.g., Resnick [8, Section 0.3]. Clearly, this is an equivalence relation between distribution functions, and hence establishes a partition of the set of distribution functions. For instance, the normal distributions form such an equivalence class. These equivalence classes may also be called location-scale families, since the equivalence class of the distribution function H consists all of the distribution functions F given in (2.1) letting run the location parameter $a \in \mathbb{R}$ and the scaling parameter $c > 0$. If X and Z are random variables with distribution functions F and H, respectively, given in (2.1), then X and $cZ + a$ have the same distribution. These equivalence classes are closed with respect to the topology of weak convergence, see the so called convergence of types theorem, e.g., Resnick [8, Proposition 0.2]. Another interesting feature is that a distribution is stable if and only if its location-scale family is closed with respect to the convolution.

If F is an absolutely continuous distribution function with density f and (2.1) holds with a distribution function H with some $a \in \mathbb{R}$, $c > 0$, then H is also absolutely continuous with some density h and

$$f(x) = h\left(\frac{x-a}{c}\right)\frac{1}{c}.$$

Now let (X, Y) be an absolutely continuous bivariate random vector such that (1.2) holds. Suppose that, in addition, $\mathbb{E}(|X|) < \infty$. Then the regression curve of X with respect to Y has the form

$$y \mapsto \mathbb{E}(X \mid Y = y) = c(y)\,\mathbb{E}(Z_1) + a(y),$$

where Z_1 is a random variable with density g_1. If, in addition, $\mathbb{E}(|X|^2) < \infty$, then the conditional standard deviation of X with respect to Y is given by

$$y \mapsto \sqrt{\operatorname{Var}(X \mid Y = y)} = c(y)\sqrt{\operatorname{Var}(Z_1)},$$

since the conditional variance of X with respect to Y is

$$\operatorname{Var}(X \mid Y = y) = \mathbb{E}[(X - \mathbb{E}(X \mid Y = y))^2 \mid Y = y] = c(y)^2\,\mathbb{E}[(Z_1 - \mathbb{E}(Z_1))^2] =$$

$$= c(y)^2 \operatorname{Var}(Z_1).$$

Clearly, the regression curve and the conditional standard deviation are linear functions of the condition if and only if the functions a and c are linear. We will restrict ourselves to this case.

3. Linear regressions and conditional standard deviations

Consider the case when the functions a, b c and d are linear functions of the form

$$a(y) = m_1 y + c_1, \quad b(x) = m_2 x + c_2, \quad c(y) = \lambda_1 (y + a_1), \quad d(x) = \lambda_2 (x + a_2).$$

Let

$$E_1 = \{x \in \mathbb{R} \,|x + a_2 > 0\}, \quad E_2 = \{y \in \mathbb{R} \,|y + a_1 > 0\},$$
$$D = E_1 \times E_2 \subset \mathbb{R}^2 \quad \text{and}$$
$$H_1 = \{x \in \mathbb{R} \,|\lambda_1 x + m_1 > 0\}, \quad H_2 = \{y \in \mathbb{R} \,|\lambda_2 y + m_2 > 0\}.$$

From (1.4) we get equation

$$\begin{aligned} g_1\left(\frac{x - m_1 y - c_1}{\lambda_1 (y + a_1)}\right) \frac{1}{\lambda_1 (y + a_1)} f_Y(y) = \\ = g_2\left(\frac{y - m_2 x - c_2}{\lambda_2 (x + a_2)}\right) \frac{1}{\lambda_2 (x + a_2)} f_X(x) \end{aligned} \tag{3.1}$$

for almost all $(x, y) \in D$, where the measurable unknown functions $g_1 : H_1 \to$ $\to \mathbb{R}$, $g_2 : H_2 \to \mathbb{R}$, $f_X : E_1 \to \mathbb{R}$, $f_Y : E_2 \to \mathbb{R}$ are non-negative, such that they are positive on some Lebesgue measurable sets of positive Lebesgue measure.

Here m_1, m_2, c_1, c_2, $a_1, a_2 \in \mathbb{R}$, λ_1, $\lambda_2 \in \mathbb{R}_+$ ($\mathbb{R}_+$ is the set of positive real numbers) are constants with the conditions $K_1 := m_1 a_1 - c_1 - a_2 \geqslant 0$, $K_2 := m_2 a_2 - c_2 - a_1 \geqslant 0$.

In this paper we consider the case, when $K_1^2 + K_2^2 \neq 0$.

Equation (3.1) is equivalent with the following one:

$$\begin{aligned} g_1\left(\frac{1}{\lambda_1}\left(\frac{x + a_2 + K_1}{y + a_1} - m_1\right)\right) \frac{f_Y(y)}{\lambda_1 (y + a_1)} = \\ = g_2\left(\frac{1}{\lambda_2}\left(\frac{y + a_1 + K_2}{x + a_2} - m_2\right)\right) \frac{f_X(x)}{\lambda_2 (x + a_2)} \end{aligned}$$

for almost all $(x, y) \in D$ and with the substitutions $x + a_2 \to x$, $y + a_1 \to y$ we get equation

$$\begin{aligned} g_1\left(\frac{1}{\lambda_1}\left(\frac{x + K_1}{y} - m_1\right)\right) \frac{1}{\lambda_1 y} f_Y(y - a_1) = \\ = g_2\left(\frac{1}{\lambda_2}\left(\frac{y + K_2}{x} - m_2\right)\right) \frac{1}{\lambda_2 x} f_X(x - a_2) \end{aligned}$$

for almost all $(x, y) \in \mathbb{R}_+^2$.

We note that since

$$\frac{1}{\lambda_1}\left(\frac{x+K_1}{y}-m_1\right)=t \Leftrightarrow 0<\frac{x+K_1}{y}=\lambda_1 t+m_1 \Longrightarrow \lambda_1 t+m_1>0,$$

the support of g_1 indeed equals to H_1. Similarly, by

$$\frac{1}{\lambda_2}\left(\frac{y+K_2}{x}-m_2\right)=s \Leftrightarrow 0<\frac{y+K_2}{x}=\lambda_2 t+m_2 \Longrightarrow \lambda_2 t+m_2>0,$$

we get, that the support of g_2 equals to H_2.

Hence we can state the following.

Lemma 1. *The functions $g_1 : H_1 \to \mathbb{R}$, $g_2 : H_2 \to \mathbb{R}$, $f_Y : E_2 \to \mathbb{R}$, $f_X : E_1 \to \mathbb{R}$ satisfy functional equation* (3.1) *for almost all $(x,y) \in D$ if and only if the functions $\bar{g}_1$, $\bar{g}_2$, $\bar{f}_1$, $\bar{f}_2 : \mathbb{R}_+ \to \mathbb{R}$ defined by*

$$\bar{g}_1(t)=g_1\left(\frac{1}{\lambda_1}(t-m_1)\right),\quad \bar{g}_2(t)=g_2\left(\frac{1}{\lambda_2}(t-m_2)\right),$$

$$\bar{f}_1(y)=f_Y(y-a_1),\quad \bar{f}_2(x)=f_X(x-a_2)$$

satisfy functional equation

$$\bar{g}_1\left(\frac{x+K_1}{y}\right)\bar{f}_1(y)=\bar{g}_2\left(\frac{y+K_2}{x}\right)\bar{f}_2(x)\frac{\lambda_1}{\lambda_2}\frac{y}{x} \tag{3.2}$$

for almost all $(x,y) \in \mathbb{R}_+^2$.

In order to determine the solutions of (3.2) (and so (3.1)) we will use the following general result (see [5]).

Let us consider the functional equation

$$f_1(x)f_2(y)=p_1(G_1(x,y))\,p_2(G_2(x,y))\,h(x,y) \tag{3.3}$$

with unknown functions $f_1 : X \to \mathbb{C}$, $f_2 : Y \to \mathbb{C}$, $p_1 : U \to \mathbb{C}$, $p_2 : V \to \mathbb{C}$ and given functions G_1, G_2 and h satisfied for almost all pairs $(x,y) \in X \times Y$ (with respect to the plane Lebesgue measure), where $X, Y, U, V \subset \mathbb{R}$ are nonvoid open intervals, h is nowhere zero on $X \times Y$, the mapping $(x,y) \mapsto G(x,y) :=$ $:= (G_1(x,y), G_2(x,y))$ is a $\mathcal{C}^1$-diffeomorphism of $X \times Y$ onto $U \times V$ with inverse $(u,v) \mapsto F(u,v) := (F_1(u,v), F_2(u,v))$, and all the partial derivatives

$$\frac{\partial G_1}{\partial x}(x,y),\quad \frac{\partial G_1}{\partial y}(x,y),\quad \frac{\partial G_2}{\partial x}(x,y),\quad \frac{\partial G_2}{\partial y}(x,y)$$

and

$$\frac{\partial F_1}{\partial u}(u,v),\quad \frac{\partial F_1}{\partial v}(u,v),\quad \frac{\partial F_2}{\partial u}(u,v),\quad \frac{\partial F_2}{\partial v}(u,v)$$

vanish nowhere on their domain.

Let us observe that substituting $u = G_1(x, y)$ and $v = G_2(x, y)$, we obtain the functional equation

$$f_1 (F_1(u, v)) f_2 (F_2(u, v)) = p_1(u)p_2(v)h (F_1(u, v), F_2(u, v)) \tag{3.4}$$

satisfied for almost all $(u, v) \in U \times V$; indeed, if (3.3) is satisfied for all $(x, y) \in \in X \times Y \setminus N$, where $N \subset X \times Y$ has plane measure zero, then (3.4) is satisfied for all $(u, v) \in U \times V \setminus M$, where $M = G(N)$ and M has plane measure zero because G is a diffeomorphism.

Theorem 1. *Suppose that the measurable functions f_1, f_2, p_1, p_2 satisfy the functional equation* (3.3) *almost everywhere. Then either one of the functions f_1 and f_2 and one of the functions p_1 and p_2 are zero almost everywhere or all of them are almost everywhere nonzero.*

Using this theorem to equation (3.2) we can state the following.

Theorem 2. *Let $\bar{g}_1$, $\bar{g}_2$, $\bar{f}_1$, $\bar{f}_2 : \mathbb{R}_+ \to \mathbb{R}$ be nonnegative measurable functions satisfying* (3.2) *for almost all $(x, y) \in \mathbb{R}^2_+$, such that they are positive on some Lebesgue measurable subsets of $\mathbb{R}_+$ of positive Lebesgue measure. Then $\bar{g}_1$, $\bar{g}_2$, $\bar{f}_1$, $\bar{f}_2$ are positive almost everywhere on $\mathbb{R}_+$.*

Proof. Let us write $xy - K_1$ instead of x in (3.2), hence we get equation

$$\bar{g}_1 (x) \bar{f}_1 (y) = \bar{g}_2 \left(\frac{y + K_2}{xy - K_1} \right) \bar{f}_2 (xy - K_1) \frac{\lambda_1}{\lambda_2} \frac{y}{xy - K_1} \tag{3.5}$$

for almost all x, $y > 0$, $xy > K_1$.

The assumptions of Theorem 1 are satisfied, for the mapping

$$(x, y) \to G(x, y) := (G_1(x, y), G_2(x, y)) = \left(\frac{y + K_2}{xy - K_1}, xy - K_1 \right)$$

we have non-vanishing partial derivatives

$$\frac{\partial G_1}{\partial x} = -\frac{(y + K_2)\, y}{(xy - K_1)^2} \neq 0, \quad \frac{\partial G_1}{\partial y} = -\frac{K_1 + K_2 x}{(xy - K_1)^2} \neq 0$$

$$\frac{\partial G_2}{\partial x} = y \neq 0, \quad \frac{\partial G_2}{\partial y} = x \neq 0.$$

The inverse of this mapping is

$$(u, v) \to F(x, y) := (F_1(u, v), F_2(u, v)) = \left(\frac{v + K_1}{uv - K_2}, uv - K_2 \right),$$

$$u, v > 0, \quad uv > K_2$$

and the partial derivatives don't vanish:

$$\frac{\partial F_1}{\partial u} = -\frac{(v+K_1)\,v}{(uv-K_2)^2} \neq 0, \quad \frac{\partial F_1}{\partial v} = -\frac{K_2+K_1 u}{(uv-K_2)^2} \neq 0$$

$$\frac{\partial F_2}{\partial u} = v \neq 0, \quad \frac{\partial F_2}{\partial v} = u \neq 0.$$

We can easily check, that all assumptions of Theorem 1 are satisfied and further none of the functions are almost everywhere zero, then all the functions are almost everywhere nonzero. Thus the nonnegativity of functions implies that $\bar{g}_1$, $\bar{g}_2$, $\bar{f}_1$, $\bar{f}_2$ are almost everywhere positive. ■

Using Theorem 2 and a general result of A. Járai [3] we can prove the following result.

Theorem 3. *Let $\bar{g}_1$, $\bar{g}_2$, $\bar{f}_1$, $\bar{f}_2 : \mathbb{R}_+ \to \mathbb{R}$ be nonnegative measurable functions, satisfying* (3.2) *for almost all $(x, y) \in \mathbb{R}_+^2$ such that they are positive on some Lebesgue measurable subsets of $\mathbb{R}_+$ of positive measure. Then there exist unique continuous functions $\widetilde{g}_1$, $\widetilde{g}_2$, $\widetilde{f}_1$, $\widetilde{f}_2 : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\widetilde{g}_1 = \bar{g}_1$, $\widetilde{g}_2 = \bar{g}_2$, $\widetilde{f}_1 = \bar{f}_1$ and $\widetilde{f}_2 = \bar{f}_2$ almost everywhere on $\mathbb{R}_+$, and if $\bar{g}_1$, $\bar{g}_2$, $\bar{f}_1$, $\bar{f}_2$ are replaced by $\widetilde{g}_1$, $\widetilde{g}_2$, $\widetilde{f}_1$, $\widetilde{f}_2$, respectively, then equation* (3.2) *is satisfied everywhere on $\mathbb{R}_+^2$.*

Proof. Theorem 2 shows that functions $\bar{g}_1$, $\bar{g}_2$, $\bar{f}_1$, $\bar{f}_2$ are positive almost everywhere on $\mathbb{R}_+$.

First we prove that there exists a unique continuous function $\tilde{g}_1$ which is equal to $\bar{g}_1$ almost everywhere on $\mathbb{R}_+$ and replacing $\bar{g}_1$ by $\tilde{g}_1$, equation (3.2) is satisfied almost everywhere.

With the substitution $t = \frac{x+K_1}{y}$ we get from (3.2) the equation

$$\bar{g}_1(t) = \frac{\bar{g}_2\left(\frac{y+K_2}{ty-K_1}\right) \bar{f}_2(ty-K_1)\, \frac{\lambda_1}{ty-K_1}}{\lambda_2 \bar{f}_1(y)\, \frac{1}{y}} \tag{3.6}$$

which is satisfied for almost all $(t, y) \in \Delta$, where $\Delta = \{(t, y)\,|t,\, y > 0,\, ty > K_1\}$. By Fubini's Theorem it follows that there exists $T' \subseteq \mathbb{R}_+$ of full measure such that for all $t \in T'$ equation (3.6) is satisfied for almost every $y \in \Delta_t$, where

$$\Delta_t = \{y \in \mathbb{R}_+ |\, (t, y) \in \Delta\}.$$

Let us define the functions $\mathtt{g}_1, \mathtt{g}_2, \mathtt{g}_3, \mathtt{h}$ in the following way:

$$\mathtt{g}_1(t, y) = \frac{y+K_2}{ty-K_1}, \quad \mathtt{g}_2(t, y) = ty - K_1,$$

$$\mathtt{g}_3(t, y) = y, \quad \mathtt{h}(t, y, z_1, z_2, z_3) = \frac{z_1 z_2}{z_3},$$

and let us now apply a theorem of Járai (see [3] Theorem 3) to (3.6) with the following casting:

$$\bar{g}_1(t) = \mathtt{f}(t), \bar{g}_2(t) = \mathtt{f}_1(t), \frac{\lambda_1 \bar{f}_2(t)}{t} = \mathtt{f}_2(t), \frac{\lambda_2 \bar{f}_1(t)}{t} = \mathtt{f}_3(t)$$

$$Z = Z_i = \mathbb{R}_+, \quad T = Y = X_i = \mathbb{R}_+, \quad (i = 1, 2, 3).$$

One can easily verify that all assumptions of Járai's Theorem are satisfied, thus we get that there exists a unique continuous function $\tilde{g}_1 : \mathbb{R}_+ \to \mathbb{R}$ which is almost everywhere equal to $\bar{g}_1$ on $\mathbb{R}_+$ and $\tilde{g}_1, \bar{g}_2, \bar{f}_1, \bar{f}_2$ satisfy equation (3.2) almost everywhere, which is equivalent to the equation

$$\widetilde{g}_1\left(\frac{x+K_1}{y}\right)\bar{f}_1(y) = \bar{g}_2\left(\frac{y+K_2}{x}\right)\bar{f}_2(x)\frac{\lambda_1}{\lambda_2}\frac{y}{x}$$

for almost all $(x, y) \in \mathbb{R}^2_+$. Furthermore, $\tilde{g}_1$ is positive for almost all $x \in \mathbb{R}_+$.

By a similar argument we can prove the same for the function $\bar{g}_2$, $\bar{f}_1$ and $\bar{f}_2$, i.e. there exist continuous functions $\tilde{g}_2 : \mathbb{R}_+ \to \mathbb{R}$, $\tilde{f}_1 : \mathbb{R}_+ \to \mathbb{R}$ and $\tilde{f}_2 : \mathbb{R}_+ \to \mathbb{R}$ which are almost everywhere equal to $\bar{g}_2$, $\bar{f}_1$ and $\bar{f}_2$ on $\mathbb{R}_+$, respectively, and the functional equation

$$\widetilde{g}_1\left(\frac{x+K_1}{y}\right)\widetilde{f}_1(y) = \widetilde{g}_2\left(\frac{y+K_2}{x}\right)\widetilde{f}_2(x)\frac{\lambda_1}{\lambda_2}\frac{y}{x} \tag{3.7}$$

is satisfied almost everywhere on $\mathbb{R}^2_+$.

Both side of (3.7) define continuous functions on $\mathbb{R}^2_+$, which are equal to each other on a dense subset of $\mathbb{R}^2_+$, therefore we obtain that (3.7) is satisfied everywhere on $\mathbb{R}^2_+$.

One can show that if the nonnegative continuous functions $\tilde{g}_1$, $\tilde{g}_2$, $\tilde{f}_1$, $\tilde{f}_2 :$ $: \mathbb{R}_+ \to \mathbb{R}$ satisfy functional equation (3.7) for all $(x, y) \in \mathbb{R}^2_+$, such that they are positive almost everywhere on $\mathbb{R}_+$, then they are positive everywhere on $\mathbb{R}_+$. ■

4. Solutions of functional equations (3.7) and (3.1), characterizations

We can take the logarithm of equation (3.7) and we get that continuous functions $\widetilde{G}_1$, $\widetilde{G}_2$, $\widetilde{F}_1$, $\widetilde{F}_2$ defined by

$$\widetilde{G}_1(t) = \ln \widetilde{g}_1(t), \quad \widetilde{G}_2(t) = \ln \widetilde{g}_2(t), \quad \widetilde{F}_1(t) = \ln \frac{\widetilde{f}_1(t)}{\lambda_1 t}, \quad \widetilde{F}_2(t) = \ln \frac{\widetilde{f}_2(t)}{\lambda_2 t}$$

satisfy functional equation

$$\widetilde{G}_1\left(\frac{x+K_1}{y}\right)+\widetilde{F}_1(y)=\widetilde{G}_2\left(\frac{y+K_2}{x}\right)+\widetilde{F}_2(x) \tag{4.1}$$

everywhere on $\mathbb{R}_+^2$.

First we prove the following

Lemma 2. *If the continuous functions $\widetilde{G}_1$, $\widetilde{G}_2$, $\widetilde{F}_1$, $\widetilde{F}_2 : \mathbb{R}_+ \to \mathbb{R}$ satisfy functional equation* (4.1) *for all $(x,y) \in \mathbb{R}_+^2$, then they are differentiable infinitely many times on $\mathbb{R}_+$.*

Proof. With the substitution $t=\frac{x+K_1}{y}$, we get from (4.1) the equation

$$\widetilde{G}_1(t)=\widetilde{G}_2\left(\frac{y+K_2}{ty-K_1}\right)+\widetilde{F}_2(ty-K_1)-\widetilde{F}_1(y), \quad (t,y)\in D, \tag{4.2}$$

where $D=\left\{(t,y)\in\mathbb{R}_+^2 | t,y\in\mathbb{R}_+, t\cdot y>K_1\right\}$.

Let $[a,b]\subset\mathbb{R}_+$ be arbitrary and choose the interval $[c,d]\subset\mathbb{R}_+$ arbitrary such that $[a,b]\times[c,d]\subset D$ holds.

Integrating (4.2) with respect to y on $[c,d]$ we get

$$(d-c)\widetilde{G}_1(t)=\int\limits_c^d \widetilde{G}_2\left(\frac{y+K_2}{ty-K_1}\right)dy+\int\limits_c^d \widetilde{F}_2(ty-K_1)\,dy-\int\limits_c^d \widetilde{F}_1(y)\,dy.$$

We use the substitutions

$$g_1(t,y)=\frac{y+K_2}{ty-K_1}=u, \qquad g_2(t,y)=ty-K_1=u$$

in the first and in the second integral, respectively.

It is easy to see that function $y\to g_1(t,y)$ is decreasing and function $y\to$ $\to g_2(t,y)$ is increasing.

Thus these equations can be solved uniquely for y if $t\in[a,b]$.

$$y=-\frac{K_1u+K_2}{1-tu}=\gamma_1(t,u), \qquad y=\frac{K_1+u}{t}=\gamma_2(t,u)$$

(Here $tu\neq 1$. $\frac{y+K_2}{ty-K_1}=u$ implies that $\frac{ty+tK_2}{ty-K_1}=tu$. Assuming that $tu=1$, we get $\frac{ty+tK_2}{ty-K_1}=1 \iff K_2t+K_1=0$, but this is impossible because of $t\in\mathbb{R}_+$, K_1, $K_2\geqslant 0$, $K_1^2+K_2^2\neq 0$.)

γ_1 and γ_2 are infinitely many times differentiable functions of t and u.

Performing the substitutions, we have

$$\widetilde{G}_1(t) = \frac{1}{d-c}\left[\int\limits_{\frac{d+K_2}{dt-K_1}}^{\frac{c+K_2}{ct-K_1}} \widetilde{G}_2(u)\, D_2\gamma_1(t,u)\, du + \int\limits_{ct-K_1}^{dt-K_1} \widetilde{F}_2(u)\, D_2\gamma_2(t,u)\, dt - C\right],$$

where $C = \int_c^d \widetilde{F}_1(y)\, dy$.

The functions $\widetilde{G}_2$, $\widetilde{F}_2$ are at least continuous. Hence, by repeated applications of the theorem concerning the differentiation of parametric integrals (see e.g. [2]) the right-hand side is differentiable infinitely many times on $[a,b]$. Since $[a,b]$ is an arbitrary interval of $\mathbb{R}_+$, we have that $\widetilde{G}_1$ is differentiable infinitely many times on $\mathbb{R}_+$.

The differentiability of $\widetilde{G}_2$ can be obtained similarly.

By the help of equation (4.1) one can easily deduce that $\widetilde{F}_1$ and $\widetilde{F}_2$ are also differentiable infinitely many times on $\mathbb{R}_+$. ■

Now, using Lemma 2.1.6. from [6], that is differentiate equation (4.1) twice we get that the functions $\widetilde{G}_1$ and $\widetilde{G}_2$ satisfy the differential equations

$$t\widetilde{G}_1''(t) + \widetilde{G}_1'(t) = \frac{\gamma}{(K_2 t + K_1)^2} \qquad (t \in \mathbb{R}_+)$$

and

$$s\widetilde{G}_2''(s) + \widetilde{G}_2'(s) = \frac{\gamma}{(K_1 s + K_2)^2} \qquad (s \in \mathbb{R}_+)$$

with some constant γ. Solving these differential equations we get the solutions for equation (4.1) and hence for equations (3.7), (3.2) and finally for (3.1), so we can state the following.

Theorem 4. *If the measurable functions g_1, g_2, f_X, f_Y satisfy equation* (3.1) *in case $K_1 > 0$, $K_2 > 0$ for almost all $(x,y) \in D$, then*

$$\begin{aligned}
g_1(x) = {} & \exp(d_1)\left(\frac{K_2}{K_1}(\lambda_1 x + m_1)\right)^{p_1}\left(\frac{K_2}{K_1}(\lambda_1 x + m_1) + 1\right)^{q} (a.a.\ x \in H_1),\\
g_2(x) = {} & \exp(d_2)\left(\frac{K_1}{K_2}(\lambda_2 x + m_2)\right)^{p_2}\left(\frac{K_1}{K_2}(\lambda_2 x + m_2) + 1\right)^{q} (a.a.\ x \in H_2),\\
f_Y(y) = {} & \exp(d_3)\frac{\lambda_1}{K_2^{p_1+q}}(y + a_1)^{p_1+q+1}\left(\frac{y+a_1}{K_2} + 1\right)^{p_2} \qquad (a.a.\ \ y \in E_2),\\
f_X(x) = {} & \exp(d_4)\frac{\lambda_2}{K_1^{p_2+q}}(x + a_2)^{p_2+q+1}\left(\frac{x+a_2}{K_1} + 1\right)^{p_1} \qquad (a.a.\ \ x \in E_1),
\end{aligned}$$

where p_i, q, $d_j \in \mathbb{R}$ $(i = 1,2;\ j = 1,2,3,4)$ *are arbitrary constants with* $d_1 + d_3 = d_2 + d_4$.

These functions are densities if and only if $-1 < p_1 < 0$, $-1 < p_2 < 0$, $-2 - \min\{p_1, p_2\} < q < -2 - p_1 - p_2$, *and* d_1, d_2, d_3 *and* d_4 *are appropriate norming constants. Consequently, in this case the marginals and the conditional distributions of the absolutely continuous random vector* (X, Y) *are beta distributions of the second kind, and the joint density function has the form*

$$f_{(X,Y)}(x,y) = \frac{1}{K}\left(\frac{x+a_2}{K_1}+1\right)^{p_1}\left(\frac{y+a_1}{K_2}+1\right)^{p_1}\left(\frac{x+a_2}{K_1}+\frac{y+a_1}{K_2}+1\right)^{q}$$

for almost all $(x,y) \in D$, *where*

$$K = K_1K_2B(p_1+1, -q-p_1-1)B(p_1+q-2, -q-p_1-p_2-2)$$

(*here* B *is the beta function*).

Theorem 5. *If the measurable functions* g_1, g_2, f_X, f_Y *satisfy equation* (3.1) *in case* $K_1 = 0$, $K_2 > 0$ *for almost all* $(x,y) \in D$, *then*

$$\begin{aligned}
g_1(x) &= A_1(\lambda_1 x + m_1)^{-b_1}\exp\left\{\frac{a}{K_2(\lambda_1 x + m_1)}\right\} && (a.a.\ \ x \in H_1),\\
g_2(x) &= A_2(\lambda_2 x + m_2)^{b_2}\exp\left\{\frac{a(\lambda_2 x + m_2)}{K_2}\right\} && (a.a.\ \ x \in H_2),\\
f_Y(y) &= \lambda_1 A_3(y+a_1)^{1-b_1}(y+a_1+K_2)^{b_2} && (a.a.\ \ y \in E_2),\\
f_X(x) &= \lambda_2 A_4(x+a_2)^{b_2-b_1+1}\exp\left\{\frac{-a}{x+a_2}\right\} && (a.a.\ \ x \in E_1),
\end{aligned}$$

where a, b_1, b_2, $A_i \in \mathbb{R}$, $i = 1,2,3,4$ *are arbitrary constants with* $A_1A_3 = A_2A_4$.

Theorem 6. *If the measurable functions* g_1, g_2, f_X, f_Y *satisfy equation* (3.1) *in case* $K_1 > 0$, $K_2 = 0$ *for almost all* $(x,y) \in D$, *then*

$$\begin{aligned}
g_1(x) &= A_1(\lambda_1 x + m_1)^{b_2}\exp\left\{\frac{a(\lambda_1 x + m_1)}{K_1}\right\} && (a.a.\ \ x \in H_1),\\
g_2(x) &= A_2(\lambda_2 x + m_2)^{-b_1}\exp\left\{\frac{a}{K_1(\lambda_2 x + m_2)}\right\} && (a.a.\ \ x \in H_2),\\
f_Y(y) &= \lambda_1 A_3(y+a_1)^{b_2-b_1+1}\exp\left\{\frac{-a}{y+a_1}\right\} && (a.a.\ \ y \in E_2),\\
f_X(x) &= \lambda_2 A_4(x+a_2)^{1-b_1}(x+a_2+K_1)^{b_2} && (a.a.\ \ x \in E_1),
\end{aligned}$$

where a, b_1, b_2, $A_i \in \mathbb{R}$, $i = 1,2,3,4$ *are arbitrary constants with* $A_1A_3 = A_2A_4$.

Remark. The functions in Theorem 5 (and in Theorem 6) can not be density functions simultaneously, so we have no solution for the original probability problem in case $K_1 = 0$, $K_2 > 0$ (and in case $K_1 > 0$, $K_2 = 0$).

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