Annales Univ. Sci. Budapest., Sect. Comp. 41 (2013) 5–13

A NOTE ON FUNCTIONAL INDEPENDENCE OF SOME ZETA-FUNCTIONS

Roma Kačinskaitė (Šiauliai, Lithuania)

Dedicated to Professors Zoltán Daróczy and Imre Kátai on their 75th birthday

Communicated by Bui Minh Phong
(Received March 28, 2013; accepted April 07, 2013)

Abstract. In the paper, we prove the joint denseness and the functional independence for the Dirichlet L-function $L(s,\chi)$ and periodic Hurwitz zeta-function $\zeta(s,\alpha;\mathfrak{b})$ with transcendental parameter α.

1. Introduction

In 1887, O. Hölder proved [3] the algebraic-differential independence for the gamma-function $\Gamma(s)$, $s=\sigma+it$, i.e. that there exists no polynomial P not identically zero such that

$$P\left(s,\Gamma(s),\Gamma'(s),\dots,\Gamma^{(n-1)}(s)\right)=0$$

for all $s\in\mathbb{C}$. In 1990, D. Hilbert noted that the Riemann zeta-function $\zeta(s)$ does not satisfy any algebraic differential equation also. Also he proposed [2] the more general case, i.e. that the function

$$\zeta(s,x)=\sum_{m=1}^{\infty}\frac{x^m}{m^s}$$

Key words and phrases: Density, Dirichlet L-function, functional independence, Hurwitz zeta-function, periodic coefficients, universality.
2010 Mathematics Subject Classification: 11M41, 11M35.
The author is partially supported by the European Commission within the 7th Framework Programme 2011–2014 project INTEGER (INstitutional Transformation for Effecting Gender Equality in Research), Grant Agreement No. 266638.

does not satisfy any algebraic-differential equation. This problem was solved by A. Ostrowski [9].

In 1973, S.M. Voronin obtained [14] the functional independence of the Riemann zeta-function (recall that the functions $f_1(s), \dots, f_m(s)$ are functionally independent if, for any continuous functions $F_0, F_1, \dots, F_n : \mathbb{C}^m \to \mathbb{C}$, not all identically vanishing, the function $\sum_{j=0}^{n} s^j F_j\left(f_1(s), \dots, f_m(s)\right)$ is non-zero for some $s \in \mathbb{C}$). He proved that $\zeta(s)$ does not satisfy any differential equation

$$\sum_{j=0}^{n} s^j F_j\left(\zeta(s), \zeta'(s), \dots, \zeta^{(N-1)}(s)\right) \equiv 0$$

with continuous functions F_j, $j = 1, \dots, n$, not all identically zero.

The Voronin result was generalized by R. Garunkštis, A. Laurinčikas, K. Matsumoto, H. Mishou, A.G. Postnikov, A. Reich, J. Steuding and many other mathematicians (see, for example, [1, 6, 7, 8, 10, 11, 12, 13]).

2. Main results

In this paper, we consider the functional independence for the Dirichlet L-function $L(s,\chi)$ and the periodic Hurwitz zeta-function $\zeta(s,\alpha;\mathfrak{b})$.

We recall that the Dirichlet L-function $L(s,\chi)$ associated with a character χ modulo q, in the half-plane $\sigma > 1$, is given by

$$L(s,\chi) = \sum_{m=1}^{\infty} \frac{\chi(m)}{m^s} = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1},$$

and is analytically continuable to the complex plane (at $s = 1$ it is regular if χ is non-principal character).

The periodic Hurwitz zeta-function $\zeta(s,\alpha;\mathfrak{b})$ with a parameter α, $0 < \alpha \leq$ ≤ 1, for $\sigma > 1$, is defined by the series

$$\zeta(s,\alpha;\mathfrak{b}) = \sum_{m=0}^{\infty} \frac{b_m}{(m+\alpha)^s},$$

and analytically continuable to whole $\mathbb{C}$, except for a possible pole at $s = 1$. Here b_m are complex numbers belonging to a periodic with a minimal period k sequence $\mathfrak{b} = \{b_m : m \in \mathbb{N} \cup \{0\}\}$.

Now we state two results for the functions $L(s,\chi)$ and $\zeta(s,\alpha;\mathfrak{b})$: the joint denseness and the joint functional independence.

Theorem 2.1. *Let α be transcendental, $0 < \alpha < 1$. For $\frac{1}{2} < \sigma_0 < 1$, positive integer N and $t \in \mathbb{R}$, the set*

$$\Big\{L(\sigma_0 + it, \chi), L'(\sigma_0 + it, \chi), \ldots, L^{(N-1)}(\sigma_0 + it, \chi), \\ \zeta(\sigma_0 + it, \alpha; \mathfrak{b}), \zeta'(\sigma_0 + it, \alpha; \mathfrak{b}), \ldots, \zeta^{(N-1)}(\sigma_0 + it, \alpha; \mathfrak{b})\Big\}$$

is dense in $\mathbb{C}^{2N}$.

Theorem 2.2. *Suppose that α is transcendental, $0 < \alpha < 1$. Let $F_j : \mathbb{C}^{2N} \to \mathbb{C}$ be a continuous function for each $j = 0, 1, \ldots, n$, and*

$$\sum_{j=0}^{n} s^j F_j \Big(L(s, \chi), L'(s, \chi), \ldots, L^{(N-1)}(s, \chi), \\ \zeta(s, \alpha; \mathfrak{b}), \zeta'(s, \alpha; \mathfrak{b}), \ldots, \zeta^{(N-1)}(s, \alpha; \mathfrak{b})\Big)$$

is identically zero. Then, for $j = 0, 1, \ldots, n$, $F_j \equiv 0$.

3. Auxiliary results

The proof of Theorem 2.1 and 2.2 is based on joint universality in the Voronin sense for the functions $L(s, \chi)$ and $\zeta(s, \alpha; \mathfrak{b})$.

Denote by meas{A} the Lebesgue measure of a measurable set $A \subset \mathbb{R}$, and let $T > 0$.

Theorem 3.1. *Let α be transcendental. Suppose that K_1 and K_2 are compact subsets of the strip $D = \{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\}$ with connected complements. Let $f_j(s)$ be a continuous function on K_j which is analytic in the interior of K_j, $j = 1, 2$. Suppose that $f_1(s)$ is non-vanishing on K_1. Then, for every positive ε, it holds that*

$$\liminf_{T\to\infty} \frac{1}{T} \text{meas} \Big\{\tau \in [0, T] : \sup_{s\in K_1} |L(s + i\tau, \chi) - f_1(s)| < \varepsilon, \\ \sup_{s\in K_2} |\zeta(s + i\tau, \alpha; \mathfrak{b}) - f_2| < \varepsilon\Big\} > 0.$$

The theorem is a special case of Theorem 1 from [4]. Joint discrete universality for the Dirichlet L-function and the periodic Hurwitz zeta-function by the author was proved in [5].

Note that in the proof of Theorem 3.1 we use joint limit theorem in the sense of the weakly convergent probability measures for the functions $L(s,\chi)$ and $\zeta(s,\alpha;\mathfrak{b})$ in the space of analytic functions.

By $\mathcal{B}(S)$ denote the class of Borel sets of the space S, and let $H(D)$ be the space of analytic functions on D equipped with the topology of uniform convergence on compacta. Let $H^2(D) = H(D) \times H(D)$. Define two tori

$$\Omega_1 = \prod_p \gamma_p \quad \text{and} \quad \Omega_2 = \prod_{m=0}^{\infty} \gamma_m,$$

where $\gamma_p = \gamma$ for all primes p, and $\gamma_m = \gamma$ for all $m \in \mathbb{N} \cup \{0\}$, and $\gamma = \{s \in \mathbb{C} : |s| = 1\}$. Then Ω_j, $j = 1,2$, is a compact Abelian topological group, and we have the probability space $(\Omega_j, \mathcal{B}(\Omega_j), m_{Hj})$ with Haar measure m_{Hj} defined on $(\Omega_j, \mathcal{B}(\Omega_j))$. Since the product $\Omega = \Omega_1 \times \Omega_2$ is a compact Abelian group also, the probability Haar measure m_H on $(\Omega, \mathcal{B}(\Omega))$ can be defined as a product of measures m_{H1} and m_{H2}. Denote by $\omega_1(p)$ the projection of $\omega_1 \in \Omega_1$ to the coordinate space γ_p, and by $\omega_2(m)$ the projection of $\omega_2 \in \Omega_2$ to the coordinate space γ_m, and, for every positive m, define

$$\omega_1(m) = \prod_{p^d \| m} \omega_1^d(p),$$

where $p^d \| m$ means the $p^d | m$ but $p^{d+1} \nmid m$.

Let, for $s \in D$, $\underline{\zeta}(s,\omega)$ be the $H^2(D)$-valued random element on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$ given by

$$\underline{\zeta}(s,\omega) = \left(L(s,\chi,\omega_1), \zeta(s,\alpha,\omega_2;\mathfrak{b})\right), \quad \omega = (\omega_1,\omega_2) \in \Omega,$$

where

$$L(s,\chi,\omega_1) = \sum_{m=1}^{\infty} \frac{\chi(m)\omega_1(m)}{m^s} = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}, \quad \omega_1 \in \Omega_1,$$

and

$$\zeta(s,\alpha,\omega_2;\mathfrak{b}) = \sum_{m=0}^{\infty} \frac{b_m \omega_2(m)}{(m+\alpha)^s}, \quad \omega_2 \in \Omega_2.$$

In view of linearly independence over the field of rational numbers $\mathbb{Q}$ of the system $\{\log p : p \text{ is prime}\} \cup \{\log(m+\alpha) : m \in \mathbb{N} \cup \{0\}\}$, it is proved that the probability measure

$$\frac{1}{T}\text{meas}\left\{\tau \in [0,T] : (L(s,\chi), \zeta(s,\alpha;\mathfrak{b})) \in A\right\}, \quad A \in \mathcal{B}(H^2(D)),$$

converges weakly to the distribution of the random element $\underline{\zeta}(s,\omega)$ as $T \to \infty$.

4. Proof of Theorem 2.1

We define the mapping $h : \mathbb{R} \to \mathbb{C}^{2N}$ by the formula

$$\begin{aligned} h(t) &= \Big(L(\sigma+it,\chi), L'(\sigma+it,\chi), \dots, L^{(N-1)}(\sigma+it,\chi), \\ &\qquad \zeta(\sigma+it,\alpha;\mathfrak{b}), \zeta'(\sigma+it,\alpha;\mathfrak{b}), \dots, \zeta^{(N-1)}(\sigma+it,\alpha;\mathfrak{b}) \Big), \end{aligned}$$

where $\frac{1}{2} < \sigma < 1$. Denote by $|\cdot|_{\mathbb{C}^{2N}}$ the distance in the space $\mathbb{C}^{2N}$. Let $\underline{z}$ be an arbitrary point of $\mathbb{C}^{2N}$ with

$$\underline{z} = (z_{10}, z_{11}, \dots, z_{1N-1}, z_{20}, z_{21}, \dots, z_{2N-1}).$$

For every $\varepsilon > 0$, we can find that there exists a sequence of real number $\{\tau_m\}$, $\lim_{m\to\infty} \tau_m = +\infty$, such that

$$|h(\tau_m) - \underline{z}|_{\mathbb{C}^{2N}} < \varepsilon.$$

To obtain this, it is sufficient to prove that there exists a sequence of real numbers $\{\tau_m\}$ such that, for $j = 0, 1, \dots, N-1$, hold inequalities

$$|L^{(j)}(\sigma + i\tau_m, \chi) - z_{1j}| < \frac{\varepsilon}{2N}, \tag{4.1}$$

and

$$|\zeta^{(j)}(\sigma + i\tau_m, \alpha; \mathfrak{b}) - z_{2j}| < \frac{\varepsilon}{2N}. \tag{4.2}$$

We consider the polynomial

$$P_{Nk}(s) = \sum_{m=0}^{N-1} \frac{z_{km} s^m}{m!}, \quad k = 1, 2.$$

Then we have that

$$P_{Nk}^{(j)}(s) = z_{km}, \quad j = 0, 1, \dots, N-1, \quad k = 1, 2.$$

Let σ_0, $\frac{1}{2} < \sigma_0 < 1$, be fixed. We choose a compact subset K of the strip D such that σ_0 belongs to the interior of D. Then by Theorem 3.1 there exists a sequence of real numbers $\{\tau_m\}$, $\lim_{m\to\infty} \tau_m = +\infty$, such that

$$\sup_{s\in K} |L(\sigma + i\tau_m, \chi) - P_{N1}(s - \sigma_0)| < \frac{\varepsilon\delta^N}{2^{N+1} N!}, \tag{4.3}$$

and

$$\sup_{s\in K}|\zeta(\sigma+i\tau_m,\alpha;\mathfrak{b})-P_{N2}(s-\sigma_0)|<\frac{\varepsilon\delta^N}{2^{N+1}N!}. \tag{4.4}$$

Here δ is the distance of σ_0 from the boundary of K. Applying Cauchy integral theorem, we obtain that, for $j=0,1,\dots,N-1$,

$$\begin{aligned}&|L^{(j)}(\sigma_0+i\tau_m,\chi)-z_{1j}|\\&\quad=\left|\frac{j!}{2\pi i}\int_{|s-\sigma_0|=\frac{\delta}{2}}\frac{L(s+i\tau_m,\chi)-P_{N1}(s-\sigma_0)}{(s-\sigma_0)^{j+1}}\mathrm{d}s\right|<\frac{\varepsilon}{2N},\end{aligned}$$

and

$$\begin{aligned}&|\zeta^{(j)}(\sigma_0+i\tau_m,\alpha;\mathfrak{b})-z_{2j}|\\&\quad=\left|\frac{j!}{2\pi i}\int_{|s-\sigma_0|=\frac{\delta}{2}}\frac{\zeta(s+i\tau_m,\alpha;\mathfrak{b})-P_{N2}(s-\sigma_0)}{(s-\sigma_0)^{j+1}}\mathrm{d}s\right|<\frac{\varepsilon}{2N}.\end{aligned}$$

Last two inequalities together with (4.3) and (4.4) yield (4.1) and (4.2), respectively. The theorem is proved.

5. Proof of Theorem 2.2

To the proof of Theorem 2.2 we need one more statement.

Lemma 5.1. *Suppose that α is transcendental. Let $F:\mathbb{C}^{2N}\to\mathbb{C}$ be a continuous function, and let*

$$\begin{aligned}F\Big(&L(s,\chi),L'(s,\chi),\dots,L^{(N-1)}(s,\chi),\\&\zeta(s,\alpha;\mathfrak{b}),\zeta'(s,\alpha;\mathfrak{b}),\dots,\zeta^{(N-1)}(s,\alpha;\mathfrak{b})\Big)\equiv 0.\end{aligned}$$

Then $F\equiv 0$.

Proof. In contrary to the assertion of lemma, let $F\not\equiv 0$. Then there exists a point $\underline{z}\in\mathbb{C}^{2N}$ such that $F(\underline{z})\neq 0$. From continuity of the function F it follows that there exists a bounded region $G\subset\mathbb{C}^{2N}$ such that $\underline{z}\in G$ and, for all $\underline{s}\in G$,

$$|F(\underline{s})|\geq c>0. \tag{5.1}$$

Let $\frac{1}{2} < \sigma < 1$. By Theorem 2.1 there exists values of t such that

$$\Big(L(\sigma+it,\chi), L'(\sigma+it,\chi), \dots, L^{(N-1)}(\sigma+it,\chi),\\ \zeta(\sigma+it,\alpha;\mathfrak{b}), \zeta'(\sigma+it,\alpha;\mathfrak{b}), \dots, \zeta^{(N-1)}(\sigma+it,\alpha;\mathfrak{b})\Big) \in G.$$

But this fact together with (5.1) contradict the assertion of the lemma. ■

Proof of Theorem 2.2. Now we suppose that $F_0 \not\equiv 0$. Then there exists a bounded region $G_0 \subset \mathbb{C}^{2N}$ such that the inequality

$$|F_0(\underline{s})| \geq c_0 > 0$$

holds for all $\underline{s} \in G_0$. Let

$$j_0 = \max_{0\leq j\leq n} \left\{ j : \sup_{\underline{s}\in G_0} |F_j(s)| \neq 0 \right\}.$$

If $j_0 = 0$ then statement of the theorem directly follows from Lemma 5.1.

Let $j_0 > 0$. Then there exists a subregion $G_0^* \subset G_0$ such that

$$\inf_{\underline{s}\in G_0^*} |F_{j_0}(\underline{s})| \geq c_1 > 0. \tag{5.2}$$

In view of Theorem 2.1, there exists a sequence of real numbers $\{\tau_m\}$, $\lim_{m\to\infty} \tau_m = +\infty$, such that

$$\Big(L(\sigma+i\tau_m,\chi), L'(\sigma+i\tau_m,\chi), \dots, L^{(N-1)}(\sigma+i\tau_m,\chi),\\ \zeta(\sigma+i\tau_m,\alpha;\mathfrak{b}), \zeta'(\sigma+i\tau_m,\alpha;\mathfrak{b}), \dots, \zeta^{(N-1)}(\sigma+i\tau_m,\alpha;\mathfrak{b})\Big) \in G_0^*$$

for $\frac{1}{2} < \sigma < 1$. Taking into account (5.2), we have

$$|\sigma+i\tau_m|^{j_0} \Big| F_{j_0}\Big(L(\sigma+i\tau_m,\chi), L'(\sigma+i\tau_m,\chi), \dots, L^{(N-1)}(\sigma+i\tau_m,\chi),\\ \zeta(\sigma+i\tau_m,\alpha;\mathfrak{b}), \zeta'(\sigma+i\tau_m,\alpha;\mathfrak{b}), \dots, \zeta^{(N-1)}(\sigma+i\tau_m,\alpha;\mathfrak{b})\Big)\Big| \to +\infty$$

as $m \to \infty$. This contradicts to the hypothesis of the theorem. Theorem 2.2 is proved. ■

6. Conclusions

The joint functional independence theorem similar to Theorem 2.2 can be obtained in more general case if we extend collection of functions namely $L(s,\chi_1), \dots, L(s,\chi_d), \zeta(s,\alpha_1;\mathfrak{b}_1), \dots, \zeta(s,\alpha_r;\mathfrak{b}_r)$.

Suppose that α_j, $j = 1, \dots, r$, be a real number, $0 < \alpha_j < 1$. Let $\mathfrak{b}_j = \{b_{mj} : m \in \mathbb{N} \cup \{0\}\}$ be a periodic sequence of complex numbers with a minimal period k_j, and suppose that $\zeta(s, \alpha_j; \mathfrak{b}_j)$ is a corresponding periodic Hurwitz zeta-function, $j = 1, \dots, r$. Denote by $k = [k_1, \dots, k_r]$ the least common multiple of the periods $k_1, \dots, k_r$, and by B denote the matrix

$$B = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1r} \\ b_{21} & b_{22} & \dots & b_{2r} \\ \dots & \dots & \dots & \dots \\ b_{k1} & b_{k2} & \dots & b_{kr} \end{pmatrix}.$$

Theorem 6.1. *Suppose that the numbers $\alpha_1, \dots, \alpha_r$ are algebraically independent over the field of rational numbers,* $\operatorname{rank}(B) = r$*, and $\chi_1, \dots, \chi_d$ are pairwise non-equivalent Dirichlet character. Let $F_j : \mathbb{C}^{N(r+d)} \to \mathbb{C}$ be continuous functions for each $j = 0, 1, \dots, n$, and the function*

$$\sum_{j=0}^{n} s^j \qquad F_j \Big(L(s,\chi_1), L'(s,\chi_1), \dots, L^{(N-1)}(s,\chi_1), \dots, \\ L(s,\chi_d), L'(s,\chi_d), \dots, L^{(N-1)}(s,\chi_d), \\ \zeta(s,\alpha_1;\mathfrak{b}_1), \zeta'(s,\alpha_1;\mathfrak{b}_1), \dots, \zeta^{(N-1)}(s,\alpha_1;\mathfrak{b}_1), \dots, \\ \zeta(s,\alpha_r;\mathfrak{b}_r), \zeta'(s,\alpha_r;\mathfrak{b}_r), \dots, \zeta^{(N-1)}(s,\alpha_r;\mathfrak{b}_r) \Big)$$

is identically zero. Then, for $j = 0, 1, \dots, n$, $F_j \equiv 0$.

References

[1] **Garunkštis, R. and A. Laurinčikas,** On one Hilbert's problem for the Lerch zeta-function, *Publ. Inst. Math., Nouv. Sr.*, **65**(79) (1999), 63–68.

[2] **Hilbert, D.,** Mathematische Probleme, *Archiv. Math. Physik*, **1** (1901), 44–63, 213–317.

[3] **Hölder, O.,** Über die Eigenschaft der Gamafunktion keiner algebraischen Differentialgleichung zu genügen, *Math. Ann.*, **28** (1887), 1–13.

[4] **Janulis, K., A. Laurinčikas, R. Macaitienė and D. Šiaučiūnas,** Joint universality of Dirichlet L-functions and periodic Hurwitz zeta-functions, *Math. Model. Anal.*, **17**(5) (2012), 673–685.

[5] **Kačinskaitė, R.,** Joint discrete universality of periodic zeta-functions, *Integral Transforms Spec. Funct.*, **22**(8) (2011), 593–601.

[6] **Kačinskaitė, R. and A. Laurinčikas,** The joint distribution of periodic zeta-functions, *Stud. Sci. Math. Hung.*, **48**(2) (2011), 257–279.

[7] **Laurinčikas, A. and K. Matsumoto,** The joint universality and the functional independence for Lerch zeta-functions, *Nagoya Math. J.*, **157** (2000), 211–227.

[8] **Mishou, H.,** The joint value-distribution of the Riemann zeta-function and Hurwitz zeta-function, *Liet Mathem. Rink.*, **47**(1) (2007), 62–80 = *Lith. Math. J.*, **47**(1) (2007), 32–47.

[9] **Ostrowski, A.,** Über Dirichletsche Reihen und algebraische Differentialgleichungen, *Math. Z.*, **8** (1920), 241–298.

[10] **Postnikov, A.G.,** On the differential independence of Dirichlet series, *Dokl. Akad. Nauk SSSR*, **66**(4) (1949), 561–564 (in Russian).

[11] **Postnikov, A.G.,** Generalization of one Hilbert's problem, *Dokl. Akad. Nauk SSSR*, **107**(4) (1956), 512–515.

[12] **Reich, A.,** Zetafunktionen und Differenzen-Differentialgleichungen, *Arch. Math.*, **38**(1) (1982), 226–235.

[13] **Steuding, J.,** *Value Distribution of L-Functions*, Springer, Berlin, 2007.

[14] **Voronin, S.M.,** On the functional independence of ζ-functions, *Sov. Math. Dokl.*, **14** (1973), 607–609 = *Dokl. Akad. Nauk SSSR*, **209** (1973), 1264–1266 (in Russian).

R. Kačinskaitė
Department of Mathematics
Faculty of Mathematics and Informatics
Šiauliai University
P. Višinskio str. 19
LT-77156 Šiauliai
Lithuania
`r.kacinskaite@fm.su.lt`