HOW LARGE ARE THE DEVIATIONS BETWEEN A DISTRIBUTION FUNCTION AND A STABLE LAW

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Dedicated to Professors Zoltán Daróczy and Imre Kátai on their 75th anniversary

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Abstract. We prove a nonuniform bound for the deviation between a distribution function and a nondegenerate stable law expressed in terms of the Lévy distance.

1. Introduction

Let F and G be two distribution functions. Then the Lévy distance $\mathcal{L}(F,G)$ between F and G is defined as follows:

$$\mathcal{L}(F,G) = \inf \mathbb{H}, \tag{1.1}$$

where $\mathbb{H} = \{h \in [0,1] : G(x-h) - h \le F(x) \le G(x+h) + h \text{ for all } x \in \mathbb{R}\}$.

The Lévy distance in the space of distribution functions is much less popular in probability theory than the uniform distance $\Delta(F,G)$ defined by

$$\Delta(F,G) = \sup_{x\in\mathbb{R}} |F(x) - G(x)|. \tag{1.2}$$

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The advantage of the Lévy distance appears in considering the weak convergence $F_n \xrightarrow{\text{w}} G$, $n \to \infty$, which is equivalent to $\mathcal{L}(F_n, G) \to 0$, $n \to \infty$ (see, for example, [4]). If G is continuous, then the weak convergence $F_n \xrightarrow{\text{w}} G$, $n \to \infty$, is also equivalent to $\Delta(F_n, G) \to 0$, $n \to \infty$, however the latter property may fail if G has discontinuities. We also recall that, in general,

$$\mathcal{L}(F, G) \le \Delta(F, G). \tag{1.3}$$

Having weak convergence in mind, we compare the deviation $|F(x) - G_\alpha(x)|$ between an arbitrary distribution function F and a nondegenerate stable law G_α of index α. The bound we obtain in Section 3 is nonuniform in x and is expressed in terms of the Lévy distance $\mathcal{L}(F, G)$. The case of a Gaussian distribution has earlier been considered in [6], [7], [8].

2. Deviation between a distribution function and a normal law

In this section, we discuss the case of $G_\alpha = \Phi$, where Φ is the standard $\mathcal{N}(0, 1)$ normal law.

Bounds for $|F(x) - \Phi(x)|$ expressed in terms of the uniform distance have been studied in many papers. The most popular case is when F corresponds to the sum of independent random variables. For the origin of this topic we refer to the paper by Esseen [3].

Kolodyazhnyi [9] extended the results of [3] by proving the following theorem.

Theorem 2.1. *Let F be an arbitrary distribution function and set $\Delta = \Delta(F, \Phi)$. Let $p > 0$ and assume that F has a finite moment of order p. Denote*

$$\lambda_p = \left| \int_{-\infty}^{\infty} |x|^p \, dF(x) - \int_{-\infty}^{\infty} |x|^p \, d\Phi(x) \right|. \tag{2.1}$$

If

$$0 < \Delta \le \frac{1}{\sqrt{e}}, \tag{2.2}$$

then there exists a universal constant c, depending only on p, such that

$$|F(x) - \Phi(x)| \le \frac{\lambda_p + c\Delta(\ln \frac{1}{\Delta})^{p/2}}{1 + |x|^p} \tag{2.3}$$

for all $x \in \mathbb{R}$.

A similar result has been obtained in [6] in terms of the Lévy distance $\mathcal{L}$ instead of the uniform distance Δ.

Theorem 2.2. *Let F be an arbitrary distribution function and set $L = \mathcal{L}(F, \Phi)$. Let $p > 0$ and assume that F has a finite moment of order p. If*

$$0 < L \leq \frac{1}{\sqrt{e}}, \tag{2.4}$$

then there exists a universal constant c, depending only on p, such that

$$|F(x) - \Phi(x)| \leq \frac{\lambda_p + cL(\ln \frac{1}{L})^{p/2}}{1 + |x|^p} \tag{2.5}$$

for all $x \in \mathbb{R}$, where λ_p is defined in (2.1).

Remark 2.1. Theorems 2.1 and 2.2 look very similar; in view of (1.3) and the monotonicity of $x \mapsto x\left(\ln \frac{1}{x}\right)^{p/2}$, the term $L(\ln \frac{1}{L})^{p/2}$ in the bound of (2.5) does not exceed the term $\Delta(\ln \frac{1}{\Delta})^{p/2}$ in (2.3). The constants on the right-hand sides of (2.3) and (2.5) are different, however their precise values do not matter in many asymptotic results (see, e.g., [8] for a further discussion of the relationship between (2.5) and (2.3)).

It turns out that the restriction (2.4) is crucial to have the term $cL(\ln \frac{1}{L})^{p/2}$ on the right-hand side of (2.5). Nevertheless, a uniform upper bound in terms of the Lévy distance is still available as proved in [6].

Theorem 2.3. *Let F be an arbitrary distribution function and set $L = \mathcal{L}(F, \Phi)$. Let $p > 0$ and assume that F has a finite moment of order p. Then there exists a universal function g, defined on $[0, 1)$, depending only on p, and such that*

$$\lim_{s \downarrow 0} g(s) = 0$$

and

$$|F(x) - \Phi(x)| \leq \frac{\lambda_p + g(L)}{1 + |x|^p} \tag{2.6}$$

for all $x \in \mathbb{R}$, where λ_p is defined in (2.1).

Several applications of Theorems 2.2 and 2.3 to prove limit theorems in probability theory, including the so-called global version of the central limit theorem and complete convergence, can be found in [7] and [8].

3. Deviation between a distribution function and a stable law

Let G_α be a nondegenerate stable law with index α. Several results are known concerning the rate of convergence of normalized sums of independent, identically distributed random variables to G_α. Most of them use the so-called pseudo-moments as a measure of divergence (see, for example, [2]).

Below is a generalization of Theorem 2.2 for an arbitrary stable law G_α. The right-hand side is expressed in terms of the Lévy distance and the difference of moments.

Theorem 3.1. *Let G_α be a nondegenerate stable law with index α. Let F be an arbitrary distribution function and set $L = \mathcal{L}(F, G_\alpha)$. Assume that $0 < p <$ $< \alpha < 2$, and that F has a finite moment of order p. Then there is a universal constant $c > 0$, depending only on p and α, such that*

(3.1) $$|F(x) - G_\alpha(x)| \le \frac{\lambda_p + cL^{1-p/\alpha}}{1 + |x|^p}$$

for all $x \in \mathbb{R}$, where λ_p is defined in (2.1)*.*

3.1. A global limit theorem

Let $\{F_n\}$ be a sequence of distribution functions and assume $r > 0$. According to Agnew [1], the *r-global limit theorem* holds for the sequence $\{F_n\}$ if

(3.2) $$\int_{-\infty}^{\infty} |F_n(x) - G(x)|^r \, dx \to 0, \qquad n \to \infty,$$

for some distribution function G.

Agnew [1] treated the case of $G = \Phi$ in detail. Some extensions have been given in [7] and [8]. Now we are able to prove an extension to the case of stable limit laws.

Theorem 3.2. *Let G_α be a stable law of index α. Let $\{F_n\}$ be a sequence of distribution functions such that*

(3.3) $$F_n \xrightarrow{w} G_\alpha, \qquad n \to \infty.$$

Let $p < \alpha$ and assume that

(3.4) $$\sup_{n \ge 1} \int_{-\infty}^{\infty} |x|^p \, dF_n(x) < \infty.$$

Then (3.2) *holds for all $r > 1/p$ with $G = G_\alpha$.*

Moreover, we can extend the result to the boundary case of $r = 1/p$.

Theorem 3.3. *Let G_α be a stable law of index α. Let $\{F_n\}$ be a sequence of distribution functions such that (3.3) holds. Let $p < \alpha$ and assume that (3.4) holds. Then*

$$\int_{-\infty}^{\infty} \frac{|F_n(x) - G_\alpha(x)|^r}{(\log(1+|x|))^{1+\delta}}\, dx \to 0, \qquad n \to \infty,$$

for all $\delta > 0$.

3.2. A weighted global limit theorem

Using the bound of (3.1) we can prove a bit more than (3.2).

Theorem 3.4. *Let G_α be a stable law of index α. Let $\{F_n\}$ be a sequence of distribution functions such that (3.3) and (3.4) hold with some $p < \alpha$. Then*

$$\int_{-\infty}^{\infty} |x|^\delta \cdot |F_n(x) - G_\alpha(x)|^r \, dx \to 0, \qquad n \to \infty,$$

for all $r > 1/p$ and $\delta < rp - 1$.

4. Proof of Theorem 3.1

We follow the lines of the proof in [6]. Without loss of generality we assume that $0 < L < 1$. Denote by $\mathcal{C}(\mathcal{F})$ the set of continuity points of F. For all $a > 0$ such that $\pm a \in \mathcal{C}(\mathcal{F})$ we have

$$\begin{aligned}
&\int_{(-a,a)} |x|^p \, dF(x) = \\
&= \int_{(-a,a)} |x|^p d[F(x) - G_\alpha(x)] + \int_{(-a,a)} |x|^p dG_\alpha(x) = \\
&= a^p[F(a) - G_\alpha(a)] - a^p[F(-a) - G_\alpha(-a)] - \\
&\quad - p \int_{(0,a)} x^{p-1}[F(x) - G_\alpha(x)]\, dx + \\
&\quad + p \int_{(-a,0)} |x|^{p-1}[F(x) - G_\alpha(x)]\, dx + \int_{(-a,a)} |x|^p \, dG_\alpha(x).
\end{aligned} \tag{4.1}$$

For all $h \in \mathbb{H}(F, G_\alpha)$ and all $x \in \mathbb{R}$ it holds that

$$F(x) - G_\alpha(x) = F(x) - G_\alpha(x-h) + h - h + G_\alpha(x-h) - G_\alpha(x) \geq \\ \geq -h - [G_\alpha(x) - G_\alpha(x-h)].$$

By the mean-value theorem and the boundedness of the density of G_α (see [10]), we conclude that there exists a constant $d > 0$ such that

$$G_\alpha(x) - G_\alpha(x-h) \leq dh,$$

whence

$$F(x) - G_\alpha(x) \geq -h\,(1+d)\,, \qquad h \in \mathbb{H}(F, G_\alpha), \tag{4.2}$$

for all $x \in \mathbb{R}$. In particular,

$$F(a) - G_\alpha(a) \geq -h\,(1+d)\,, \qquad h \in \mathbb{H}(F, G_\alpha). \tag{4.3}$$

Similarly,

$$F(x) - G_\alpha(x) = F(x) - G_\alpha(x+h) - h + h + G_\alpha(x+h) - G_\alpha(x) \leq \\ \leq h + [G_\alpha(x+h) - G_\alpha(x)]$$

and thus

$$F(x) - G_\alpha(x) \leq h\,(1+d)\,, \qquad h \in \mathbb{H}(F, G_\alpha), \tag{4.4}$$

for all $x \in \mathbb{R}$. In particular,

$$F(-a) - G_\alpha(-a) \leq h\,(1+d)\,, \qquad h \in \mathbb{H}(F, G_\alpha). \tag{4.5}$$

Applying (4.4) we obtain, for every $h \in \mathbb{H}(F, G_\alpha)$, that

$$\int\limits_{(0,a)} |x|^{p-1}[F(x) - G_\alpha(x)]\,dx \leq \int\limits_{(0,a)} |x|^{p-1} h\,(1+d)\ dx = \\ = \frac{1}{p} a^p h\,(1+d)\,.$$

Similarly, from (4.2), we derive

$$\int\limits_{(-a,0)} |x|^{p-1}[F(x) - G_\alpha(x)]\,dx \geq -\frac{1}{p} a^p h\,(1+d)\,.$$

Combining the latter estimates with (4.3), (4.5) and inserting them into (4.1), we obtain

$$\int\limits_{(-a,a)} |x|^p dF(x) \geq -hB_p + \int\limits_{(-a,a)} |x|^p dG_\alpha(x),$$

where

$$B_p = 4\,(1+d)\,a^p. \tag{4.6}$$

Recalling the definition of λ_p, we have

$$\begin{aligned}\lambda_p \geq \int\limits_{|x|<a} |x|^p\, dF(x) - \int\limits_{|x|<a} |x|^p\, G_\alpha(x) + \\ + \int\limits_{|x|\geq a} |x|^p\, dF(x) - \int\limits_{|x|\geq a} |x|^p\, dG_\alpha(x) \geq \\ \geq -hB_p + \int\limits_{|x|\geq a} |x|^p\, dF(x) - \int\limits_{x\geq a} |x|^p\, dG_\alpha(x),\end{aligned}$$

whence

$$\int\limits_{|x|\geq a} |x|^p dF(x) \leq \lambda_p + hB_p + \int\limits_{|x|\geq a} |x|^p dG_\alpha(x).$$

Further, if $x \geq a$, then

$$\int\limits_{|y|\geq a} |y|^p dF(y) \geq \int\limits_{y\geq x} y^p\, dF(y) \geq x^p(1-F(x)) \geq x^p[G_\alpha(x) - F(x)].$$

Therefore, for every $h \in \mathbb{H}(F, G_\alpha)$,

$$x^p[G_\alpha(x) - F(x)] \leq \lambda_p + hB_p + \int\limits_{|y|\geq a} |y|^p dG_\alpha(y). \tag{4.7}$$

Now,

$$\begin{aligned}F(x) - G_\alpha(x) \leq 1 - G_\alpha(x) \leq \\ \leq \int\limits_{|y|\geq x} dG_\alpha(y) \leq \frac{1}{x^p} \int\limits_{|y|\geq x} |y|^p dG_\alpha(y), \qquad x \in \mathbb{R},\end{aligned} \tag{4.8}$$

whence, for all $x \geq a$,

$$x^p[F(x) - G_\alpha(x)] \leq \int\limits_{|y|\geq a} |y|^p dG_\alpha(y).$$

Combining this bound with (4.7) we get, for $x \geq a$ and $h \in \mathbb{H}(F, G_\alpha)$, that

$$|x|^p|F(x) - G_\alpha(x)| \leq \lambda_p + hB_p + \int\limits_{|y|\geq a} |y|^p dG_\alpha(y). \tag{4.9}$$

A similar bound holds for $x \le -a$. Finally, in view of (4.2) and (4.4), it also holds for $|x| < a$. Therefore (4.9) holds for all $x \in \mathbb{R}$. The same reasoning applies for $p = 0$. Note that $\lambda_0 = 0$. Thus

$$(4.10)\qquad \begin{aligned}&(1+|x|^p)|F(x) - G_\alpha(x)| \le \\ &\quad \le \lambda_p + hB_p + hB_0 + \int\limits_{|y|\ge a} |y|^p \, dG_\alpha(y) + \int\limits_{|y|\ge a} dG_\alpha(y).\end{aligned}$$

The right-hand side of this estimate is a continuous function of a (see (4.6)), therefore one can remove the assumption that $\pm a \in \mathcal{C}(\mathcal{F})$. Thus (4.10) holds for all $a > 0$. Moreover, on taking the infimum with respect to $h \in \mathbb{H}(F, G_\alpha)$, we have, for all $x \in \mathbb{R}$ and all $a > 0$, that

$$(4.11)\qquad \begin{aligned}&(1+|x|^p)|F(x) - G_\alpha(x)| \le \\ &\quad \le \lambda_p + LB_p + LB_0 + \int\limits_{|y|\ge a} |y|^p \, dG_\alpha(y) + \int\limits_{|y|\ge a} dG_\alpha(y),\end{aligned}$$

where L is the Lévy distance between F and G_α. Since there is a constant $\kappa > 0$ such that

$$1 - G_\alpha(x) + G_\alpha(-x) \sim \frac{\kappa}{x^\alpha}, \qquad x \to \infty,$$

we get

$$\int\limits_{|y|\ge a} |y|^p \, dG_\alpha(y) \asymp \frac{1}{a^{p-\alpha}}, \qquad \int\limits_{|y|\ge a} dG_\alpha(y) \asymp \frac{1}{a^\alpha}.$$

Substituting $a = L^{-1/\alpha}$ in (4.11) we see that its right-hand side is

$$\asymp \lambda_p + L^{1-p/\alpha} + L + L^{(\alpha-p)/\alpha} + L \asymp \lambda_p + L^{1-p/\alpha} + L \asymp \lambda_p + L^{1-p/\alpha},$$

where we also made use of $0 < L < 1$. ■

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