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UNIQUENESS OF SOLUTIONS OF SIMULTANEOUS DIFFERENCE EQUATIONS

Dedicated to Professor Zoltán Daróczy and Professor Imre Kátai on the occasion of theirs 75th birthdays

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Abstract. We study the uniqueness of continuous, continuous at a point as well as monotonic solutions of some simultaneous linear difference equations, in the case when individual equations usually have a lot of solutions in the considered class of functions. Also the general solution on cosets of the group generated by the set of numbers parametrizing the equations has been given.

1. Introduction

Studying some weak generalized stabilities of random variables the authors of the paper [4] came naturally to the simultaneous equations

$$\varphi(nx) = \varphi(x) + c(n)x^p, \quad n \in \mathbb{N},$$

and to the problem of determining their continuous solutions $\varphi : (0, \infty) \to \mathbb{R}$. Here we consider a more general situation of the simultaneous equations

$$\varphi(tx) = \varphi(x) + c(t)x^p, \quad t \in T, \tag{1.1}$$

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indexed with members of an arbitrary set $T \subset (0,\infty)$. We are interested mainly in continuous solutions. However, we discuss (1.1) also in other function classes.

We start with the well studied situation when T is a singleton, that is we deal with a single equation of the form

$$\varphi(tx) = \varphi(x) + cx^p \tag{1.2}$$

with a fixed $t \in (0,\infty)$. Clearly, if $t = 1$ then (1.2) has no solution φ at all in the case $c \neq 0$, and any $\varphi : (0,\infty) \to \mathbb{R}$ satisfies (1.2) when $c = 0$. So in what follows we assume that $t \in (0,\infty) \setminus \{1\}$.

If we are interested in continuous solutions $\varphi : (0,\infty) \to \mathbb{R}$ of equation (1.2) we have the following result coming immediately from a theorem of J. Kordylewski and M. Kuczma (see [5]; also [8, Thm. 2.1] or [9, Thm. 3.1.1]). It turns out that equation (1.2) has a lot of continuous solutions $\varphi : (0,\infty) \to \mathbb{R}$.

Theorem A. *Let $t \in (0,\infty) \setminus \{1\}$, $c \in \mathbb{R}$, and $p \in \mathbb{R}$. Equation* (1.2) *has a continuous solution depending on an arbitrary function: for any $x_0 \in (0,\infty)$ every continuous function φ, defined on the interval with the endpoints x_0 and tx_0, satisfying the condition*

$$\varphi(tx_0) = \varphi(x_0) + cx_0^p,$$

can be uniquely extended on $(0,\infty)$ *to a continuous solution of equation* (1.2).

A similar situation is while looking for monotonic solutions of equation (1.2) with $p = 0$, that is the equation

$$\varphi(tx) = \varphi(x) + c. \tag{1.3}$$

This can be derived from a result of J. Burek and M. Kuczma [2] (see also [8, Thm. 5.5 and Lemma 5.1] or [9, Thm. 2.3.8]; cf. [12], too).

Theorem B. *Let $t \in (0,\infty) \setminus \{1\}$ and $c \in \mathbb{R}$. Equation* (1.3) *has a monotonic solution depending on an arbitrary function: if $c(t-1) \geq 0$ [$c(t-1) \leq 0$], then every increasing [decreasing] function φ, defined on the interval with the endpoints x_0 and tx_0, satisfying the condition*

$$\varphi(tx_0) = \varphi(x_0) + c,$$

can be uniquely extended on $(0,\infty)$ *to an increasing* [*decreasing*] *solution of equation* (1.3).

Contrary to the situation described in Theorems A and B we have the uniqueness of monotonic solutions of equation (1.2) with $p \neq 0$ as well as solutions of equation (1.2) in the class of functions which are convex or concave. The following result deals with monotonic solutions and is a particular case of a theorem of M. Kuczma (see [7]; cf. [8, Thm. 5.3]; [9, Thm. 2.3.6], also [12]).

Theorem C. *Let* $t \in (0,\infty) \setminus \{1\}$, $c \in \mathbb{R}$, *and* $p \in \mathbb{R} \setminus \{0\}$. *A function* $\varphi : (0,\infty) \to \mathbb{R}$ *is a monotonic solution of equation* (1.2) *if and only if it is of the form*

$$\varphi(x) = ax^p + b \tag{1.4}$$

with $a = c/\left(t^p - 1\right)$ *and some* $b \in \mathbb{R}$.

What concerns convex and concave solutions of (1.2) we have the following result being an immediate corollary from a theorem of W. Krull [6] (see also [8, Thm. 5.11] and [9, Thm. 2.4.2]).

Theorem D. *Let* $t \in (0,\infty) \setminus \{1\}$, $c \in \mathbb{R}$, *and* $p \in \mathbb{R}$.

(i) *Assume that* $p \neq 0$. *A function* $\varphi : (0,\infty) \to \mathbb{R}$ *is a convex or concave solution of equation* (1.2) *if and only if it is of form* (1.4) *with* $a = c/(t^p - 1)$ *and some* $b \in \mathbb{R}$.

(ii) *Assume that* $p = 0$. *A function* $\varphi : (0,\infty) \to \mathbb{R}$ *is a convex or concave solution of equation* (1.2) *if and only if it is of the form*

$$\varphi(x) = a \log x + b \tag{1.5}$$

with $a = c/\log t$ *and some* $b \in \mathbb{R}$.

In the next section we formulate theorems showing that if we replace an individual equation (1.2) by simultaneous equations (1.1) with T being not a singleton, then in some cases we may expect uniqueness of solutions also in the classes of monotonic functions (when $p = 0$) and continuous functions.

2. Main results

It turns out that in the case $p \neq 0$ we can determine the general solution of simultaneous equations (1.1), defined on an arbitrary coset of the group $\langle T \rangle$ generated by T. We have the following uniqueness result.

Theorem 2.1. *Let* T *be a set of positive numbers and* $p \in \mathbb{R} \setminus \{0\}$. *If simultaneous equations* (1.1) *with some* $c : T \to \mathbb{R}$ *have a solution defined on a coset* *of the group* $\langle T\rangle$*, then there is a number* $a \in \mathbb{R}$ (*not depending on the coset*) *such that*

$$c(t) = a\,(t^p - 1) \tag{2.1}$$

for every $t \in T$*. If the function* $c : T \to \mathbb{R}$ *is given by* (2.1) *with that* a *and* $x_0 \in (0,\infty)$*, then* $\varphi : x_0\,\langle T\rangle \to \mathbb{R}$ *satisfies equations* (1.1) *if and only if it is of the form* (1.4) *with some* $b \in \mathbb{R}$*.*

We should not expect a similar uniqueness result in the case $p = 0$ as then condition (1.1), postulated on an individual coset, gives too little information. Indeed, then (1.1) applied to the coset $x_0\,\langle T\rangle$ implies

$$\varphi(tx_0) = \varphi\,(x_0) + c(t), \quad t \in T,$$

which allows only to determine φ if we know c, and vice versa.

What concerns the form of subgroups of the multiplicative group of positive numbers, the situation is strongly polarized: every such a group is either discrete, i.e. of the form $\{t^n : n \in \mathbb{Z}\}$ with some $t \in (0,\infty)$, or is a dense subset of $(0,\infty)$.

If the group $\langle T\rangle$ is dense in $(0,\infty)$ we can give the form of all continuous solutions $\varphi : (0,\infty) \to \mathbb{R}$ of simultaneous equations (1.1), also in the case $p = 0$ which is quite different from that when $p \neq 0$.

Theorem 2.2. *Let* T *be a set of positive numbers such that the group* $\langle T\rangle$ *is dense in* $(0,\infty)$ *and let* $p \in \mathbb{R}$*.*

(i) *Assume that* $p \neq 0$*. If simultaneous equations* (1.1) *with some* $c : T \to \mathbb{R}$ *have a solution defined on* $(0,\infty)$*, then there is a number* $a \in \mathbb{R}$ *such that* c *is of form* (2.1)*. If* $a \in \mathbb{R}$ *and the function* $c : T \to \mathbb{R}$ *is given by* (2.1)*, then* $\varphi : (0,\infty) \to \mathbb{R}$ *is a continuous at a point solution of equations* (1.1) *if and only if it is of form* (1.4) *with some* $b \in \mathbb{R}$*.*

(ii) *Assume that* $p = 0$*. If simultaneous equations* (1.1) *with some* $c : T \to \mathbb{R}$ *have a continuous or monotonic solution defined on* $(0,\infty)$*, then there is a number* $a \in \mathbb{R}$ *such that*

$$c(t) = a \log t \tag{2.2}$$

for every $t \in T$*. If* $a \in \mathbb{R}$ *and the function* $c : T \to \mathbb{R}$ *is given by* (2.2)*, then* $\varphi : (0,\infty) \to \mathbb{R}$ *is a continuous solution of equations* (1.1) *if and only if it is of form* (1.5) *with some* $b \in \mathbb{R}$*.*

The proofs of both the theorems are postponed to Section 3.

Below are two simple implementations of the density of a multiplicative subgroup of the group G.

Example 2.3. Let α, β be positive numbers with noncommensurable logarithms: $\alpha, \beta : (0, \infty) \setminus \{1\}$ and $\log \alpha / \log \beta \notin \mathbb{Q}$. Then, as follows from the Kronecker theorem (see, for instance, [3, Sec. 16, Thm. C]), the group generated by α and β, that is $\{\alpha^m \beta^n : \ m, n \in \mathbb{Z}\}$, is dense in $(0, \infty)$.

Example 2.4. Clearly $\exp \mathbb{Q}$, endowed with the usual multiplication, is a dense subgroup of $(0, \infty)$. Note, however, that any two members of it are logarithmically commensurable, so the reasons of the density here are completely different from those occurring in Example 2.3.

3. Concluding remarks and problems

In the special case of a two-elementary set $T \subset (0, \infty)$, generating the dense group $\langle T \rangle$, the simultaneous equations (1.1) with $p = 0$ were solved by J. Matkowski under the assumption of the continuity of φ at at least one point (see [11, Thm. 1 and Cor. 1]). Thus the following question arises naturally which we cannot answer at the moment.

Problem 3.1. *Is it possible to relax the assumption of the continuity in Theorem* 2.2 (ii)*, replacing it by the continuity at a point?*

We know nothing on the possible forms of measurable solutions of the simultaneous equations (1.1). For instance the following question would be of interest.

Problem 3.2. *Describe Lebesgue measurable as well as Baire measurable solutions of equations* (1.1) *or give examples showing that the task is hopeless.*

Finally observe that if $c : (0, \infty) \to \mathbb{R}$ is a solution of the Cauchy equation

$$f(tx) = f(t) + f(x), \tag{3.1}$$

then it satisfies also equations (1.1) with $p = 0$. Taking Lebesgue non-measurable c (cf. [1, Sec. 2.1, Thm.1, and the comment after it] or [10, Thm. 9.4.3 and Cor. 5.2.2]) we see that equations (1.1) have non-measurable solutions, at least in the case $p = 0$.

Problem 3.3. *Are there non-measurable solutions of simultaneous equations* (1.1) *with* $p \neq 0$*?*

4. Proofs

Proof of Theorem 2.1. If $T = \{1\}$ then, by taking $a = 0$ and $b = \varphi(x_0)$, we get the assertion. So, in what follows, we may assume that $T \setminus \{1\} \neq \emptyset$.

Let $x_0 \in (0, \infty)$ and let $\varphi : x_0 \langle T \rangle \to \mathbb{R}$ be a solution of equations (1.1). Thus

$$\varphi(tx_0) = \varphi(x_0) + c(t)x_0^p, \quad t \in T.$$

The formula

$$c(t) = \frac{\varphi(tx_0) - \varphi(x_0)}{x_0^p}$$

defines an extension of c to the set $\langle T \rangle$. Clearly

$$\varphi(tx_0) = \varphi(x_0) + c(t)x_0^p, \quad t \in \langle T \rangle, \tag{4.1}$$

and, in particular, $c(1) = 0$.

We claim that

$$c(st) = c(t) + c(s)t^p \tag{4.2}$$

for every $s, t \in \langle T \rangle$. At first take any $s \in T$ and $t \in \langle T \rangle$. Then, by (4.1), (2.1) and again (4.1), we get

$$\varphi(x_0) + c(st)x_0^p = \varphi(stx_0) = \varphi(tx_0) + c(s)(tx_0)^p = \varphi(x_0) + c(t)x_0^p + c(s)t^p x_0^p,$$

and (4.2) follows. Thus we have shown that the set

$$S = \{s \in \langle T \rangle : c(st) = c(t) + c(s)t^p \quad \text{for every } t \in \langle T \rangle\}$$

contains T. To prove the claim it is enough to check that S is a subgroup of $\langle T \rangle$. If $s_1, s_2 \in S$ and $t \in \langle T \rangle$, then

$$\begin{aligned} c(s_1 s_2 t) &= c(s_2 t) + c(s_1)(s_2 t)^p = c(t) + c(s_2)t^p + c(s_1)s_2^p t^p \\ &= c(t) + (c(s_2) + c(s_1)s_2^p)t^p = c(t) + c(s_1 s_2)t^p, \end{aligned}$$

whence $s_1 s_2 \in S$. Moreover, if $s \in S$ then

$$c(s^{-1}) + c(s)(s^{-1})^p = c(ss^{-1}) = c(1) = 0,$$

so for every $t \in \langle T \rangle$ we have

$$c(s^{-1}t) = c(ss^{-1}t) - c(s)(s^{-1}t)^p = c(t) - c(s)(s^{-1})^p t^p = c(t) + c(s^{-1})t^p,$$

that is $s^{-1} \in S$. Consequently, $S = \langle T \rangle$ and (4.2) holds for every $s, t \in \langle T \rangle$.

Now, by virtue of (4.2), we obtain

$$c(t) + c(s)t^p = c(s) + c(t)s^p, \quad s, t \in \langle T \rangle ,$$

whence

$$\frac{c(s)}{s^p - 1} = \frac{c(t)}{t^p - 1}, \quad s, t \in \langle T \rangle \setminus \{1\}.$$

This means that there is a number $a \in \mathbb{R}$ such that (2.1) holds for every $t \in \langle T \rangle$. Taking any $t_0 \in T \setminus \{1\}$ we have

$$a = \frac{c(t_0)}{t_0^p - 1}$$

and see that a actually depends on the original c, not on x_0 and φ, and, consequently, not on the extended c.

Since (2.1) is true for every $t \in \langle T \rangle$ it follows from (4.1) that

$$\varphi(tx_0) = \varphi(x_0) + a(t^p - 1)x_0^p, \quad t \in \langle T \rangle ,$$

and thus for any $x \in x_0 \langle T \rangle$ we get

$$\begin{aligned} \varphi(x) &= \varphi\left(\frac{x}{x_0}x_0\right) = \varphi(x_0) + a\left(\left(\frac{x}{x_0}\right)^p - 1\right)x_0^p \\ &= \varphi(x_0) + a(x^p - x_0^p) = ax^p + b \end{aligned}$$

with $b = \varphi(x_0) - ax_0^p$.

The rest of the assertion is obvious. ■

To prove Theorem 2.2 we need the following auxiliary result.

Lemma 4.1. *Let D be a dense subset of $(0, \infty)$ and let $f : (0, \infty) \to \mathbb{R}$ be monotonic and satisfy (3.1) for all $t \in D$ and $x \in (0, \infty)$. Then the function f is continuous.*

Proof. Since f is monotonic it has finite one-sided limits at every point. Taking into account that D is dense in $(0, \infty)$ and (3.1) holds for all $t \in D$ and $x \in (0, \infty)$, we infer that

$$f(tx+) = f(t+) + f(x) \quad \text{and} \quad f(tx-) = f(t-) + f(x),$$

whence

$$f(tx+) - f(tx-) = f(t+) - f(t-), \quad t, x \in (0, \infty).$$

It follows that either f is continuous, or f is discontinuous at every point of $(0,\infty)$. However, because of the monotonicity of f the second possibility cannot occur. ■

Proof of Theorem 2.2. Let $\varphi:(0,\infty)\to\mathbb{R}$ be a solution of equations (1.1). By Theorem 2.1 there is a number a, not depending on the cosets of the group $\langle T\rangle$, such that c is of form (2.1). Moreover, we can find a function $b:(0,\infty)\to\mathbb{R}$ giving the formula

$$\varphi(x)=ax^p+b(y),\quad x\in y\langle T\rangle,$$

for every $y\in(0,\infty)$. This means that the function $x\longmapsto\varphi(x)-ax^p$ is constant on every coset of the group $\langle T\rangle$. But all of them are dense subsets of $(0,\infty)$. Thus, if φ is continuous at a point, then the function $x\longmapsto\varphi(x)-ax^p$ is constant on $(0,\infty)$.

To prove (ii) assume that $p=0$, let $c:T\to\mathbb{R}$ be a function and $\varphi:(0,\infty)\to\mathbb{R}$ be a solution of simultaneous equations (1.1). Then

$$\varphi(tx)=\varphi(x)+c(t),\quad x\in(0,\infty),\ t\in T, \tag{4.3}$$

and a simple induction shows that for given $k\in\mathbb{N}$ and $t_1,\dots,t_k\in T$ we have

$$\varphi(t_1\cdot\ldots\cdot t_kx)=\varphi(x)+c(t_1)+\ldots+c(t_k),\quad x\in(0,\infty).$$

If $1\in T$ then (1.1) forces $c(1)=0$. Otherwise put $c(1)=0$. Thus, if $k\in\mathbb{N}$, $t_1,\dots,t_k,s_1,\dots,s_k\in T\cup\{1\}$ and $t_1\cdot\ldots\cdot t_k=s_1\cdot\ldots\cdot s_k$, then

$$c(t_1)+\ldots+c(t_k)=c(s_1)+\ldots+c(s_k).$$

It follows that the formula

$$c(t)=c(t_1)+\ldots+c(t_k),$$

where $t=t_1\cdot\ldots\cdot t_k$, $t_1,\dots,t_k\in T\cup\{1\}$, defines a function which is an extension of c to the semigroup $S(T)$ generated by T. Moreover, (4.3) yields

$$\varphi(tx)=\varphi(x)+c(t),\quad x\in(0,\infty),\ t\in S(T).$$

Now take any $x\in(0,\infty)$. Then, for all $s,t\in S(T)$,

$$\varphi\left(\frac{s}{t}x\right)=\varphi\left(s\frac{x}{t}\right)=\varphi\left(\frac{x}{t}\right)+c(s)$$

and

$$\varphi(x)=\varphi\left(t\frac{x}{t}\right)=\varphi\left(\frac{x}{t}\right)+c(t).$$

Hence

$$\varphi\left(\frac{s}{t}x\right) = \varphi(x) + c(s) - c(t), \quad s,t \in S(T).$$

Setting here $x = 1$ we see that

$$c(s) - c(t) = \varphi\left(\frac{s}{t}\right) - \varphi(1), \quad s,t \in S(T),$$

and, consequently,

$$\varphi\left(\frac{s}{t}x\right) = \varphi(x) + \varphi\left(\frac{s}{t}\right) - \varphi(1), \quad s,t \in S(T).$$

Thus we have proved that

$$\varphi(tx) - \varphi(1) = (\varphi(x) - \varphi(1)) + (\varphi(t) - \varphi(1)),$$

that is equality (3.1), holds for all $t \in \langle T \rangle$ and $x \in (0,\infty)$, where $f = \varphi - \varphi(1)$.

Assume that φ is continuous or monotonic. Then, by Lemma 4.1, the function f is continuous. This means that f is a continuous solution of the Cauchy equation (3.1), and thus (see [1, Sec.2.1, Thm. 2] or [10, Thm. 13.1.5])

$$\varphi(x) = a\log x + \varphi(1), \quad x \in (0,\infty),$$

with some $a \in \mathbb{R}$. Moreover, setting $x = 1$ in (4.3), we have

$$c(t) = \varphi(t) - \varphi(1) = a\log t$$

for every $t \in (0,\infty)$.

The remaining assertion is clear. ■

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