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# FIX-POINT FREE AFFINE TRANSFORMATIONS HAVING INVARIANT LINES

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*Dedicated to Professors Zoltán Daróczy and Imre Kátai on their 75th anniversary*

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**Abstract.** We extend a result of E. Kasparek characterizing continuous affine transformations without fixed points $f : X \to X$ having invariant straight line to the case when $X$ is an arbitrary real topological linear space.

In this note $X$ always denotes a real topological linear space and $f : X \to X$ a continuous affine transformation having no fixed point. In our considerations an important role is played by transformation $g : X \to X$ given by the formula

$$g(x) = f(x) - f(0) - x,\ x \in X. \tag{1}$$

Note that $g$ is a linear function transforming $X$ into itself. We characterize transformations $f$ in the class of functions having an invariant straight line. Our main result extends an analogous theorem obtained by Erwin Kasparek [1] who has proved it in the case $X = \mathbb{R}^n$. As usual for each nonnegative integer $n$ by $f^n$ we mean the $n-th$ iterate of $f$, i.e., $f^0(x) = x,\ f^{n+1}(x) = f(f^n(x))$. In a similar way, in the case if $f$ is invertible we may define the $n-th$ iterates for arbitrary integer $n$. Moreover symbol $\mathbb{Z}$ stands for the set of all integers. We start with some basic remarks.

**Remark 1.** *If $l \subset X$ is a straight line and $f(l) \subset l$ then $f(l) = l$.*

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**Proof.** Assume that $f(l)$ has exactly one point $z \in l$. In particular, $f(z) = z$, which is not the case. Therefore $f(l)$ contain at least two different point $u$ and $v$. Let $u = f(x),\ v = f(y), x \neq y, x, y \in l$. Now, affinity of $f$ implies that

$$\lambda u + (1-\lambda)v = f(\lambda x + (1-\lambda)y) \in l,$$

which together with our assumption $f(l) \subset l$ proves that $f(l) = l$. ■

**Remark 2.** *If $f$ has an invariant straight line $l$ then the restriction $f$ to $l$, i.e., function $f|_l$ is invertible.*

**Proof.** If $x, y \in l,\ x \neq y$ and $f(x) = f(y) = z$ then $z \in l$ and taking $\lambda \in \mathbb{R}$ such that $z = \lambda x + (1-\lambda)y$ we get

$$f(z) = f(\lambda x + (1-\lambda)y) = \lambda f(x) + (1-\lambda)f(y) = z,$$

a contradiction. ■

In the reminder we use the following lemma.

**Lemma 3.** *If $l$ is an invariant straight line of $f$ then $f|_l$ is a translation, i.e., there exists a $v \in X \setminus \{0\}$ such that for each $x \in l$ we have*

$$f(x) = x + v. \tag{2}$$

**Proof.** For every $x \in l$ the points $f(x),\ f^2(x)$ also belong to $l$. Thus there exists a function $\varphi : l \to \mathbb{R} \setminus \{0\}$ such that

$$\varphi(x)[f(x) - x] = f^2(x) - f(x), \qquad x \in l. \tag{3}$$

We shall show that for each $x \in l$ we have $\varphi(x) = 1$. Firstly, we observe that $\varphi(x) \neq -1$. If fact, the condition $\varphi(x) = -1$ implies that $f^2(x) = x$ and consequently,

$$f\left(\frac{x + f(x)}{2}\right) = \frac{f(x) + f^2(x)}{2} = \frac{f(x) + x}{2},$$

which means that $f$ has a fixed-point, a contradition. Let's rewrite (3) to the following form

$$f(x) = \frac{\varphi(x)}{1+\varphi(x)}x + \frac{1}{1+\varphi(x)}f^2(x).$$

It follows from affinity of $f$ that

$$f^2(x) = \frac{\varphi(x)}{1+\varphi(x)}f(x) + \frac{1}{1+\varphi(x)}f^3(x),$$

which is equivalent to the following condition

$$\varphi(x)[f^2(x) - f(x)] = f^3(x) - f^2(x).$$

Hence and by (3) we infer that

$$\varphi(x) = \varphi(f(x)),$$

and, consequently,

$$\varphi(f^n(x)) = \varphi(x), \tag{4}$$

for all integer $n$. By virtue of (3) and (4) we obtain

$$\varphi(x)^{k+1}(f(x) - x) = f^{k+2}(x) - f^{k+1}(x), \qquad k \in \mathbb{Z},$$

whence

$$\sum_{k=0}^{n} \varphi(x)^{k+1}(f(x) - x) = f^{n+2}(x) - f(x), \qquad n \in \mathbb{Z}. \tag{5}$$

Assume now that $\varphi(x) \neq 1$. Then (5) has a form

$$\varphi(x)\frac{1 - \varphi(x)^{n+1}}{1 - \varphi(x)}(f(x) - x) = f^{n+2}(x) - f(x). \tag{6}$$

If $|\varphi(x)| < 1$, then we tend with $n$ to infinity, and if $|\varphi(x)| > 1$, we tend with $n$ to minus infinity. In both cases the sequence $f^n(x)$ is convergent. It is easy to see that its limit point has to be a fixed-point of $f$. This shows that

$$\varphi(x) = 1, \qquad x \in l.$$

According to (3) we get

$$f(x) - x = f^{k+1}(x) - f^k(x), \qquad x \in l,\ k \in \mathbb{Z}. \tag{7}$$

Let us fix $x, y \in l$. Chose an integer $k$ and $\lambda \in \mathbb{R}$ such that

$$y = \lambda f^k(x) + (1 - \lambda) f^{k+1}(x).$$

Then

$$f(y) = \lambda f^{k+1}(x) + (1 - \lambda) f^{k+2}(x),$$

and using also (7)

$$\begin{aligned} f(y) - y &= \lambda(f^{k+1}(x) - f^k(x)) + (1 - \lambda)(f^{k+2}(x) - f^{k+1}(x)) \\ &= \lambda(f(x) - x) + (1 - \lambda)(f(x) - x) = f(x) - x. \end{aligned}$$

Setting $v := f(y) - y$ we obtain $v \neq 0$ and

$$f(x) = x + v, \qquad x \in l.$$

This ends the proof of Lemma 3. ■

**Lemma 4.** *If $f$ has an invariant straight line and $g$ is defined by* (1) *then $f(0) = u + v$, where $u \in Im\ g$, $v \in ker\ g \setminus \{0\}$.*

**Proof.** Let $l$ be an invariant straight line of $f$. On account of Lemma 3

$$f(x) = x + v, \qquad x \in l,$$

where $v$ is a fixed nonzero vector of $X$. For each $x \in l$ we have $x + v \in l$. Therefore $f(x) = x + v$ and $f(x + v) = x + v + v$, whence $f(x + v) - f(0) - (f(x) - f(0)) = v,\ x \in l$. By linearity of $f - f(0)$ on $X$ we get

$$f(v) - f(0) = v.$$

Finally, $g(v) = 0$, which means that $v \in ker\ g$. Setting $u := f(0) - v$ and taking an $x \in l$ we get

$$g(-x) = -g(x) = -f(x) + x + f(0) = -v + f(0) = u,$$

whence $u \in Im\ g$, and the proof of Lemma 4 is complete. ■

**Lemma 5.** *If $f(0) = u + v$, where $u \in Im\ g$ and $v \in ker\ g \setminus \{0\}$ then $f$ has an invariant straight line.*

**Proof.** By our assumptions

$$f(v) - v - f(0) = 0 \tag{8}$$

and $-u \in Im\ g$. Let $x_1 \in X$ be such that

$$f(x_1) - x_1 - f(0) = -u. \tag{9}$$

Let us put

$$H := \{x \in X; f(x) = x + v\}.$$

According to (9) we get

$$f(x_1) = x_1 + f(0) - u = x_1 + v,$$

whence $x_1 \in H$. Moreover, by the linearity of $f - f(0)$ and (8) we get

$$\begin{aligned} f(x_1 + v) &= f(x_1 + v) - f(0) + f(0) = f(x_1) - f(0) + f(v) = \\ &= x_1 + v - f(0) + v + f(0) = x_1 + v + v, \end{aligned}$$

whence $x_1 + v \in H$. Therefore $H$ contains at least two different points. Let $l$ be the straight line generated by this points. For an arbitrary $w \in l$ there exists a $\lambda \in \mathbb{R}$ such that $w = \lambda x_1 + (1-\lambda)(x_1 + v)$. Then

$$\begin{aligned} f(w) &= f(\lambda x_1 + (1-\lambda)(x_1+v)) = \lambda f(x_1) + (1-\lambda) f(x_1+v) = \\ &= \lambda(x_1+v) + (1-\lambda)(x_1+2v) = x_1 + (2-\lambda)v = \\ &= (\lambda-1)x_1 + (2-\lambda)(x_1+v). \end{aligned}$$

This means that $f(l) \subset l$ and now our assertion follows from Remark 1. ∎

From Lemmas 4 and 5 the following theorem easy follows.

**Theorem 6.** *Transformation $f$ has an invariant straight line if and only if there exist a $u \in Im\ g$ and $v \in ker\ g \setminus \{0\}$ such that $f(0) = u + v$.*

**Corollary 7.** *Transformation $f$ has an invariant straight line if and only if $g(f(0)) \in Im\ g^2$.*

**Proof.** If $f$ has an invariant straight line then on account of Lemma 2 $f(0) = u + v$, where $u \in Im\ g$ and $v \in ker\ g \setminus \{0\}$. Therefore $g(f(0)) = g(u) \in Im\ g^2$. On the other hand, if $g(f(0)) \in Im\ g^2$ then there exists a $w \in X$ such that $g(f(0)) = g(g(w))$. Therefore $g(f(0) - g(w)) = 0$, and hence $f(0) - g(w) = v$, where $v \in ker\ g$. To end the proof it is enough to show that $v \neq 0$. Suppose $v = 0$. Then $f(0) = g(w)$ and according to (1)

$$-g(w) = g(-w) = f(-w) + w - f(0),$$

or equivalently,

$$f(-w) = -w,$$

a contradiction. ∎

**Corollary 8.** *If $g$ transforms $X$ onto $X$ and $g^2 = g$ then $f$ has an invariant straight line.*

**Proof.** It follows from our assumptions that

$$X = ker\ g \oplus Im\ g.$$

Thus $f(0) = v + u$, where $v \in ker\ g$ and $u \in Im\ g$. Moreover $v \neq 0$ because otherwise $f$ would have a fixed point. Now, it is enough to apply Theorem 6. ∎

## References

[1] **Kasparek, E.,** The invariant straight lines of an affine transformation in $\mathbb{R}^n$ without fixed points, *Ann. Math. Sil.*, **24** (2010), 35–37.

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