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SIMULTANEOUS NUMBER SYSTEMS IN THE LATTICE OF EISENSTEIN INTEGERS

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Dedicated to Professors Zoltán Daróczy and Imre Kátai on their 75th anniversary

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Abstract. The notion of simultaneous number systems was introduced by Kátai et al. in [1]. In this paper simultaneous number systems constructions are investigated in the lattice of Eisenstein integers. We show that except for 8 trivial cases for each Eisenstein integer η_1 there is an Eisenstein integer η_2 and some appropriate digit set D such that (η_1, η_2, D) forms a simultaneous number system.

1. Introduction

The *Eisenstein integers* (sometimes also called *Eulerian integers*) are complex numbers of the form $\eta = a + b\omega$ where $a, b \in \mathbb{Z}$ and $\omega = \frac{-1+i\sqrt{3}}{2} = \exp(2\pi i/3)$ is a cube root of unity. They form a triangular lattice Λ_ε in the complex plane, or in a different view, they form a commutative ring of the algebraic integers in the third cyclotomic field. Equivalently, we can consider the Eisenstein integers as linear operators of the form

$$M_\varepsilon = M_\varepsilon(a,b) = \begin{pmatrix} a & -b \\ b & a-b \end{pmatrix} \tag{1.1}$$

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acting on $\mathbb{Z}^2$, where $a, b \in \mathbb{Z}$. It is known that the norm of η is $N(a + b\omega) = a^2 - ab + b^2$. The group of units in the ring of Eisenstein integers is the cyclic group formed by the complex sixth roots of unity.

Let Λ be a lattice in $\mathbb{R}^n$, $M : \Lambda \to \Lambda$ be a linear operator such that $\det(M) \neq 0$, and let D be a finite subset of Λ containing 0.

Definition 1.1. The triple (Λ, M, D) is called a *generalized number system* (GNS) if every element x of Λ has a unique, finite representation of the form

$$x = \sum_{i=0}^{l} M^i d_i$$

where $d_i \in D$ and $l \in \mathbb{N}, d_l \neq 0$.

Clearly, Λ is a finitely generated free abelian group with addition. If two elements of Λ are in the same coset of the factor group $\Lambda/M\Lambda$ then they are said to be *congruent modulo* M.

Theorem 1.1 ([2])**.** *If* (Λ, M, D) *is a number system then*

(1) D *must be a complete residue system modulo* M*,*

(2) M *must be expansive and*

(3) $\det(I - M) \neq \pm 1$.

If a system fulfills these conditions then it is a *radix system* and the operator M is called a *radix base.*

Let $\phi : \Lambda \to \Lambda$, $x \overset{\phi}{\mapsto} M^{-1}(x - d)$ for the unique $d \in D$ satisfying $x \equiv d \pmod{M}$. Since M^{-1} is contractive and D is finite, there exists a norm $\|.\|$ on $\mathbb{R}^n$ and a constant $C \in \mathbb{R}$ such that the orbit of every $x \in \Lambda$ eventually enters the finite set $\{x \in \Lambda \ : \ \|x\| < C\}$ for the repeated application of ϕ. This means that the sequence (path) $x, \phi(x), \phi^2(x), \dots$ is eventually *periodic* for all $x \in \Lambda$. If a points $p \in \Lambda$ is periodic then $\|p\| \leq L = Kr/(1 - r)$, where $r = \|M^{-1}\| = \sup_{\|x\| \leq 1} \|M^{-1}x\| < 1$ and $K = \max_{d \in D} \|d\|$ (see [3]). Let us denote the set of periodic elements by $\mathcal{P}$. The paths of all periodic elements constitute a finite number of disjoint *cycles* $\mathcal{C}_i$. Then, the number system property is equivalent to $\mathcal{P} = \{0\}$, or with the situation that the system has only one cycle $\mathcal{C}_1 = \{0 \to 0\}$.

In this paper we consider special *block diagonal systems* $(\mathbb{Z}^2 \otimes \mathbb{Z}^2, M_1 \oplus M_2, D)$, where M_1 and M_2 are Eisenstein operators, $d_j = (v^T || v^T)^T \in D$ $(v \in \mathbb{Z}^2)$, $\otimes, \oplus$ and $||$ denote the direct product, the direct sum, and the concatenation respectively, furthermore v^T (transpose of v) denotes a row vector. These 4-dimensional systems can be considered as *simultaneous systems of the Eisenstein integers.* We emphasize that in our case the digits are in the subspace $W = \{(x, y, x, y)^T \ : x, y \in \mathbb{Z}\} \leq \mathbb{Z}^4$.

Lemma 1.1. *The only possible radix bases of simultaneous Eisenstein number systems are*

$$M_A(a,b) = \begin{pmatrix} M_\varepsilon & 0 & 0 \\ 0 & a+1 & -b \\ 0 & b & a-b+1 \end{pmatrix}, M_B(a,b) = \begin{pmatrix} M_\varepsilon & 0 & 0 \\ 0 & a & -b-1 \\ 0 & b+1 & a-b-1 \end{pmatrix},$$

$$M_C(a,b) = \begin{pmatrix} M_\varepsilon & 0 & 0 \\ 0 & a+1 & -b-1 \\ 0 & b+1 & a-b \end{pmatrix},$$

where $a, b \in \mathbb{Z}$ and M_ε was defined in (1.1).

Proof. The lemma is a direct consequence of [4, Theorem 3.2]. ■

The first operator is called an A-type, the second one a B-type and the third one a C-type base.

Lemma 1.2. *Let M be the operator of type A, B or C. If $\|M_\varepsilon\|_2 > 4 + \frac{5}{2}\sqrt{2} + \frac{1}{2}\sqrt{98 + 72\sqrt{2}}$ (≈ 14.6) then there is always some digit set D for which the system $(\mathbb{Z}^4, M, D)$ is a simultaneous Eisenstein number system.*

Proof. The proof goes exactly in the same way as was shown in [5, Theorem 3.1]. ■

There are at least two possibilities for digit set constructions in Lemma 1.2.

1. If D_1 and D_2 are adjoint digit sets (containing elements from the fundamental set of the appropriate adjoint lattice) belonging to the blocks of M (to M_1 and to M_2, resp.) then $D = \cup_{d \in D_2}(D_1 + M_1 d)$ is suitable [5].
2. D is dense in W, i.e. it consists of elements with the smallest norm from each congruent class [6].

Lemma 1.2 shows that for all operators M having "big enough norm" there are suitable digit sets D for which the systems $(\mathbb{Z}^4, M, D)$ are simultaneous Eisenstein number systems. This paper deals with the remaining cases:

- examining the number system property using dense digit sets,
- constructing appropriate digit sets algorithmically, when necessary, or
- proving the non-existence of such sets.

Since $\|M^{-1}\|_2 < 1$ always holds (except a few obvious cases) therefore in the following the norm $\|.\|$ means the 2-norm.

2. The searching methods

In order to decide the number system property of a given radix system in general two methods can be considered: the method of Brunotte [7, 8], or the variants of the covering methods [3, 2, 7]. However, for simultaneous systems a much faster algorithm is known.

Lemma 2.1. *Let* $M = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}$ *be the operator of type* A, B *or* C*. For a given simultaneous Eisenstein system* $(\mathbb{Z}^4, M, D)$ *with dense digit set* D *let* $K = \max\{\|d\| : d \in D\}$, $K^* = \max\{\|d\| : d = (x,y)^T, (x,y,x,y)^T \in D\}$, $r = \|M^{-1}\|$, $r_i = \|M_i^{-1}\|$, $L = K\frac{r}{1-r}$, $L_i = K^*\frac{r_i}{1-r_i}$ $(i = 1, 2)$ *and let*

$$\kappa = \frac{\|M_1^{-1}\| \cdot \|M_2^{-1}\| \cdot (L_2 + K^*)}{1 - \|M_1^{-1}\|} = K^* \frac{r_1 r_2}{(1-r_1)(1-r_2)} = L_1 L_2 / K^* \,.$$

If (1) $\kappa < 1$ *and all the elements of* $W \cap L \setminus \{\underline{0}\}$ *are non-periodic or* (2) $\kappa \geq 1$ *all the elements* $v = (x,y,z,w)^T$ $(v \neq \underline{0})$ *for which*

$$\|v\| \leq L, \quad \|(x,y)^T - (z,w)^T\| < \kappa \tag{2.1}$$

are non-periodic then $(\mathbb{Z}^4, M, D)$ *is a simultaneous number system.*

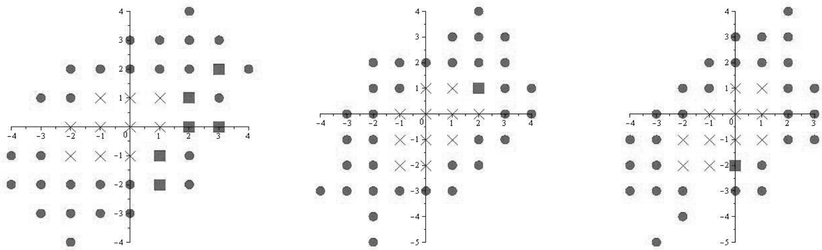
Proof. The proof is analogous to [9, Lemma 3.4]. ■

If the application of Lemma 2.1 shows that the system $(\mathbb{Z}^4, M, D)$ is not a number system then the simultaneous GNS construction algorithm [9] can be used. The algorithm has 3 different possible outcomes. It terminates

- either with an output that the construction is not possible at all (in which case the witness $\gamma \in W$ shows that any digit $\beta \in W$ congruent with γ would constitute a loop in the system), or
- with an appropriate digit set (in which case the two dimensional projection $S = \{(x,y)^T\}$ of the digit set $D = \{(x,y,x,y)^T : x, y \in \mathbb{Z}\}$ is drawn), or
- with a remark of an unsuccessfully construction attempt.

3. The results of the computations

Figure 1 shows the GNS computation results for various operator types. All computations were performed on a laptop with Intel Core i5-2520M, 2.5GHz



(a) Type-A cases (b) Type-B cases (c) Type-C cases

Figure 1: Computation results of simultaneous Eisenstein number system analysis. A point (a, b) in the figure means the appropriate $M(a, b)$ basis. The diagonal-cross points denote the non-radix bases. The circles denote the bases for which simultaneous number systems can be constructed. The boxed points means the bases for which suitable digit sets can not be constructed at all.

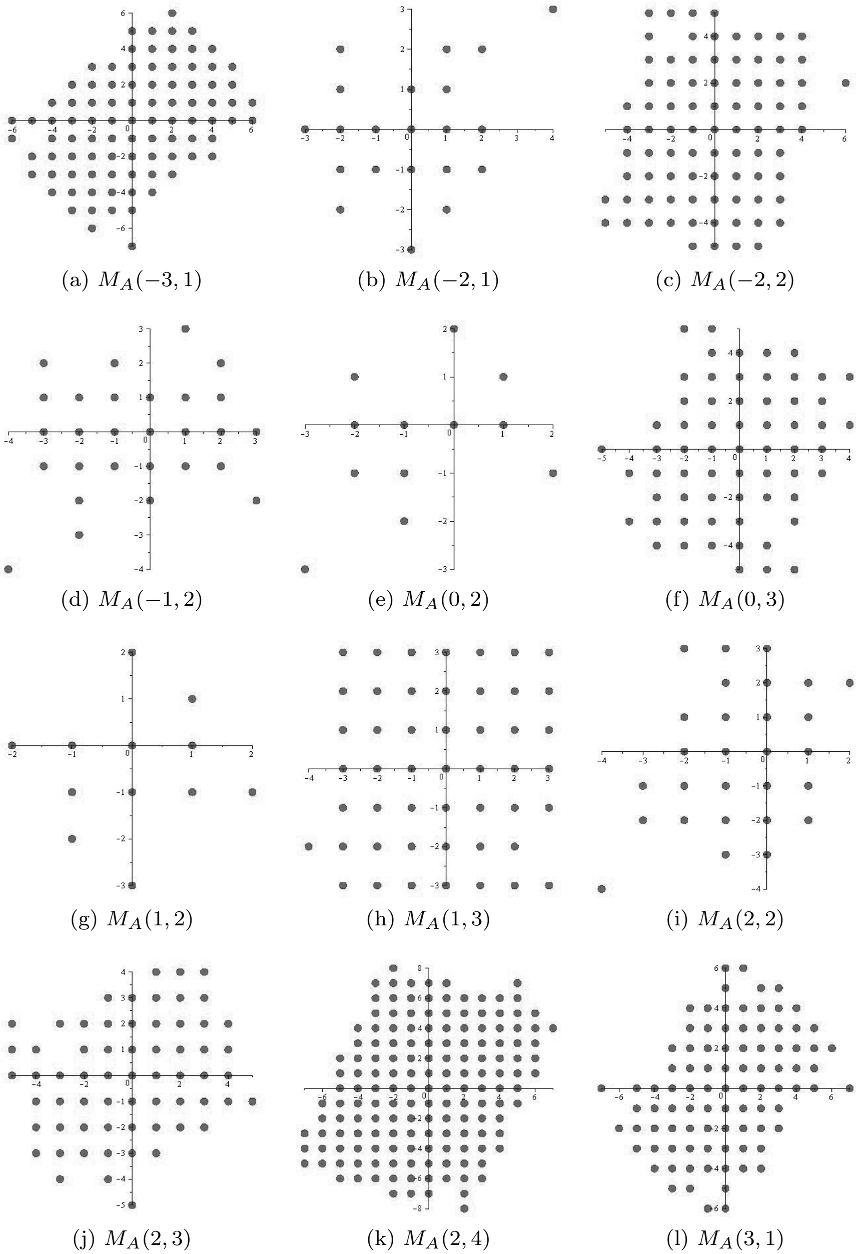
CPU, 8MB RAM, programmed in Maple. We note that the points without any marks denote the bases for which the dense digit sets are suitable, i.e. these systems are Eisenstein number systems.

3.1. Type-A Cases

First we can observe that the dynamics of the systems $\left(\mathbb{Z}^4, M_A(a,b), D\right)$ and $\left(\mathbb{Z}^4, M_A(a-b,-b), D\right)$ are exactly the same. Hence, it is enough to examine the cases $b \geq 0$. Figure 2 and Figure 3 (a)–(b) show the cases when the digit sets with the appropriate operators form simultaneous Eisenstein number systems. The marked points $(x, y) \in S$ in the pictures denote that the vecor $(x, y, x, y)^T$ belongs to the appropriate digit set. Table 1 contains the radix systems which can not be simultaneous Eisenstein number systems for any digit set D. The table contains the γ residue class representants as well.

3.2. Type-B Cases

Figure 3 (c)–(l), Figure 4 and Figure 5 (a)–(h) show the GNS computation results, i.e. the cases when the digit sets with the appropriate operators form simultaneous Eisenstein number systems. Observe that there are some operators for which the same digit set satisfies the number system property at the same time. This is not a coincidence. For the operator $M = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}$ let us



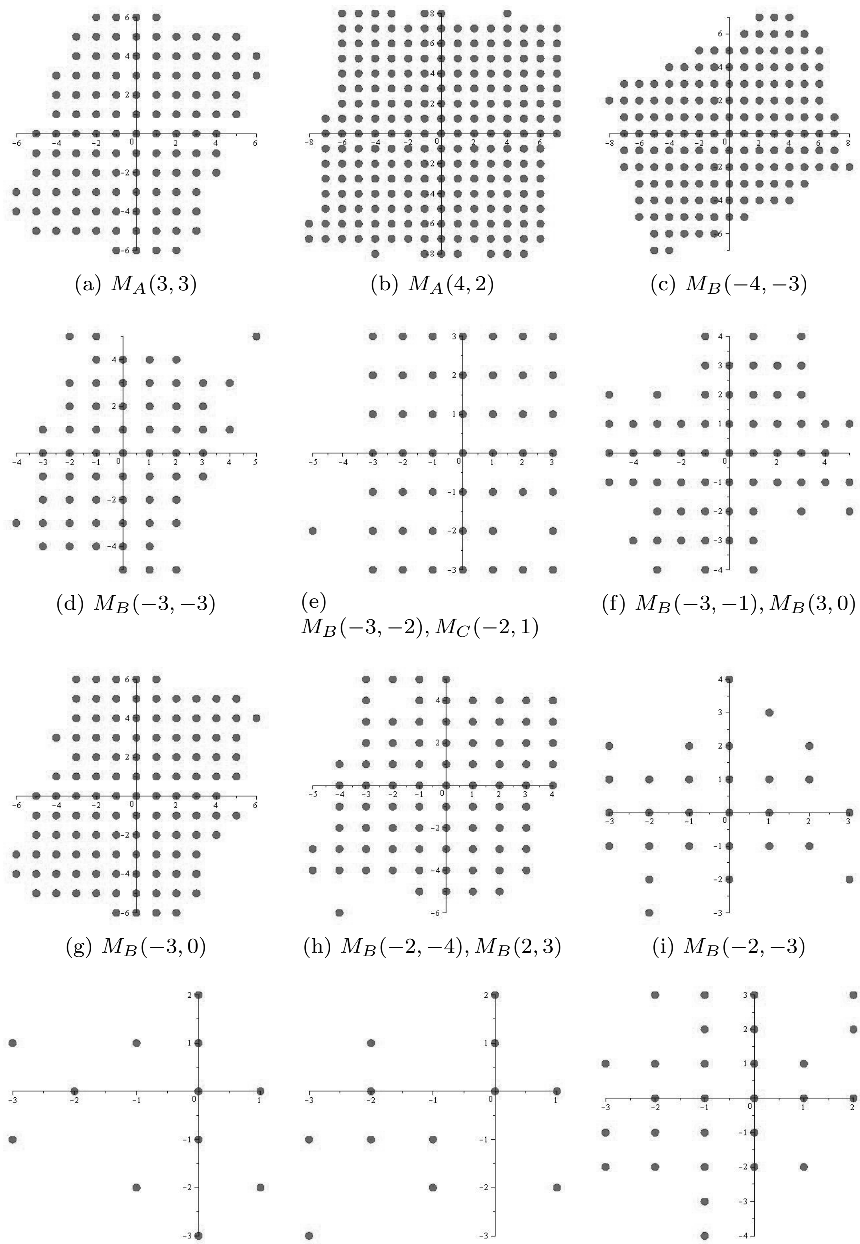
(a) $M_A(-3,1)$ (b) $M_A(-2,1)$ (c) $M_A(-2,2)$

(d) $M_A(-1,2)$ (e) $M_A(0,2)$ (f) $M_A(0,3)$

(g) $M_A(1,2)$ (h) $M_A(1,3)$ (i) $M_A(2,2)$

(j) $M_A(2,3)$ (k) $M_A(2,4)$ (l) $M_A(3,1)$

Figure 2: Digit sets of simultaneous Eisenstein number systems.



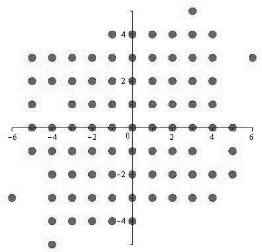
(a) $M_A(3,3)$ (b) $M_A(4,2)$ (c) $M_B(-4,-3)$

(d) $M_B(-3,-3)$ (e) $M_B(-3,-2), M_C(-2,1)$ (f) $M_B(-3,-1), M_B(3,0)$

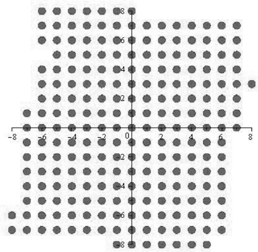
(g) $M_B(-3,0)$ (h) $M_B(-2,-4), M_B(2,3)$ (i) $M_B(-2,-3)$

(j) $M_B(-2,-2)$ (k) $M_B(-2,-1)$ (l) $M_B(-2,0)$

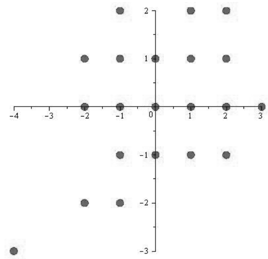
Figure 3: Digit sets of simultaneous Eisenstein number systems.



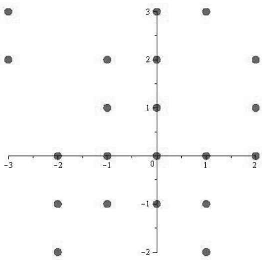
(a) $M_B(-2,1), M_B(2,-2)$



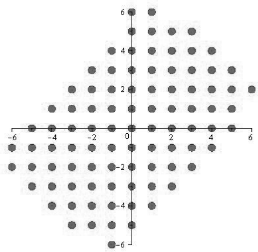
(b) $M_B(-2,2)$



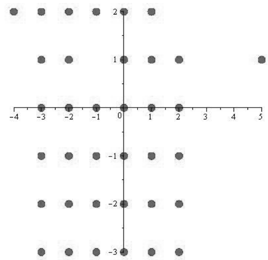
(c) $M_B(-1,-3)$



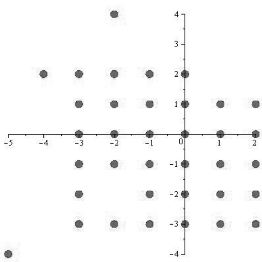
(d) $M_B(-1,1)$



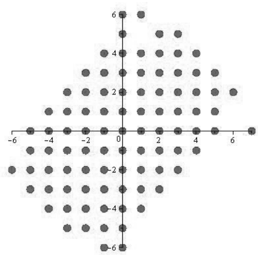
(e) $M_B(-1,2)$



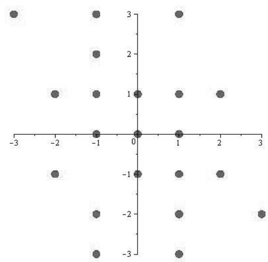
(f) $M_B(0,-3)$



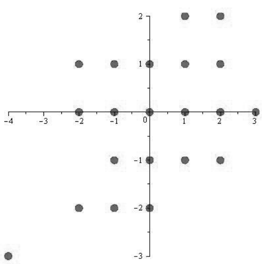
(g) $M_B(0,2)$



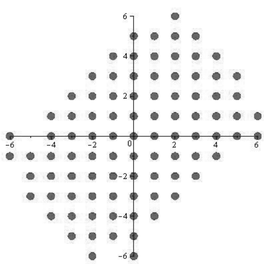
(h) $M_B(1,-3)$



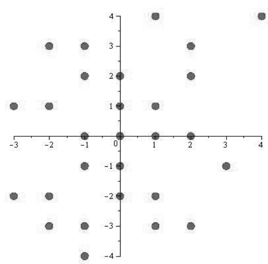
(i) $M_B(1,-2)$



(j) $M_B(1,2)$

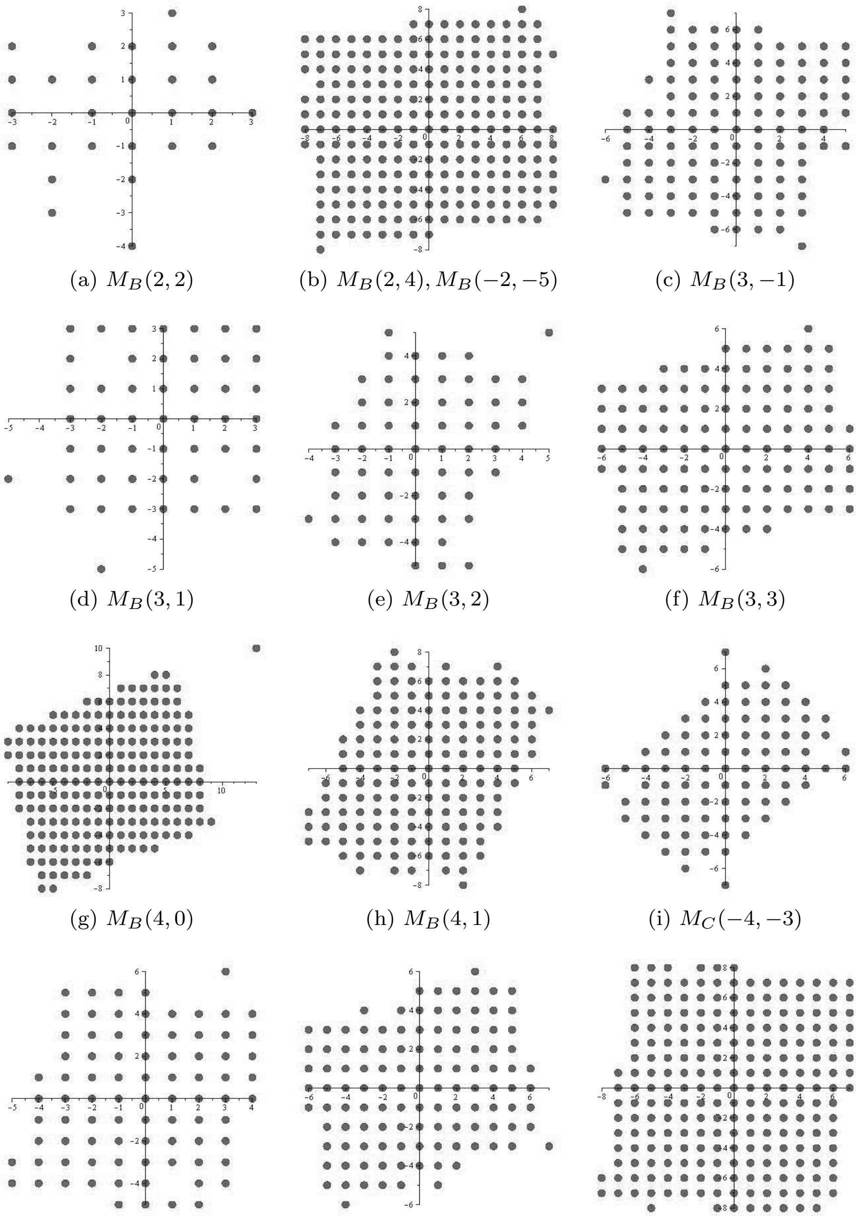


(k) $M_B(1,3)$



(l) $M_B(2,-1)$

Figure 4: Digit sets of simultaneous Eisenstein number systems.



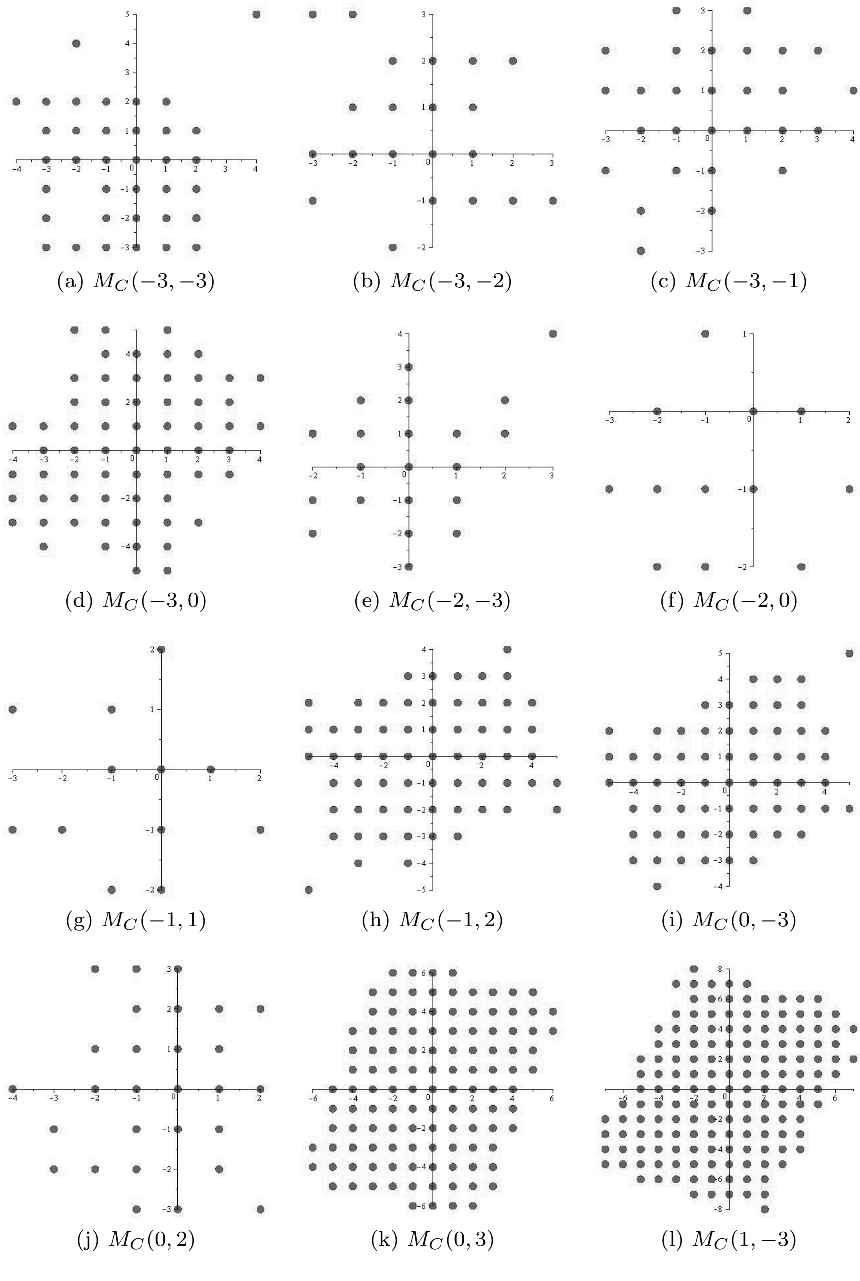
(a) $M_B(2,2)$ (b) $M_B(2,4), M_B(-2,-5)$ (c) $M_B(3,-1)$

(d) $M_B(3,1)$ (e) $M_B(3,2)$ (f) $M_B(3,3)$

(g) $M_B(4,0)$ (h) $M_B(4,1)$ (i) $M_C(-4,-3)$

(j) $M_C(-4,-2)$ (k) $M_C(-4,-1)$ (l) $M_C(-3,-5)$

Figure 5: Digit sets of simultaneous Eisenstein number systems.



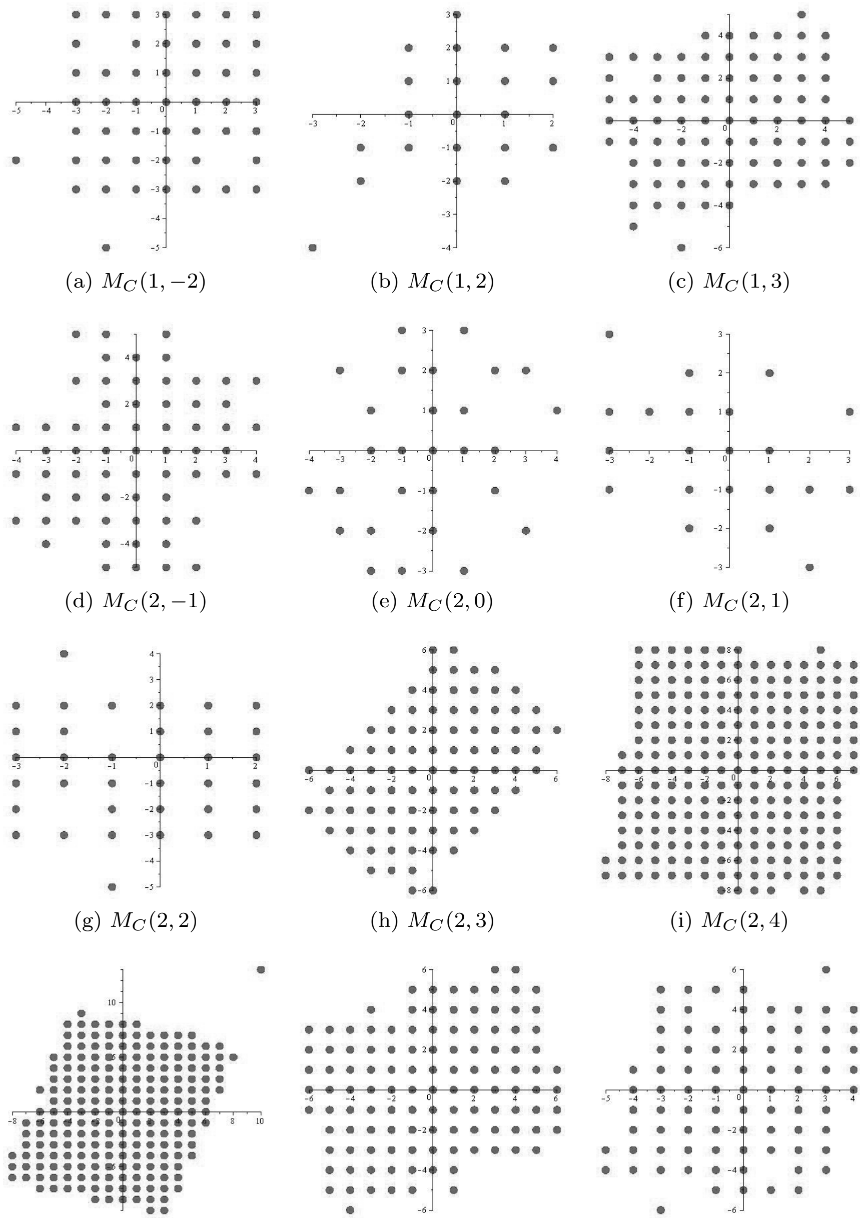
(a) $M_C(-3,-3)$ (b) $M_C(-3,-2)$ (c) $M_C(-3,-1)$

(d) $M_C(-3,0)$ (e) $M_C(-2,-3)$ (f) $M_C(-2,0)$

(g) $M_C(-1,1)$ (h) $M_C(-1,2)$ (i) $M_C(0,-3)$

(j) $M_C(0,2)$ (k) $M_C(0,3)$ (l) $M_C(1,-3)$

Figure 6: Digit sets of simultaneous Eisenstein number systems.



(a) $M_C(1,-2)$ (b) $M_C(1,2)$ (c) $M_C(1,3)$

(d) $M_C(2,-1)$ (e) $M_C(2,0)$ (f) $M_C(2,1)$

(g) $M_C(2,2)$ (h) $M_C(2,3)$ (i) $M_C(2,4)$

(j) $M_C(3,-1)$ (k) $M_C(3,0)$ (l) $M_C(3,1)$

Figure 7: Digit sets of simultaneous Eisenstein number systems.

Base	γ	Base	γ
$M_A(2,0)$	$(2,2,2,2)^T$	$M_A(2,1)$	$(1,2,1,2)^T$
	$(2,0,2,0)^T$		$(1,-1,1,-1)^T$
	$(2,-2,2,-2)^T$		$(-1,1,-1,1)^T$
	$(0,2,0,2)^T$		$(2,1,2,1)^T$
	$(0,-2,0,-2)^T$		$(-2,-1,-2,-1)^T$
	$(-2,0,-2,0)^T$		$(-1,-2,-1,-2)^T$
	$(-2,2,-2,2)^T$	$M_B(2,1)$	$(-1,-2,-1,-2)^T$
	$(-2,-2,-2,-2)^T$		$(-2,-1,-2,-1)^T$
$M_A(3,2)$	$(-4,2,-4,2)^T$		$(1,-1,1,-1)^T$
	$(4,-2,4,-2)^T$		$(-1,1,-1,1)^T$
$M_A(3,0)$	$(-6,-6,-6,-6)^T$	$M_C(0,-2)$	$(1,2,1,2)^T$
	$(-6,0,-6,0)^T$		$(2,1,2,1)^T$
	$(0,-6,0,-6)^T$		$(1,-1,1,-1)^T$
			$(-1,1,-1,1)^T$

Table 1: Radix bases for which there does not exist any digit set constituting an Eisenstein GNS. The γ values represents the cosets for which the elements produce loops in the system.

interchange the blocks and let us denote the result by M^*, i.e. $M^* = \begin{pmatrix} M_2 & 0 \\ 0 & M_1 \end{pmatrix}$. Then the following can easily be proved:

- $M_B(a,a) = -M_B^*(a,a)$,
- $M_B(a,2a) = -M_B^*(a,2a)$,
- $M_B(a,-a) = -M_B^*(a,-a)$.

Table 1 contains the only B-type radix system which can not be simultaneous Eisenstein number system for any digit set D.

3.3. Type-C Cases

Figure 5 (i)–(l) and Figures 6 − 7 show the cases when the digit sets with the appropriate operators form simultaneous Eisenstein number systems. Again, Table 1 shows the only C-type radix system which can not be simultaneous Eisenstein number systems for any digit set D. We can observe that $M_B(2,1)$ is $\mathbb{Z}$-similar to $M_C(0,-2)$ which explains their similar behaviour.

4. Summary

In this paper we proved the following

Theorem 4.1. *For every Eisenstein integer* $\eta_1 = a_1 + b_1\omega$ *except* $-1, 0, 1$, ω, $1+\omega$, $-\omega$, $-1-\omega$, *and* $1-\omega$ *there is some Eisenstein integer* $\eta_2 = a_2 + b_2\omega$ *such that the system* (η_1, η_2, D) *is a simultaneous number system for some digit set* D.

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