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SCORE SETS IN MULTITOURNAMENTS I. MATHEMATICAL RESULTS

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Dedicated to the 75th birthday of Professors
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Abstract. Let a, b, m, and n be integers ($0 \le a \le b$, $1 \le m \le n$). An (a, b, n)*-tournament* [9] is a directed loopless multigraph $T = (V, A)$, where $V = \{V_1, \ldots, V_n\}$ and if $1 \le i < j \le n$, then V_i and V_j are connected with at least a and at most b arcs. The *score sequence* of T is the nondecreasing sequence of its outdegrees and the *score set* $D = \{d_1, \ldots, d_m\}$ of T is the increasingly ordered set of its outdegrees. We propose four algorithms generating score sequences corresponding to any D: Balancing reconstructs the majority of the score sets; Shortening reconstructs all score sets containing at most seven elements and so improves the theorem of Hager [7]; Sequencing finds a shortest score sequence corresponding to D, while Diophantine generates all score sequences corresponding to D. The algorithms are based on a new, extended version of the Reid-Yao theorem [25, 34].

1. Introduction

Let $a,\ b$, and n integers with $0 \le a \le b$ and $1 \le n$. An (a, b, n)*-tournament* is a loopless directed multigraph $T = (V, A)$ on vertices $V_1\ \ldots,\ V_n$ in which if $1 \le$

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$\leq i < j \leq n$, then V_i and V_j is connected with at least a and at most b arcs. If n is not relevant or the context determines it then we use the simpler notation (a, b)-tournament. We remark that a usual tournaments [33] is a $(1, 1)$-tournament and *simple directed* graphs [32] are such $(0, 2)$-tournaments in which parallel arcs are not allowed.

An (a, b)-tournament is called *complete* if in the case $a \leq c \leq b$ all the results $0 : c,\ 1 : c - 1,\ \dots,\ c : 0$ are allowed and is *incomplete* otherwise. For example the old football (where 2:0, 1:1 and 0:2 are the permitted results) is a complete $(2, 2)$-tournament, while the modern football (where 3:0, 1:1 and 0:3 are permitted, but 2:0, 2:1, 1:2 and 0:2 not) is incomplete.

Let l, m and u be integers, further $1 \leq m$ and $l \leq u$. A sequence of integers $F = f_1,\ \dots,\ f_m$ and a set of integers $D = \{f_1,\ \dots,\ f_m\}$ is called (l, u, m)-*bounded* if $l \leq f_i \leq u$ for $i = 1,\ \dots,\ m$. If an (l, u, m)-bounded sequence or set is monotone, then it is called (l, u, m)-*regular.* In this paper we deal first of all with $(a(m - 1), b(m - 1), m)$-regular and $(0, m - 1, m)$-regular sequences and sets.

The *greatest common divisor* is denoted by gcd, the binomial coefficient $\binom{n}{2}$ by B_n, and a number x repeated y times by $x^{<y>}$.

The structure of the paper is the following. In Section 2 the results on the score sequences of (a, b)-tournaments are summarized, while in Section 3 four construction algorithms of $(1, 1)$-tournaments having prescribed score sets are presented.

2. Score sequences of complete (a, b)-tournaments

The following theorem of Landau allows the quick testing of the potential score sequences of $(1, 1)$-tournaments.

Theorem 2.1. (Landau [17]) *If $n \geq 1$ then an $F = f_1,\ \dots,\ f_n$ nondecreasing $(0, n - 1, n)$-regular sequence is the score sequence of a $(1, 1, n)$-tournament if and only if*

$$\sum_{i=1}^{k} f_i \geq \binom{k}{2} \tag{2.1}$$

for $1 \leq k < n$ with equality for $k = n$.

Proof. See [5, 8, 17, 22, 26]. ∎

Beineke and Eggleton [26] noted in the 1970's that not all of the Landau inequalities need to be checked when testing a sequence F for realizability as a score sequence of some tournament. One only need check

$$\sum_{i=1}^{k} f_i \geq \binom{k}{2} \tag{2.2}$$

for those values of k for which $f_k < f_{k+1}$.

In 1963 Moon generalized the theorem of Landau to (b, b)-tournaments.

Theorem 2.2. (Moon [21]) *If b and n are positive integers, then an $F = f_1, \ldots, f_n$ nondecreasing $(0, b(n-1), n)$-regular sequence is the score sequence of a (b, b)-tournament if and only if*

$$\sum_{i=1}^{k} f_i \geq b\binom{k}{2} \tag{2.3}$$

for all $1 \leq k < n$ index, with equality for $k = n$.

Proof. See [2, 15, 21]. ■

The following extension of the theorem of Moon appeared in 2009.

Theorem 2.3. (Iványi [9]) *If a, b and n are integers with $0 \leq a \leq b$ and $1 \leq n$, then an $F = (f_1, \ldots, f_n)$ nondecreasing $(a(n-1), b(n-1), n)$-regular sequence is the score sequence of an (a, b, n)-tournament if and only if*

$$aB_k \leq \sum_{i=1}^{k} f_i \leq bB_n - L_k - (n-k)f_k \tag{2.4}$$

for all $1 \leq k \leq n$ index, where $L_0 = 0$ and if $1 \leq k \leq n$, then

$$L_k = \max\left(L_{k-1}, \binom{k}{2} - \sum_{i=1}^{k} f_i\right). \tag{2.5}$$

Proof. See [9]. ■

There are algorithms which reconstruct the score sequences of $(1, 1, n)$-tournaments as the algorithm of Guiduli et al. [6], of Gervacio [4], of Kleitman and Wang [16], and of Ryser [28]. Also there are algorithms which construct all score sequences of $(1, 1, n)$-tournaments as the algorithm of Hemasinha [8] or of Ruskey et al. [27].

In [11] we presented algorithm Score-Sequences. This algorithm is based on Theorem 2.3 and generates all score sequences of (a, b, n)-tournaments.

3. Score sets of $(1, 1)$-tournaments

Algorithm Score-Set [11] produces the score set D corresponding to a given score sequence S in linear time.

Figure 1 shows a $(1, 1, 4)$-tournament with score sequence $S = 0,\ 2,\ 2,\ 2$ and score set $D = \{0,\ 2\}$.

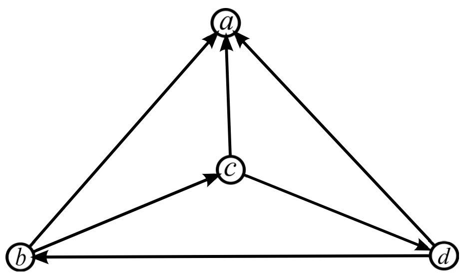

Figure 1. A tournament with score sequence $0,\ 2,\ 2,\ 2$ and score set $\{0, 2\}$

Constructing a tournament with a prescribed score set is more difficult than determining the score set. Quite surprisingly, if sufficiently many players participate in a tournament then any finite set of nonnegative integers is a score set. This was conjectured by K. B. Reid in 1978. In his paper Reid proved several special cases of the conjecture: for sets consisting of one, two or three elements, and also for the case when the set contains the elements of an arithmetic or geometric progression.

Theorem 3.1. (Reid [25]) *If $m \geq 1$ and $D = \{d_1,\ \ldots,\ d_m\}$ is a set of nonnegative integers, further*

1. *either $m = 1,\ 2$ or 3,*
2. *or $D = \{a,\ a + d,\ \ldots,\ a + pd\}$, where a and p are nonnegative integers and d is a positive integer,*
3. *or $D = \{a,\ ad,\ \ldots,\ ad^p\}$, where $a,\ d$ and p are positive integers and $d > 1$,*

then there exists at least one $(1, 1)$-tournament whose score set is D.

Proof. See [25]. ■

In 1986 Hager settled the cases $|D| = 4$ and $|D| = 5$.

Theorem 3.2. (Hager [7]) *If $4 \leq m \leq 5$ and $D = \{d_1,\ \ldots,\ d_m\}$ is a set of nonnegative integers, then there exists at least one $(1, 1)$-tournament whose score set is D.*

Proof. See [7]. ■

In 2006 Pirzada and Naikoo gave a constructive proof of a new special case of Theorem 3.4.

Theorem 3.3. (Pirzada and Naikoo [24]) *If $a_1, \ldots, a_m$ are nonnegative integers with $a_1 < \cdots < a_m$, then there exists at least one $(1,1)$-tournament T with score set*

$$D = \left\{ d_1 = a_1, d_2 = \sum_{i=1}^{2} a_i, \ldots, d_m = \sum_{i=1}^{m} a_i \right\}. \tag{3.1}$$

Proof. See [24]. ■

Finally in 1989 Yao gave an existence proof based on arithmetic analysis.

Theorem 3.4. (Yao [34]) *Every finite nonempty set of nonnegative integers is the score set of at least one $(1,1)$-tournament.*

Proof. See [34]. ■

Taking into account the remark of Beineke and Eggleton [26, page 180] we can formulate Reid's conjecture as an arithmetical statement without the terms of the graph theory. Let $D = \{d_1, \ldots, d_m\}$ be an increasingly ordered set of nonnegative integers. According to the conjecture there exist positive integer exponents $x_1, \ldots, x_m$ such that

$$S = d_1^{<x_1>}, \ldots, d_m^{<x_m>} \tag{3.2}$$

is the score sequence of some $(1, 1, \sum_{i=1}^{m} x_i)$-tournament. Using Landau's theorem it can be easily seen that Reid's conjecture is equivalent to the following statement [13, 22, 23, 34].

For every $(0, d_m, m)$-regular set $D = \{d_1, \ldots, d_m\}$ there exist positive integers $x_1, \ldots, x_m$, such that

$$\sum_{i=1}^{k} x_i d_i \geq \binom{\sum_{i=1}^{k} x_i}{2} \quad \text{for } k = 1, \ldots, m-1, \tag{3.3}$$

and

$$\sum_{i=1}^{m} x_i d_i = \binom{\sum_{i=1}^{m} x_i}{2}. \tag{3.4}$$

Commenting Yao's proof Qiao Li wrote in 1989 [18]: *Yao's proof is the first proof of the conjecture, but I do not think it is the last one. I hope a shorter and simpler new proof will be coming in the near future.*

However, the constructive proof has not been discovered yet.

We propose four algorithms Balancing, Shortening, Sequencing, and Diophantine to reconstruct score sets. These algorithms are based on Theorem 3.5. The proof of Theorem 3.5 is based on the following three lemmas.

Since there are quick (quadratic) algorithms constructing $(1, 1, n)$-tournaments corresponding to a given score sequence, our algorithms construct a suitable score sequence.

If the score sequence of a tournament is S and the score set is D, then we say, that S *generates* D, or D *corresponds* to S.

We formulate a stronger assertion, than the theorem of Yao and analyze four construction methods based on this assertion. The proof is based on the following three lemmas.

Lemma 3.1. (Iványi et al. [11, 13], Iványi and Phong [12]) *If $b \geq 1$, $d_1 \geq 0$ and $D = \{d_1\}$, then there exists a (b, b)-tournament T with score set D if and only if b is a divisor of $2d_1$. If there exists a corresponding T, then it has the unique score sequence $S = d_1^{<1+2d_1/b>}$.*

Proof. If $|S| = n$ and S generates D then the sum of the elements of S equals to nd_1 and also to $bn(n-1)/2$ implying $n = 2^{<1+2d_1/b>}$. Such tournament is realizable for example so, that any player P_i gathers b points against players $\mathrm{P}_{i+1}, \ldots, \mathrm{P}_{i+(n-1)/2}$ b and zero points against the remaining players (the indices are taken mod n). ■

Lemma 3.2. (Iványi et al. [11, 13], Iványi and Phong [12]) *If the score sequence $S = s_1, \ldots, s_n$ of a T tournament generates the score set $D = \{d_1, \ldots, d_m\}$, then $n \geq \lceil d_m/b \rceil + 1$.*

Proof. d_m points require at least $\lceil d_m/b \rceil$ opponents, therefore T contains at least $\lceil d_m/b \rceil + 1$ players. ■

Lemma 3.3. (Iványi et al. [11, 13], Iványi and Phong [12]) *If $b \geq 1$, $m \geq 2$ and the score sequence $S = s_1, \ldots, s_n$ of a (b, b, n)-tournament generates the score set $D = \{d_1, \ldots, d_m\}$, then*

$$\frac{2d_1}{b} + 1 < n < \frac{2d_m}{b} + 1, \tag{3.5}$$

and both bounds are sharp.

Proof. All elements of D appear in S at least once. Therefore the average of the elements of S is between d_1 and d_m. Since a (b, b, n)-tournament consists of B_n matches, the average of the scores is $bB_n/n = b(n-1)/2$. So

$$d_1 < \frac{b(n-1)}{2} < d_m, \tag{3.6}$$

implying (3.5).

If $k \geq 0$ and $D = \{k, k+1\}$, then according to (3.6) $n = 2k + 2$ shows the sharpness of the lower bound.

And if $k > 1$, then the score set $D = \{0, k\}$ can be generated only by the score sequence $S = \{0, k^{<2k>}\}$ since S can contain only one zero therefore the upper bound is also sharp. ■

This lemma has a simple, but useful consequence.

Corollary 3.1. *If $b \geq 1$, $m \geq 2$ and the score sequence $S = s_1, \ldots, s_n$ of a (b, b, n)-tournament generates the score set $D = \{d_1, \ldots, d_m\}$, further $2d_1/b$ and $2d_m/b$ are integers, then*

$$\frac{2d_1}{b} + 2 \leq n \leq \frac{2d_m}{b}, \tag{3.7}$$

and both bounds are sharp.

Proof. The fact that the fractions are integers implies (3.7). The proof of the sharpness is similar as it was in the proof of Lemma 3.3. ■

The following assertion gives a stronger version of the theorem of Yao.

Theorem 3.5. *If $m \geq 1$ and $D = \{d_1, \ldots, d_m\}$ is an increasing $(0, m-1, m)$-regular sequence, then*

1. *there exist a $(1, 1, n)$-tournament with score set D and score sequence $S = s_1, \ldots, s_n$ such that*
2. *if $m = 1$, then $S = d_1^{<1+2d_1>}$;*
3. *if $m \geq 2$, then*

$$\max(d_m + 1, 2d_1 + 2) \leq n \leq 2d_m, \tag{3.8}$$

4. *and the bounds in (3.8) are sharp.*

Proof. This theorem is the consequence of Theorem 3.4, and Lemmas 3.1, 3.2, 3.3.

Part 1 of the theorem is equivalent with the Theorem of Yao.

Part 2 of the theorem is equivalent with Lemma 3.1.

Part 3 of the theorem is equivalent with Lemma 3.2.

Part 4 of the theorem is equivalent with Lemma 3.3. ■

We remark that in 1977 Kapoor, Polimeni and Wall proved the following characterization for simple graphs. Let the set of the distinct degrees of a simple graph G denoted by D_G, and if $D = \{d_1, \ldots, d_m\}$ $(m \geq 1)$ is an increasingly ordered set of positive integers, then denote by $\mu(D)$ and $\mu(d_1, \ldots, d_m)$ the number of vertices of a smallest (considering the number of vertices) simple graph whose score set is D.

Theorem 3.6. (Kapoor, Polimeni, Wall [14]) *If* $m \geq 1$ *and* $D = \{d_1, \ldots, d_m\}$ *is a* (d_1, d_m, m)*-regular set then* $\mu(D) = d_m + 1$.

Proof. See [14]. ∎

In 2006 Ahuja and Tripathi [1] determined the possible orders (number of vertices) of simple graphs having prescribed score sets. In the same year Tripathi and Vijay [29] characterized the possible sizes (number of edges) of simple graphs having prescribed degree set.

3.1. Construction of score sets of $(1,1)$-tournaments by balancing

The basic idea of Balancing is that every match increases the number of wins and also the number of losses by one in a tournament.

At a fixed number of players Balancing divides the players (and their scores) of a tournament into 3 classes: *winners* are the players having more wins than losses, *losers* are the players having more losses than wins, and *balanced* are the players having the same number of wins and losses. In the case of the winners the difference of the number of wins and losses $d_i - (n-1-d_i)$ is called their *plus*, while in the case of the losers the difference $(n-1-d_i) - d_i$ is called their *minus*.

Algorithm Balancing works as follows. Its input is a positive integer $m \leq 7$ and the set of the prescribed scores $D = \{d_1, \ldots, d_m\}$, which is an increasingly ordered $(0, m-1, m)$-regular set. The output is a score sequence $S = s_1, \ldots, s_n$ corresponding to D.

If $m > 1$ then there is at least one loser and at least one winner. We denote the number of winners by w and the number of losers by l.

The algorithm is described in details and analyzed by simulation in [11].

Table 1 contains the constructed score sequences for the zerofree score sets containing seven elements. The base of considering only the zerofree sets is the following assertion.

Lemma 3.4. *Let* $m \geq 2$. *A sequence* $S = s_1^{<y_1>}, \ldots, s_n^{<y_n>}$ *is the score sequence corresponding to the score set* $D = \{0, d_2, \ldots, d_m\}$ *if and only if the sequence* $S' = s_2^{<y_2-1>}, \ldots, s_n^{<y_n-1>}$ *is the score sequence corresponding to* $D' = \{d_2 - 1, \ldots, d_m - 1\}$.

Proof. If S is the score sequence corresponding to D then $s_1 = 0$ and $y_1 = 0$ that is all other players won against the player having the score $s_1 = 0$ so S' corresponds to D'.

If S' does not correspond to D' then we add a new score $d_1 = 0$ to D', increase the multiplicity of other scores by 1 and get D which does not correspond to D'. ∎

n	D	Balancing	Shortening	Sequencing	$\lvert S\rvert - \lvert S_{\min}\rvert$
1	$\{6\}$	6^{13}	same	same	$13-13=0$
2	$\{5,6\}$	$5^6,6^6$	same	same	$12-12=0$
3	$\{4,6\}$	$4^3,6^9$	same	same	$12-12=0$
4	$\{4,5,6\}$	$4^5,5,6^5$	same	$4^7,5,6^2$	$11-10=1$
5	$\{3,6\}$	$3^6,6^3$	same	same	$9-9=0$
6	$\{3,5,6\}$	$3^3,5^2,6^6$	same	same	$11-11=0$
7	$\{3,4,6\}$	$3^6,4,6$	same	same	$8-8=0$
8	$\{3,4,5,6\}$	$3^3,4^4,5,6$	same	$3^5,4,5,6^2$	$9-9=0$
9	$\{2,6\}$	$2^5,6^3$	same	same	$8-8=0$
10	$\{2,5,6\}$	$2^5,5,6$	same	same	$7-7=0$
11	$\{2,4,6\}$	$2^4,4^2,6^2$	same	same	$8-8=0$
12	$\{2,4,5,6\}$	$2^3,4^2,5^2,6^2$	same	$2^4,4^1,5^2,6^1$	$9-8=1$
13	$\{2,3,6\}$	$2^3,3^3,6^1$	same	same	$7-7=1$
14	$\{2,3,5,6\}$	$2,3,5^4,6^5$	same	$2,3^5,5,6$	$11-8=3$
15	$\{2,3,4,6\}$	$2^4,3,4,6$	same	same	$7-7=0$
16	$\{2,3,4,5,6\}$	$2^2,3^2,4,5^2,6^2$	same	same	$9-9=0$
17	$\{1,6\}$	$1^3,6^7$	same	$1^4,6^4$	$10-8=2$
18	$\{1,5,6\}$	$1^3,5^3,6^3$	same	same	$9-9=0$
19	$\{1,4,6\}$	$1^2,4^4,6^3$	same	same	$9-9=0$
20	$\{1,4,5,6$	$1^2,4^3,5^2,6^2$	same	same	$9-9=0$
21	$\{1,3,6\}$	no solution	$1^3,3,6^5$	$1^3,3^2,6^2$	$9-7=2$
22	$\{1,3,5,6\}$	$1,3,5^3,6^6$	same	$1^2,3,5^5,6$	$11-9=2$
23	$\{1,3,4,6\}$	$1^2,3^3,4,6$	same	same	$7-7=0$
24	$\{1,3,4,5,6\}$	$1^3,3,4,5,6$	same	same	$7-7=0$
25	$\{1,2,6\}$	$1^2,2^2,6^5$	same	same	$9-9=0$
26	$\{1,2,5,6\}$	$1^2,2^2,5^2,6^2$	same	$1^3,2,5^2,6$	$8-7=1$
27	$\{1,2,4,6\}$	$1,2,4,6^8$	same	$1^3,2,4,6^2$	$11-7=4$
28	$\{1,2,4,5,6\}$	$1^2,2^2,4,5,6$	same	same	$7-7=0$
29	$\{1,2,3,6\}$	$1^2,2^2,3,6^2$	same	same	$7-7=0$
30	$\{1,2,3,5,6\}$	no solution	$1^2,2,3^2,5,6$	same	$7-7=0$
31	$\{1,2,3,4,6\}$	$1,2,3,4^3,6^3$	same	$1^2,2,3,4^2,6$	$9-7=2$
32	$\{1,2,3,4,5,6\}$	$1,2,3,4^2,5^2,6^2$	same	$1^2,2,3,4,5,6^2$	$9-8=1$

Table 1. Indices, prescribed score sets, reconstructed score sequences produced by Balancing, Shortening, Sequencing, and the difference between the lengths of the score sequences produced by Shortening and Sequencing for 7 players and score sets beginning with $d_1 > 0$. To save space in the exponents we omit the symbols $<$ and $>$.

3.2. Construction of score sets of $(1,1)$-tournaments by shortening

Shortening works in three rounds:

1. in the first round Shortening repeatedly deletes the leading zero element of D and decrease the remaining elements by one, then try to reconstruct the shortened set using Balanced and if the reconstruction is successful then adds the deleted zeros at the same time increasing the remaining elements by one and tests the so received sequences using by Landau theorem;

2. if the first round does not result a score sequence corresponding to D then while $d_m = n - 1$, Shortening provisionally deletes the last element of D and decrease m and n by 1, then tries to reconstruct the shortened sequence, adds the deleted elements, and finally tests the received sequence using algorithm by Landau.

3. if the first and second rounds are unsuccessful then Shortening deletes the last element of D and tries to reconstruct it, then tests if addition of $1, \ldots,\ n - m$ copies of the deleted element results a score sequence corresponding to D.

According to the computer experiments Balanced reconstructed all score sets containing at most six element. There are two score sets with $m = 7$ not reconstructed by Balancing: $D_1 = \{1, 3, 6\}$ and $D_2 = \{1, 2, 3, 5, 6\}$.

In the case of D_2 in the second round the permitted values of n are $7,\ \ldots,\ 12$. Shortening deletes 6 and 5, reconstructs the shortened set, $\{1, 2, 3\}$ receiving the sequence $1^{<2>},\ 2,\ 3^{<2>}$. Adding 5 and 6 the algorithm gets the score sequence $S_2 = 1^{<2>},\ 2,\ 3^{<2>},\ 5,\ 6$ corresponding to the prescribed D_2. Deleting 5 and 6 Balancing reconstructs the shortened set resulting the sequence $1^2, 2, 3^2$, then adding one 5 and one 6 resulting is $S_2 = 1^{<2>}, 2, 3^{<2>}, 5, 6$.

The reconstruction of D_1 requires the third round of Shortening: after the deletion of 6 Balancing results $1^{<3>}, 3$ and adding five times 6 we get the score sequence $S_1 = 1^{<3>}, 3, 6^{<5>}$ corresponding to D_1.

3.3. Construction of score sets of $(1,1)$-tournaments by sequencing

The basic idea of Sequencing is that we gradually generate and store the score sets belonging to the score sequences of $(1, 1, n)$-tournaments for

$n = 1, \ldots, N$ and using Theorem 3.5 and we can find all score sequences corresponding to the prescribed score sets with $d_m \leq N/2$.

The program and simulation results (enumeration results and running time) of SEQUENCING can be found in [11].

Here we present only the results of the investigated algorithms for zerofree score sets containing at most seven elements.

3.4. Construction of score sets of $(1,1)$-tournaments using Diophantine equations

The base of algorithm DIOPHANTINE is the reformulated version of Yao's theorem described in (3.3) and (3.4). If n is fixed, then we consider all partitions of n into *positive integers* and test the corresponding sequences, if they are score sequences.

The program and simulation results (enumeration results and running time) of DIOPHANTINE can be found in [11].

For example BALANCING could not reconstruct the score set $\{1, 2, 3, 5, 6\}$. DIOPHANTINE finds eleven solutions.

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