Annales Univ. Sci. Budapest., Sect. Comp. 40 (2013) 269–274

ALIENATION OF ADDITIVE AND LOGARITHMIC EQUATIONS

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Dedicated to Professors Zoltán Daróczy and Imre Kátai on their 75th birthday

Communicated by Antal Járai

(Received March 27, 2013; accepted May 10, 2013)

Abstract. Let $(R, +, \cdot)$ stand for an Archimedean totally ordered unitary ring and let $(H, +)$ be an Abelian group. Denote by C the cone of all strictly positive elements in R. We study the solutions $f, g : C \longrightarrow H$ of a Pexider type functional equation

$$\text{(E)} \qquad f(x+y) + g(xy) = f(x) + f(y) + g(x) + g(y)$$

resulting from summing up the additive and logarithmic equations side by side. We show that modulo an additive constant equation (E) forces f and g to split back to the system of two Cauchy equations

$$\begin{cases} f(x+y) = f(x) + f(y) \\ g(xy) = g(x) + g(y) \end{cases}$$

for every $x, y \in C$ (alienation phenomenon).

In what follows $(C, +, \cdot)$ will stand for the cone of all positive elements in an Archimedean totally ordered unitary ring $(R, +, \cdot)$ and $(H, +)$ will be an Abelian group. We deal with the very classical Cauchy functional equations defining additivity and logarithmic functions, i.e.

$$a(x+y) = a(x) + a(y)$$

Key words and phrases: Additivity, logarithmic equation, alienation, Archimedean ring.
2010 Mathematics Subject Classification: 39B52, 13J25.

and

$$\ell(xy) = \ell(x) + \ell(y),$$

respectively, where a and ℓ are functions mapping C into H. These equations have very rich literature; the basic facts concerning that topic may be found (among others) in the well known monographs of János Aczél [1] and Marek Kuczma [7]. The question we are faced to is to examine whether or not the so called *alienation* phenomenon in the sense of J. Dhombres does occur. Roughly speaking, summing up these two basic functional equations side by side we ask whether the resulting equation

$$a(x+y) + \ell(xy) = a(x) + a(y) + \ell(x) + \ell(y)$$

forces the functions a and ℓ to be additive and logarithmic, respectively. Although, at first glance, it seems hardly likely, bearing in mind the results obtained by J. Dhombres [2] (where the alienation idea comes from) and by the present author in [3] and [4] (see also R. Ger & L. Reich [6]), such a conjecture becomes more reasonable. In contrast to the papers just quoted, following the case of additivity and exponentiality dealt with in author's paper [5] we have decided to discuss a Pexider version of the problem from the very beginning, because the case where $a = \ell$, i.e. in the case of equation

$$a(x+y) + a(xy) = 2a(x) + 2a(y)\,,$$

we are faced to a very special form of the general functional equation studied in [6]; on the other hand there are no nontrivial mappings that would be both additive and logarithmic.

Our result presented below establishes the alienation (up to an additive constant) of the additive and logarithmic Cauchy equations.

Theorem. *Given an Archimedean totally ordered unitary ring $(R, +, \cdot)$ and an Abelian group $(H, +)$ denote by C the positive cone in R. Then functions $f, g : C \longrightarrow H$ satisfy equation*

(E) $$f(x+y) + g(xy) = f(x) + f(y) + g(x) + g(y)$$

for all $x, y \in C$, if and only if there exist: an additive map $a : R \longrightarrow H$, a logarithmic map $\ell : C \longrightarrow H$ and a constant $c \in H$ such that

$$f(x) = a(x) + c \quad \textit{and} \quad g(x) = \ell(x) - c \quad \textit{for all } x \in C.$$

Proof. Assume that functions $f, g : C \longrightarrow H$ satisfy equation (E) for all $x, y \in C$ and put $h := f + g$. Let $F : C \times C \longrightarrow H$ stand for the Cauchy kernel of h, i.e.

$$F(x, y) = h(x+y) - h(x) - h(y) \quad \text{for all} \quad x, y \in C.$$

Then F satisfies the cocycle equation

$$\text{(Co)} \qquad F(x+y,z)+F(x,y)=F(x,y+z)+F(y,z)$$

for all $x,y,z\in C$. On the other hand, by means of (E) and the definition of h, one has

$$F(x,y)=h(x+y)-f(x+y)-g(xy)=g(x+y)-g(xy)$$

provided that x,y are in C. Inserting that form of F into (Co) we get the equality

$$(1) \qquad g((x+y)z)+g(xy)-g(x+y)=g(x(y+z))+g(yz)-g(y+z)$$

valid for every triple $(x,y,z)\in C^3$. Now, setting here $y=x$ gives

$$(2)\ \ g(2xz)+g\left(x^2\right)-g(2x)=g(x(x+z))+g(xz)-g(x+z) \quad \text{for all } x,z\in C,$$

and putting $z=e$, the identity element of R, into (1) leads to

$$g(xy)=g(xy+x)+g(y)-g(y+e) \quad \text{for all } x,y\in C.$$

With $y=e$ this implies the equality

$$g(x)=g(2x)+g(e)-g(2e),$$

and on setting $\alpha:=g(2e)-g(e)$ we arrive at

$$(3) \qquad g(2x)=g(x)+\alpha \quad \text{for all } x\in C.$$

Applying (3) in (2) we infer that

$$u(x):=g\left(x^2\right)-g(x)=g(x(x+z))-g(x+z)$$

or, equivalently,

$$(4) \qquad g(x(x+z))=u(x)+g(x+z) \quad \text{for all } x,z\in C.$$

With $y=x+z$ equation (4) may equivalently be rewritten in the form

$$x\preceq y \Longrightarrow g(xy)=u(x)+g(y) \quad \text{for all } x,y\in C.$$

Now, going back to (E), we deduce that

$$x\preceq y \Longrightarrow f(x+y)+u(x)=f(x)+f(y)+g(x).$$

In other words,

$$x \preceq y \Longrightarrow f(x+y) = A(x) + f(y) \quad \text{for all } x, y \in C, \tag{5}$$

where we have put $A := f - u + g$. Now, we are going to show that that map A is additive. To this end, observe that due to the inequality $x \preceq x + y$ valid for all $x, y \in C$, relation (5) implies that

$$f(2x+y) = A(x) + f(x+y) \quad \text{for all } x, y \in C. \tag{6}$$

Replacing here y by $y + z$ we get

$$f(2x+y+z) = A(x) + f(x+y+z) \quad \text{for all } x, y, z \in C,$$

whence, by setting here $2y$ in place of y one obtains

$$f(2(x+y)+z) = A(x) + f(2y+x+z) \quad \text{for all } x, y, z \in C,$$

which, with the aid of a double use of (6), gives

$$A(x+y) + f(x+y+z) = f(2(x+y)+z) =$$
$$= A(x) + f(2y+x+z) = A(x) + A(y) + f(x+y+z)$$

for all $x, y, z \in C$, proving the additivity of A, as claimed. It is well known and easily verifiable that the formula

$$a(x) := \begin{cases} A(x) & \text{whenever } x \in C \\ 0 & \text{for } x = 0 \\ -A(-x) & \text{whenever } x \in -C \end{cases}$$

uniquely extends A to an additive map $a : R \longrightarrow H$.

Observe that, on account of (6) and the additivity of A, for arbitrary x, y from C one has

$$f(2x+y) - A(2x+y) = A(x) + f(x+y) - 2A(x) - A(y) = f(x+y) - A(x+y),$$

which on setting $c := f - A$ states that

$$c(2x+y) = c(x+y) \quad \text{for all } x, y \in C. \tag{7}$$

Fix arbitrarily an $s \in C$ and a t such that $s \prec t \prec 2s$. With $x := t - s$ and $y := 2s - t$ we have then $x, y \in C$ as well as

$$s = x + y \quad \text{and} \quad t = 2x + y,$$

which jointly with (7) implies that $c(t) = c(s)$ for all t from the order segment $[s, 2s)$. As a matter of fact, we have also $c(2s) = c(s)$; indeed, fix a t_0 from the segment $(s, 2s)$ to get $s \prec t_0 \prec 2s \prec 2t_0$ (without loss of generality we may assume that $(s, 2s) \neq \emptyset$) whence $c(2s) = c(t_0) = c(s)$. Clearly, now we can conlude that for every $s \in C$ and each positive integer n the restriction $c|_{[s,2^n s]}$ is constant. Consequently, c is globally constant on C. In fact, fix arbitrarily two elements $a, b, a \prec b$, (recall that the ordering is total) from C. Since the ring R is supposed to be Archimedean there exists a positive integer n such that $a \prec b \prec 2^n a$ whence $c(b) = c(a)$.

To finish the proof, it remains to observe that the equality $f = A + c$ forces the map $g + c$ to be logarithmic, by means of (E). Since the reverse implication is fairly straightforward, the proof has been completed. ■

We terminate this paper with two short remarks:

1. Admitting 0 to belong to the cone C considered retains the alienation phenomenon but in a trivial way (the corresponding logarithmic function vanishes identically on C).

2. It seems to me that the assumption that the ring considered is Archimedean is inessential but at present I am unable to settle it.

References

[1] **Aczél, J.,** *Lectures on Functional Equations and their Applications*, Academic Press, New York and London, 1966.

[2] **Dhombres, J.,** Relations de dépendance entre équations fonctionnelles de Cauchy, *Aequationes Math.*, **35** (1988), 186–212.

[3] **Ger, R.,** On an equation of ring homomorphisms, *Publ. Math. Debrecen*, **52** (1998), 397–417.

[4] **Ger, R.,** Ring homomorphisms equation revisited, *Rocznik Nauk.-Dydakt. Prace Mat.*, **17** (2000), 101–115.

[5] **Ger, R.,** Additivity and exponentiality are alien to each other, *Aequationes Math.*, **80** (2010), 111–118.

[6] **Ger, R. and L. Reich,** A generalized ring homomorphisms equation, *Monatsh. Math.*, **159** (2010), 225–233 .

[7] **Kuczma, M.,** *An Introduction to the Theory of Functional Equations and Inequalities*, Polish Scientific Publishers and Silesian University Press, Warszawa-Kraków-Katowice, 1985.

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