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ON THE FEJÉR KERNEL FUNCTIONS WITH RESPECT TO THE CHARACTER SYSTEM OF THE GROUP OF 2-ADIC INTEGERS

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Dedicated to Professors Zoltán Daróczy and Imre Kátai on the occasion of their 75th birthday

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Abstract. It was a long time open problem the almost everywhere convergence of the Cesàro $(C,1)$-means of integrable functions $\sigma_n f \to f$ for $f \in L^1(I)$, where I is the group of 2-adic integers. In his paper the author verified [2] this a.e. relation with the help of an inequality for the integral of the maximal function of the Fejér kernels. Later, the author also discussed [3] the almost everywhere convergence of the Cesàro (C,α)-means for every $\alpha > 0$. In general, in the investigation of the Fejér means with respect to convergence or divergence issue, a formula for the kernel function plays an important role. In the Walsh-Paley case see [1] or alternatively [7]. The Walsh-Kaczmarz version is due to Skvortsov [10], [9]. For the time being, there is no known formula for this 2-adic integers case. This paper fills this gap.

1. Introduction

We follow the standard notions of dyadic analysis introduced by mathematicians F. Schipp, P. Simon, W. R. Wade (see e.g. [7]) and others. Denote

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by $\mathbb{N} := \{0, 1, ...\}, \mathbb{P} := \mathbb{N} \setminus \{0\}$, the set of natural numbers, the set of positive integers and $I := [0, 1)$ the unit interval. Denote by $\lambda(B) = |B|$ the Lebesgue measure of the set $B (B \subset I)$. Denote by $L^p(I)$ the usual Lebesgue spaces and $\|.\|_p$ the corresponding norms $(1 \le p \le \infty)$. Set

$$\mathcal{I} := \left\{ \left[\frac{p}{2^n}, \frac{p+1}{2^n} \right) : p, n \in \mathbb{N} \right\},$$

the set of dyadic intervals and for given $x \in I$ and $n \in \mathbb{N}$ let $I_n(x)$ denote the interval $I_n(x) \in \mathcal{I}$ of length 2^{-n} which contains x. Also use the notion $I_n := I_n(0)\ \ (n \in \mathbb{N})$. Let

$$x = \sum_{n=0}^{\infty} x_n 2^{-(n+1)}$$

be the dyadic expansion of $x \in I$, where $x_n = 0$ or 1 and if x is a dyadic rational number $(x \in \{\frac{p}{2^n} : p, n \in \mathbb{N}\})$ we choose the expansion which terminates in 0 's. For $n \in \mathbb{N}$ let n_k be the kth coordinate of n with respect to number system based 2. That is,

$$n = \sum_{k=0}^{\infty} n_k 2^k,$$

where $n_k \in \{0, 1\}\ (k \in \mathbb{N})$. Also use the notation

$$n^{(s)} = \sum_{k=s}^{\infty} n_k 2^k \quad (n, s \in \mathbb{N}).$$

The notion of the Hardy space $H(I)$ is introduced in the following way [7, 8]. A function $a \in L^\infty(I)$ is called an atom, if either $a = 1$ or a has the following properties: supp $a \subseteq I_a, \|a\|_\infty \le |I_a|^{-1}, \int_I a = 0$, for some $I_a \in \mathcal{I}$. We say that the function f belongs to H, if f can be represented as $f = \sum_{i=0}^\infty \lambda_i a_i$, where a_i's are atoms and for the coefficients (λ_i) the inequality $\sum_{i=0}^\infty |\lambda_i| < \infty$ is true. It is known that H is a Banach space with respect to the norm

$$\|f\|_H := \inf \sum_{i=0}^{\infty} |\lambda_i|,$$

where the infimum is taken over all decompositions $f = \sum_{i=0}^\infty \lambda_i a_i \in H$.

The 2-adic (or arithmetic) sum $a + b := \sum_{n=0}^\infty r_n 2^{-(n+1)}\ (a, b \in I)$, where bits $q_n, r_n \in \{0, 1\} (n \in \mathbb{N})$ are defined recursively as follows : $q_{-1} := 0$, $a_n + b_n + q_{n-1} = 2q_n + r_n$ for $n \in \mathbb{N}$. (Since q_n, r_n take on only the values $0, 1$, these equations uniquely determine the coefficients q_n and r_n.) The group $(I, +)$ is called the group of 2-adic integers. Set

$$\epsilon(t) := \exp(2\pi \imath t) \quad (t \in \mathbb{R}),$$

where $\imath = (-1)^{\frac{1}{2}}$. Set

$$v_{2^n}(x) := \epsilon\left(\frac{x_n}{2} + \cdots + \frac{x_0}{2^{n+1}}\right) \quad (x \in I, n \in \mathbb{N})$$

and

$$v_n := \prod_{n=0}^{\infty} v_{2^j}^{n_j},$$

where $\mathbb{N} \ni n = \sum_{i=0}^{\infty} n_i 2^i \, (n_i \in \{0,1\} \, (i \in \mathbb{N}))$. It is known [4] that the system $(v_n, n \in \mathbb{N})$ is the character system of $(I, +)$. Denote by

$$\hat{f}(n) := \int_I f \bar{v}_n d\lambda \, , \, D_n := \sum_{k=0}^{n-1} v_k \, , \, K_n := \frac{1}{n+1} \sum_{k=0}^{n} D_k$$

the Fourier coefficients, the Dirichlet and the Fejér or $(C,1)$ kernels, respectively. It is also known that the Fejér or $(C,1)$ means of f is

$$\sigma_n f(y) := \frac{1}{n+1} \sum_{k=0}^{n} S_k f(y) = \int_I f(x) K_n(y-x) d\lambda(x) =$$
$$= \frac{1}{n+1} \sum_{k=0}^{n} \int_I f(x) D_k(y-x) d\lambda(x) \quad (n \in \mathbb{N}, \; y \in I).$$

It is known that [5, 6, 2] for $n \in \mathbb{N}, \; x \in I$

$$D_{2^n}(x) = \begin{cases} 2^n, & \text{if } x \in I_n, \\ 0, & \text{if } x \notin I_n \end{cases}$$

and also that [5]

$$D_n(x) = v_n(x) \sum_{k=0}^{\infty} D_{2^k}(x) n_k (-1)^{x_k}.$$

Denote by K_n^α the kernel of the summability method (C, α), and call it the (C, α) kernel, or the Cesàro kernel for $\alpha \in \mathbb{R}$:

$$K_n^\alpha = \frac{1}{A_n^\alpha} \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} D_\nu, \quad A_k^\alpha = \frac{(\alpha+1)(\alpha+2)\cdots(\alpha+k)}{k!} \quad (\alpha \neq -k).$$

It is well-known [12, Ch. 3] that $A_n^\alpha = \sum_{k=0}^{n} A_{n-k}^{\alpha-1}, \quad A_n^\alpha - A_{n-1}^\alpha = A_n^{\alpha-1}$, $A_n^\alpha \sim n^\alpha$. The (C, α) Cesàro means of the integrable function f is

$$\sigma_n^\alpha f(y) := \frac{1}{A_n^\alpha} \sum_{k=0}^{n} A_{n-k}^{\alpha-1} S_k = \int_I f(x) K_n^\alpha (y-x) d\lambda(x).$$

It was a question of Taibleson [11] open for a long time, that does the Fejér-Lebesgue theorem, that is the a.e. convergence $\sigma_n f \to f$ hold for all integrable function f. In 1992 Schipp and Wade proved [5] the L^1 norm convergence $\sigma_n f \to f$ for all $f \in L^1$.

In 1997 Gát proved [2] the a.e. convergence $\sigma_n f \to f$ for every integrable function. He also proved that the maximal operator of the Fejér means, that is, $\sigma^* f$ is of type (H, L^1). In other words, there exists an absolute constant $C > 0$ such that $\|\sigma^* f\|_H \le C\|f\|_{L^1}$ for all $f \in H$.

Later, Gát proved [3] for all $\alpha > 0$ the a.e. relation $\sigma_n^\alpha f \to f$ for each integrable function f. In the proofs of these results a key lemma was [2, Lemma 3]: Let $B \ge t$ be fixed natural numbers. Then,

$$\int_{I_t \setminus I_{t+1}} \sup_{N \ge 2^B} |K_N(z)| d\lambda(z) \le C \left(2^{t-B}\right)^{\frac{1}{2}}. \tag{1.1}$$

In this paper we essentially improve inequality (1.1). Moreover, for the time being there is no formula for the Fejér kernels. This paper aims to fill this gap. For more details on the Fourier theory with respect to the character system of the group of 2-adic integers see for instance [5, 6]. In this paper C denotes an absolute constant which may be different from line to line.

2. The results

For $k \in \mathbb{N}$ and $x \in I$ let product $< k, x >$ be defined as

$$< k, x >:= \frac{k_0 x_0}{2^1} + \frac{k_1 x_1}{2^1} + \frac{k_1 x_0}{2^2} + \frac{k_2 x_2}{2^1} + \frac{k_2 x_1}{2^2} + \frac{k_2 x_0}{2^3} + \cdots =$$

$$= \sum_{i=0}^{\infty} k_i \sum_{j=0}^{i} \frac{x_j}{2^{i-j+1}}.$$

For any $x \in I$ and $s \in \mathbb{N}$ use the notation

$$\tau_s x = \frac{x_{s-1}}{2^1} + \frac{x_{s-2}}{2^2} + \cdots + \frac{x_0}{2^s}.$$

The main tool in the proof of the main result is the following lemma.

Lemma 2.1. *Let $x \in I$ and $s \in \mathbb{N}$. Then the following equality holds:*

$$\sum_{k=0}^{2^s-1} v_k(x) \epsilon\left(\frac{< k, \frac{1}{2} >}{2}\right) = 1 + \imath \cot \pi \left(\tau_s x + \frac{1}{2^{s+1}}\right).$$

Proof. Since

$$v_k(x) = \epsilon\,(< k, x >) = \\ = \epsilon\left(\frac{k_0x_0}{2^1} + \frac{k_1x_1}{2^1} + \frac{k_1x_0}{2^2} + \cdots + \frac{k_{s-1}x_{s-1}}{2^1} + \cdots + \frac{k_{s-1}x_0}{2^s}\right),$$

then by $< k, 1/2 >= k_0/2^1 + k_1/2^2 + \cdots + k_{s-1}/2^s$ for $k < 2^s$ we have

$$\sum_{k=0}^{2^s-1} v_k(x)\epsilon\left(\frac{< k, \frac{1}{2} >}{2}\right) = \\ = \sum_{k_0,\dots,k_{s-1}\in\{0,1\}} \epsilon\Bigg(\frac{1}{2^{s+1}}\Big(k_{s-1}(2^sx_{s-1} + 2^{s-1}x_{s-2} + \cdots + 2^1x_0 + 2^0) + \\ + k_{s-2}(2^sx_{s-2} + 2^{s-1}x_{s-3} + \cdots + 2^2x_0 + 2^1) + \cdots + \\ + k_1(2^sx_1 + 2^{s-1}x_0 + 2^{s-2}) + k_0(2^sx_0 + 2^{s-1})\Big)\Bigg).$$

Set $L_x = 2^{s+1}\tau_s x + 1 = 2^sx_{s-1} + 2^{s-1}x_{s-2} + \cdots + 2^1x_0 + 2^0$ which is an odd natural number. The function $\epsilon(\frac{1}{2^{s+1}}\cdot)$ is periodic modulo 2^{s+1}. The equalities

$$L_x = 2^sx_{s-1} + 2^{s-1}x_{s-2} + \cdots + 2^1x_0 + 2^0,$$
$$2L_x = 2^sx_{s-2} + 2^{s-1}x_{s-3} + \cdots + 2^2x_0 + 2^1 \pmod{2^{s+1}},$$
$$2^2L_x = 2^sx_{s-3} + 2^{s-1}x_{s-4} + \cdots + 2^3x_0 + 2^2 \pmod{2^{s+1}},$$
$$\dots,$$
$$2^{s-2}L_x = 2^sx_1 + 2^{s-1}x_0 + 2^{s-2} \pmod{2^{s+1}},$$
$$2^{s-1}L_x = 2^sx_0 + 2^{s-1} \pmod{2^{s+1}}$$

give

$$\sum_{k=0}^{2^s-1} v_k(x)\epsilon\left(\frac{< k, \frac{1}{2} >}{2}\right) = \\ = \sum_{k_0,\dots,k_{s-1}\in\{0,1\}} \epsilon\left(\frac{L_x}{2^{s+1}}(k_{s-1} + 2^1k_{s-2} + \cdots + 2^{s-1}k_0)\right) = \\ = \sum_{l=0}^{2^s-1} \epsilon\left(\frac{L_x}{2^{s+1}}l\right) = \\ = \frac{\exp\left(2\pi\imath\frac{L_x}{2}\right) - 1}{\exp\left(2\pi\imath\frac{L_x}{2^{s+1}}\right) - 1} = \\ = \frac{2}{1 - \epsilon\left(\frac{L_x}{2^{s+1}}\right)}.$$

Recall that L_x is odd and consequently $\epsilon(L_x/2) = -1$. Since $\frac{1}{1-\exp(\imath\phi)} = 1/2 + \imath/2 \cot\phi/2$, then

$$\sum_{k=0}^{2^s-1} v_k(x)\epsilon\left(\frac{< k, \frac{1}{2} >}{2}\right) = 1 + \imath\cot\left(\pi\frac{L_x}{2^{s+1}}\right).$$

This completes the proof of Lemma 2.1. ∎

The following theorem is the main result of the paper. It gives a formula for the Fejér kernel functions with respect to the character system of the group of 2-adic integers.

Theorem 2.1. *For $n, t \in \mathbb{N}$, $x \in I_t \setminus I_{t+1}$ we have*

$$nK_n(x) = \sum_{s=0}^{t} n_s 2^s \left(D_{n^{(s+1)}}(x) + v_{n^{(s+1)}}(x)\frac{2^s-1}{2}\right) +$$
$$+ \sum_{s=t+1}^{\infty} n_s 2^{2t}(1 + \imath\cot(\pi\tau_s x))v_{n^{(s+1)}}(x).$$

Proof. Let $2^A \le n < 2^{A+1}$. Then we have

$$\begin{aligned} &nK_n(x) = \\ &= \sum_{s=0}^{A} \sum_{k=n^{(s+1)}}^{n^{(s)}} D_k(x) = \\ &= \sum_{s=0}^{A} n_s \sum_{k=n^{(s+1)}}^{n^{(s+1)}+2^s-1} D_k(x) = \\ &= \sum_{s=0}^{A} n_s \sum_{k=0}^{2^s-1} \left(D_{n^{(s+1)}}(x) + v_{n^{(s+1)}}(x)D_k(x)\right). \end{aligned}$$

For $s \le t$ and $k < 2^s$ $x \in I_t$ gives $D_k(x) = k$ and consequently

$$\begin{aligned} &\sum_{s=0}^{t} n_s \sum_{k=0}^{2^s-1} \left(D_{n^{(s+1)}}(x) + v_{n^{(s+1)}}(x)D_k(x)\right) \\ &= \sum_{s=0}^{t} n_s 2^s \left(D_{n^{(s+1)}}(x) + v_{n^{(s+1)}}(x)\frac{2^s-1}{2}\right). \end{aligned}$$

In the sequel we turn our attention to the case $s > t$. Use the formula for the

Dirichlet kernel $D_n(x)$ given by Schipp ([5]).

$$K_{n^{(s+1)},2^s}(x) := \sum_{k=n^{(s+1)}}^{n^{(s+1)}+2^s-1} D_k(x) =$$

$$= \sum_{k=n^{(s+1)}}^{n^{(s+1)}+2^s-1} (\sum_{j=0}^{t-1} k_j 2^j) v_k(x) + \sum_{k=n^{(s+1)}}^{n^{(s+1)}+2^s-1} k_t 2^t (-1) v_k(x) =: \sum^{1} + \sum^{2}.$$

Since 2^{s+1} is a divisor of $n^{(s+1)}$, then $k = n^{(s+1)}+l$ $(0 \le l < 2^s)$. Consequently, we have

$$v_k(x) = v_{n^{(s+1)}+l}(x) = v_{n^{(s+1)}}(x) v_l(x).$$

Thus, (in the case of $s > t$)

$$\sum^{1} = v_{n^{(s+1)}}(x) \sum_{l_{s-1}=0}^{1} \cdots \sum_{l_0=0}^{1} (\sum_{j=0}^{t-1} k_j 2^j) v_l(x) = v_{n^{(s+1)}}(x) \sum_{l_t=0}^{1} v_{2^t}^{l_t}(x) \phi(x),$$

where ϕ does not depend on l_t. Since $x \in I_t \setminus I_{t+1}$, $v_{2^t}^{l_t}(x) = \epsilon(l_t(x_t/2 + \cdots + +x_0/2^{t+1})) = \epsilon(l_t/2) = (-1)^{l_t}$, then we get $\sum^1 = 0$. Consequently,

$$\sum_{k=n^{(s+1)}}^{n^{(s+1)}+2^s-1} D_k(x) = v_{n^{(s+1)}}(x) \sum_{l_{s-1}=0}^{1} \cdots \sum_{l_0=0}^{1} l_t 2^t (-1) v_l(x).$$

For $j < t$ (recall that $x \in I_t \setminus I_{t+1}$) we have

$$v_{2^j}^{l_j}(x) = \epsilon\left(l_j \left(\frac{x_j}{2^1} + \frac{x_{j-1}}{2^2} + \cdots + \frac{x_{t+1}}{2^{j-t}}\right)\right) \epsilon\left(l_j \frac{1}{2^{j-t+1}}\right).$$

These assumptions imply the equality

$$\sum_{k=n^{(s+1)}}^{n^{(s+1)}+2^s-1} D_k(x) =$$

$$= v_{n^{(s+1)}}(x) 2^{2t} \sum_{l_{s-1}=0}^{1} \sum_{l_{s-2}=0}^{1} \cdots \sum_{l_{t+1}=0}^{1} \epsilon\left(l_{t+1} \frac{x_{t+1}}{2^1}\right) \epsilon\left(l_{t+1} \frac{1}{2^2}\right) \times$$

$$\times \epsilon\left(l_{t+2}\left(\frac{x_{t+2}}{2^1} + \frac{x_{t+1}}{2^2}\right)\right) \epsilon\left(l_{t+2} \frac{1}{2^3}\right) \cdots \times$$

$$\times \epsilon\left(l_{s-1}\left(\frac{x_{s-1}}{2^1} + \cdots + \frac{x_{t+1}}{2^{s-t-1}}\right)\right) \epsilon\left(l_{s-1} \frac{1}{2^{s-t}}\right)$$

Use the notation

$$k = l_{t+1} 2^0 + l_{t+2} 2^1 + \cdots + l_{s-1} 2^{s-t-2} < 2^{s-t-1}$$

and
$$\frac{x_{t+1}}{2^1}+\frac{x_{t+2}}{2^2}+\cdots+\frac{x_{s-1}}{2^{s-t-1}}+\frac{x_s}{2^{s-t}}\cdots=\{2^{t+1}x\}.$$
Since
$$\epsilon\left(l_{t+1}\frac{1}{2^2}\right)\epsilon\left(l_{t+2}\frac{1}{2^3}\right)\cdots\epsilon\left(l_{s-1}\frac{1}{2^{s-t}}\right)=$$
$$=\epsilon\left(k_0\frac{1}{2^2}\right)\epsilon\left(k_1\frac{1}{2^3}\right)\cdots\epsilon\left(k_{s-t-2}\frac{1}{2^{s-t}}\right)=\epsilon(<k,1/2>/2)$$
and since all functions v_{2^j} is periodic modulo 1, then
$$\sum_{k=n^{(s+1)}}^{n^{(s+1)}+2^s-1}D_k(x)=$$
$$=v_{n^{(s+1)}}(x)2^{2t}\sum_{k=0}^{2^{s-t-1}-1}v_k(2^{t+1}x)\epsilon(<k,1/2>/2).$$
By Lemma 2.1 we get
$$\sum_{k=0}^{2^{s-t-1}-1}v_k(2^{t+1}x)\epsilon(<k,1/2>/2)=$$
$$=1+\imath\cot\pi\left(\frac{(2^{t+1}x)_{s-t-2}}{2^1}+\cdots+\frac{(2^{t+1}x)_0}{2^{s-t-1}}+\frac{1}{2^{s-t}}\right)=$$
$$=1+\imath\cot\pi\left(\frac{x_{s-1}}{2^1}+\cdots+\frac{x_{t+1}}{2^{s-t-1}}+\frac{1}{2^{s-t}}\right)=$$
$$=1+\imath\cot\pi\left(\frac{x_{s-1}}{2^1}+\cdots+\frac{x_{t+1}}{2^{s-t-1}}+\frac{x_t}{2^{s-t}}+\dots\right)=$$
$$=1+\imath\cot(\pi\tau_s x).$$
Consequently,
$$\sum_{k=n^{(s+1)}}^{n^{(s+1)}+2^s-1}D_k(x)=v_{n^{(s+1)}}(x)2^{2t}(1+\imath\cot\pi(\tau_s x)).$$
Finally, for $s>t$ we write
$$\sum_{s=t+1}^{A}n_s\sum_{k=n^{(s+1)}}^{n^{(s+1)}+2^s-1}D_k(x)=$$
$$=\sum_{s=t+1}^{A}n_s 2^{2t}v_{n^{(s+1)}}(x)(1+\imath\cot\pi(\tau_s x)).$$
This completes the proof of Theorem 2.1. ■

Theorem 2.1 immediately gives the following estimation for the absolute value of the kernel.

Corollary 2.1. *For $x \in I_t \setminus I_{t+1}$, $n > 2^t$, $n, t \in \mathbb{N}$ we have the estimation for the Fejér kernel:*

$$|nK_n(x)| \le C2^{2t} \sum_{s=t+1}^{\infty} n_s(1 + |\cot \pi\tau_s x|).$$

Besides, Corollary 2.1 implies an estimation for the kernel depending only on the integer part of the binary logarithm of n, that is, on $A = \lfloor \log_2 n \rfloor$ and not depending on the coordinates of n.

Corollary 2.2. *For $x \in I_t \setminus I_{t+1}$, $n > 2^t$, $n, t \in \mathbb{N}$ we have the estimation for the Fejér kernel:*

$$|nK_n(x)| \le C2^{2t} \sum_{s=t+1}^{A} (1 + |\cot \pi\tau_s x|).$$

The following corollary of Theorem 2.1 improves the result in paper [2] which played a key role in the proof of the a.e. convergence of the $(C, 1)$ means of the Fourier series of integrable functions on the group of 2-adic integers. This result is essentially better than inequality (1.1), that is, in paper [2, Lemma 3].

Corollary 2.3. *For $B \ge t$, $B, t \in \mathbb{N}$ we have the estimation for the Fejér kernel:*

$$\int_{I_t \setminus I_{t+1}} \sup_{n \ge 2^B} |K_n(x)| d\lambda(x) \le C \frac{(B-t)^2}{2^{B-t}}.$$

Proof. In the proof we use the inequality in Corollary 2.2. For $s > t$ and $j = 1, \dots, s-t-1$ set

$$J_j = (I_t \setminus I_{t+1}) \cap \{x \in I : x_{s-1} = 0, \dots, x_{s-j} = 0, x_{s-j-1} = 1\}.$$

For $j = 0$ let

$$J_0 = (I_t \setminus I_{t+1}) \cap \{x \in I : x_{s-1} = 1\}.$$

Then we have

$$I_t \setminus I_{t+1} = \bigcup_{j=0}^{s-t-1} J_j.$$

On the set J_j the following inequality holds:

$$\begin{aligned} |\cot \pi\tau_s x| &\le \frac{C}{\tau_s x} = \\ &= \frac{C2^s}{2^{s-1}x_{s-1} + 2^{s-2}x_{s-2} + \dots + 2^{s-j-1}x_{s-j-1} + \dots + 2^t x_t} \le \\ &\le \frac{C2^s}{2^{s-j-1}} \le C2^j. \end{aligned}$$

Consequently, we get

$$\int_{I_t\setminus I_{t+1}} |1+\cot \pi\tau_s x| d\lambda(x) \le$$
$$\le C\sum_{j=0}^{s-t-1}\left(\frac{1}{2^t}+2^j\frac{1}{2^{j+t}}\right)\le C(s-t)\frac{1}{2^t}.$$

This inequality by Corollary 2.2 gives

$$\int_{I_t\setminus I_{t+1}} \sup_{n\ge 2^B}|K_n(x)|d\lambda(x)\le$$
$$\le \sum_{A=B}^{\infty}\int_{I_t\setminus I_{t+1}}\sup_{\{n:2^A\le n<2^{A+1}\}}|K_n(x)|d\lambda(x)\le$$
$$\le C\sum_{A=B}^{\infty}2^{2t-A}\sum_{s=t+1}^{A}\int_{I_t\setminus I_{t+1}} 1+|\cot \pi\tau_s x|d\lambda(x)\le$$
$$\le C\sum_{A=B}^{\infty}2^{2t-A}\sum_{s=t+1}^{A}(s-t)\frac{1}{2^t}\le C\sum_{A=B}^{\infty}\frac{(A-t)^2}{2^{A-t}}\le C\frac{(B-t)^2}{2^{B-t}}.$$

This completes the proof of Corollary 2.3. ∎

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