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ESTIMATION OF THE KOLMOGOROV ENTROPY IN THE GENERALIZED NUMBER SYSTEM

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Dedicated to Professors Zoltán Daróczy and Imre Kátai on their 75th anniversary

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Abstract. We study the dynamical properties of the complex number systems. A map is introduced on the transition graph by analogy of statistical physics systems. We gain an estimation of the Kolmogorov entropy by the Grassberger-Procaccia algorithm for a finite approximation of B_γ.

1. Introduction

The entropy plays important role in physics, mathematics and informatics.

The concept of entropy emerged in the thermodynamics by Clausius R., who also gave a mathematical form of entropy [1] in 1865. Boltzmann's L. and Ehrenfest's T. results showed that the thermodynamic and statistical physics entropy is identical.

In the 1950s, Kolmogorov A.N. proposed the notion of the entropy to characterize dynamical systems [2], he applied probability theory to solve the problem of isomorphism of dynamical systems and point systems. Important question was at this time that dynamical systems (deterministic time dependent flow)

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arising in probability theory are different from the metric point of view Shannon C. introduced the form of information entropy [3].

From the mathematical point of view, Rényi A. investigated generalized entropies of order q to use arbitrary probability distribution within the information theory [4].

In this article we apply the generalized order-q Rényi entropies $K(q)$, where $K(1)$ is the Kolmogorov-Sinai entropy in the limit $q \to 1$. The order-2 Rényi entropy $K(2)$ is studied by Grassberger P. and Procaccia I. [5]. They introduced a method to obtain this quantity from correlation sums of time series. It gives an approximation to the Kolmogorov-Sinai Entropy $K(1)$, where $\lim_{q\to 1^+} K(q) = K(1)$.

The goal of this article is to study some dynamical system by Rényi entropy which contains the element of B_γ with increasing the digits. We would know the chaotic property of it.

We investigate the time dependent behaviour of generalized number systems by applying a map on the transition graph, and determine a numerical estimation of the generalized entropy of order-2.

2. Generalized number systems

Kátai I. introduced the concept of generalized number systems [6] in the 1970s. This idea is considered and settled for Gaussian integers [7], real quadratic fields [8], imaginary quadratic fields [9] and other important algebraic structures [10],[11],[12].

We give here the basic definitions.

In the case $\mathbb{Z}_k$ is a ring of integer vectorial in $\mathbb{R}_k$ $(k \geq 1)$. Let M be a $k \times k$ type matrix with integer entries and $\mathcal{L} = M\mathbb{Z}_k$. Then $\mathcal{L}$ is a subgroup in $\mathbb{Z}_k$, $O(\mathbb{Z}_k/\mathcal{L}) = t$ the order, when $t = |\det M|$. Let $\mathcal{A} = \{a_0 = 0, a_1, \dots, a_{t-1}\}$ denote a complete set of the representation of the residue classes mod M for $(t \geq 2)$, it is called as digit set. We say that $(\mathcal{A}, M)$ is a number system, if every $n \in \mathbb{Z}_k$ has a finite representation in the following form:

$$n = a_0 + Ma_1 + \cdots + M^{h-1}a_{h-1}, \qquad a_j \in \mathcal{A} \quad \text{for} \quad h > 0. \tag{2.1}$$

Let us define the function $J : \mathbb{Z}_k \to \mathbb{Z}_k$ in the following way. For every $n \in \mathbb{Z}_k$ there exists a unique $a_0 \in \mathcal{A}$ and $n_1 \in \mathbb{Z}_k$ such that $n = a_0 + Mn_1$, i.e. let $J(n) = n_1$ be. Let us define the set H:

$$H = \Big\{ z \big| z = \sum_{i=1}^{\infty} M^{-i} f_i, \ \ f_i \in \mathcal{A} \Big\}. \tag{2.2}$$

The set H is compact, it plays fundamental role in the investigation of the generalized number systems $(\mathcal{A}, M)$.

If $(\mathcal{A}, M)$ is number systems then

$$\cup_{n \in \mathbb{Z}_k}(H + n) = \mathbb{R}, \tag{2.3}$$

and $n_1, n_2 \in \mathbb{Z}_k$, $n_1 \neq n_2$ furthermore:

$$\lambda(H + n_1 \cap H + n_2) = 0, \tag{2.4}$$

where λ is the Lebesques measure. We define a set S, which consists of those $\gamma \in \mathbb{Z}_k$, $\gamma \neq 0$ for which:

$$H \cap H + \gamma \neq \emptyset. \tag{2.5}$$

The set (2.5) is denoted by B_γ and

$$B = \cup_\gamma B_\gamma. \tag{2.6}$$

We would like to determinate the Kolmogorov entropy of B_γ. We shell do it in the ring of quadratic integers in a given algebraic number fields. The detailed description of it can be found in the section 4.

In the next subsection we introduce the transition graph that will help us to investigate a walk along the graph which is analogous to dynamical systems.

2.1. Transition graph of number systems

We construct a finite directed labelled graph $G(S)$, according to the article [6], to use the function $Q : S \to S$, where $S \subseteq \mathbb{Z}_k \setminus \{0\}$, define a walk P along the transition graph.

Let $S \subseteq \mathbb{Z}_k$ be the set and $n \in S$, if $n \neq 0$ and $H + n \cap H \neq 0$. S can be computed by the following way:

Let $S_0 = \{n | \parallel n \parallel \leq 2L, \ \ n \neq 0\}$, i.e. n satisfies this condition, where L is an appropriate upper limit.

We can draw an edge from each $n \in S_0$ to n_1, if $n = b + Mn_1$ holds with a suitable $b \in \mathcal{B}(= \mathcal{A} - \mathcal{A})$.

$n \xrightarrow{\ \ b\ \ } n_1.$

At the end of this construction we delete all those nodes from which no edge goes out i.e. $\deg^-(n) = 0$ or ends, remove all coinciding edges as well. Each elements of the set S satisfy: $\deg^+(n) > 0$ and $\deg^-(n) > 0$. Finitely many repeating these steps we obtain the directed transition graph $G(S)$.

Because $B_\gamma = \{z | z \in H, z \in H + \gamma\}$ we can characterize the element z by their expansions.

Let $P := \gamma_1 \xrightarrow{\delta_1} \gamma_2 \xrightarrow{\delta_2} \gamma_3 \dots \gamma_{r-1} \xrightarrow{\delta_{r-1}} \gamma_r \dots$ be a walk on the graph $G(S)$, it is labelled by $(\delta_1, \delta_2, \dots \delta_{r-1} \dots)$, (i.e. for finite path of length r $Q^{(r-1)}(\gamma_1) = \gamma_r$). Each walk P along the transition graph can be represented by a sequence of labels: $\delta_1, \delta_2 \dots \delta_{r-1} \dots$. The equation $\delta_j = f_j - f'_j$ is satisfied by appropriate $f_i, f'_i \in \mathcal{A}$. Thus $z \in B_\gamma$, if

$$z = \sum_{i=1}^{\infty} M^{-i} f_i, \quad f_i \in \mathcal{A}. \tag{2.7}$$

for at least one sequence $f_1, f_2, \dots$

3. Kolmogorov-Sinai entropy

We will investigate the dynamical properties of measured time series which is denoted as $y_1, \dots, y_n$ and y_i means the measurement of the quantity y at time $t_i = t_0 + i\Delta t$, where the time interval $\Delta t > 0 \in \mathbb{R}$. It can be characterized by the generalized entropy of order q in natural sciences [14], [15], [16], [17].

We apply this idea on a lattice, similar to the sandbox method [18].

Let us consider a lattice C in the $\mathbb{R}^d$ with linear size ε ($\varepsilon > 0$, $\varepsilon \in \mathbb{R}$) such that C_j denotes the elementary box of lattice C, which fulfills the next conditions:

$$C = \cup_j C_j, \text{ and } C_j \cap C_i = \emptyset \text{ where } j, i \in \{0, \dots, N^d - 1\}. \tag{3.1}$$

The set of C_j provides a partition of $[0, N\varepsilon[^d \subset \mathbb{R}^d$. Let K be a compact set,

$$T_j = K \cap C_j \neq \emptyset, \text{ where } j = 1, \dots, M \text{ and } K \cap C_j = \emptyset \text{ for any other } C_j.$$

$$T = \cup_{j=1}^{M} T_j, \text{ where } T_j \cap T_i = \emptyset, \quad i \neq j.$$

The $\underline{x}^{(n)}$ denotes the path of length n in $\mathbb{R}^d$ i.e. time series of the measurement. Let the kth point of the path of length n be denoted by $x_k^{(n)} (k = 1, \dots n)$. We shall consider the sequence of $x_1^{(n)}, x_2^{(n)}, \dots, x_n^{(n)}$ as a time series on the lattice. The set K contains the points of some trajectories $x_k^{(n)} (k = 1, \dots, n)$.

We will use the notation of symbolic dynamics, because the concept of entropy is applied in different sciences, and it can be formulated on lattice.

Each path $\underline{x}^{(n)}$ for a finite n can be associated to a symbolic sequence $O_n = (o_1, o_2, \dots o_n)$, where the symbol $o_k \in \{1, \dots, M\}$ is represented by the index of the lattice element T_k.

A given sequence O_n occurs with probability $P(O_n)$. This quantity is independent of time, and fulfils:

$$\sum_{o_{n+1}=1}^{M} P(o_1, o_2, \dots o_n, o_{n+1}) = P(O_n), \tag{3.2}$$

where M is the number of the boxes with non-zero probability.

The information of the sequences O_n is:

$$I_n(1) = -\sum_{O_n} P(O_n) \ln P(O_n), \tag{3.3}$$

where n means the length of the symbol series and the summation is taken over all sequences with non-zero probabilities ($P(O_n) > 0$). We can get back the Rényi information of order-q $I_1(q)$ for $n = 1$ [4] and Shannon information $I_1(1)$ in a special case $q = 1$ [3].

Sinai introduced the mean rate of the created information, called the Kolmogorov-Sinai entropy as follows

$$K(1) = \limsup_{n\to\infty} I_n(1)/n. \tag{3.4}$$

Further dynamical properties of the symbol sequences are described by the Rényi information of order-q:

$$I_n(q) = \frac{1}{1-q} \ln \sum_{O_n} P(O_n)^q, \quad q \neq 1. \tag{3.5}$$

It leads to the order-q Rényi entropy [4]:

$$K(q) = \limsup_{\epsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \frac{1}{1-q} \ln \sum_{O_n} P(O_n)^q, \quad q \neq 1, \tag{3.6}$$

where $-\infty < q < \infty$, the lattice size $\varepsilon \to 0$, the length of the symbol sequent n goes to ∞ and the unit of the time interval Δt equals a constant in this definition.

By writing $p^q = p \exp(q-1) \ln p$ and expanding the exponent, the quantity $\lim_{q\to 1^+} K(q) = K(1)$ and $\lim_{q\to 0^+} K(q) \leq h$, where h is the topological entropy [5].

We can conclude by the Rényi entropy that the investigated system behaves chaotically or not.

It is known $K(1) = 0$ in an ordered system, but $K(1)$ is constant$\neq 0$ in chaotic (deterministic) system and $K(1)$ is infinite in a random system. Note $K(2) > 0$ is sufficient condition for chaos [15].

4. Numerical procedure

We determine the generalized entropy of order 2 denoted by $K(2)$ which provides an estimation of the Kolmogorov-Sinai entropy.

The idea of the generalized number system for $\mathbb{Z}_k$, introduced in the section (2), is equivalent to the ring of the integer $\mathbb{Z}[\theta]$ in $\mathbb{Q}[\theta]$, where θ is an algebraic integer and the element of the set forms $f(\theta) = v_0 + v_1\theta + \cdots + v_{n-1}\theta^{n-1}$, $v_j \in \mathbb{Z}$.

There exists a $\sigma \in \mathbb{Z}[\theta]$ and $I_\theta = \{\sigma\theta \big| \sigma \in \mathbb{Z}[\theta]\}$, supposing $\gamma_1 - \gamma_2 = \theta\sigma$, for $\gamma_1, \gamma_2 \in \mathbb{Z}[\theta]$ i.e. γ_1, γ_2 congruent modulo θ. The digit set is defined as $\mathcal{A} = \{a_0, a_0, \ldots, a_{t-1}\}$ $(\subset \mathbb{Z}[\theta])$.

The map $J : \mathbb{Z}[\theta] \to \mathbb{Z}[\theta]$ is introduced as $J(\alpha) = \alpha_1$, where there exists a unique $b \in \mathcal{A}$ in $(\mathcal{A}, \theta)$ and a unique $\alpha_1 \in \mathbb{Z}[\theta]$ for wich $\alpha = b + \theta\alpha_1$ and the expansion of α is defined by $\alpha_l = J^{(l)}(\alpha)$.

This identity between the $\mathbb{Z}_k$ and $\mathbb{Z}[\theta]$ was proved [6].

Kátai I. and Szabó J. studied the canonical number system for Gaussian complex integers [7]. They proved that $(\theta, \mathcal{A})$ is a canonical number system if and only if $\Re\theta < 0$ and $\Im\theta = \pm 1$, where θ is a Gaussian integer and $\mathcal{A} = \{0, 1, \ldots, N(\theta) - 1\}$ $(N(\theta) = \theta\overline{\theta})$.

The definition of fundamental set H (2.2), introduced in section (2), holds in this field. Let $\rho = 1/\theta$, where $\rho \in \mathbb{C}$, $0 < |\rho| < 1$ and $\mathcal{A} = \{0, 1\}$. Then the analogue set H:

$$H = \Big\{z \big| z = \sum_{i=1}^{\infty} \rho^i f_i, \quad f_i \in \mathcal{A}\Big\}. \tag{4.1}$$

Because $B_\gamma = H \cap H + \gamma$,

$$B_\gamma = \Big\{z \big| z \in H, z - \gamma \in H\Big\}. \tag{4.2}$$

Therefore all expansions of γ appear as

$$\gamma = \rho^1 e_1 + \rho^2 e_2 \ldots, \tag{4.3}$$

where $e_1, e_2 \cdots \in \mathcal{B} = \mathcal{A} - \mathcal{A}$. Then $e_i = f_i - f'_i$ holds, where $f_i, f'_i \in \mathcal{A}$, $(i = 1 \ldots)$. We determine all of possible values of the digit set f_i which follows from expressions (4.1),(4.2):

$$z = \rho^1 f_1 + \rho^2 f_2 \ldots. \tag{4.4}$$

The important results were published in the articles Indlekofer K.-H., Kátai I., Racsko P. [11] and Indlekofer K.-H., Járai A., Kátai I. [12].

The elements of the set B_γ are defined by (4.4), which contains z over infinite sums, we will approximate them with finite sums. The set $\widetilde{B}_\gamma$ contains these elements for some fixed k and γ, which are written as

$$x = \sum_{i=1}^{k} \rho^i f_i, \quad f_i \in \mathcal{A}. \tag{4.5}$$

We applied the Grassberger-Procaccia method for $\widetilde{B}_\gamma$, whose every element corresponds to a subset of B_γ.

In the article [18] we determined the sandbox dimension of the set B_γ approximately. Then we calculated the digits of the element of the set B_γ for finite limit.

The question arises, whether our calculation how reliable. Because the system shows chaotic behaviour, than the result of our calculation could lead to inaccurate.

4.1. Grassberger-Procaccia method for finite set

We shall give an estimation of the Kolmogorov-Sinai entropy on the time series to use the article [5]. This method is introduced on finite sets to find a good approximation of the quantity $K(2)$.

The compact set $K \subset \mathbb{R}^d$ consists of each point of some orbit $x_k^{(n)}$ of length n $(k = 1 \dots n)$. Each path corresponds to the series of the indices j for which $x_k^{(n)} \in T_j$ and $j \in \{1, \dots, M\}$.

Let us define the distance on this lattice, where the linear size ε of a box is taken as the unit length i.e. usually euclidean distance times $1/\varepsilon$. We introduce two constant values a and L, where a means the minimal distance between two points $a = \min\{|x - y| : x \neq y, x, y \in K\}$ and L is the diameter of the set K i.e. $L = \max\{|x - y| : x, y \in K\}$. Let us introduce a closed ball with center $x \in K$ and the radius r $(a \leq r \leq L, r \in \mathbb{R})$:

$$B(x, r) = \{y \big| |x - y| \leq r, y \in K\}. \tag{4.6}$$

We denote by $D(\underline{x}^{(n)}, r)$ the set which includes that path whose distance from the $\underline{x}^{(n)}$ of length n is less than or equals to r $(a \leq r \leq L)$. It is defined as

$$\begin{aligned} D(\underline{x}^{(n)}, r) = \{\underline{y}^{(n)} \mid |x_1^{(n)} - y_1^{(n)}| \leq r,\ |x_2^{(n)} - y_2^{(n)}| \leq r,\ \dots, \\ \dots,\ |x_n^{(n)} - y_n^{(n)}| \leq r\}, \end{aligned} \tag{4.7}$$

where $x_i^{(n)}$ and $y_i^{(n)}$ are the ith points of trajectories $\underline{x}^{(n)}$ and $\underline{y}^{(n)}$ $(i = 1, \dots n)$.

Let I' denote all sequences of the indices, where $\underline{n}' = (n_1', n_2' \dots n_n') \in I'$ and $n_1', n_2', \dots n_n' \in \{1, \dots, M\} \subset \{1, \dots, N^d\}$. The set $T_{\underline{n}'}^{(n)}$ contains the orbit of length n:

$$T_{\underline{n}'}^{(n)} = \{(x_1^{(n)}, x_2^{(n)} \dots x_n^{(n)}) | x_1^{(n)} \in T_{n_1'}, x_2^{(n)} \in T_{n_2'}, \dots x_n^{(n)} \in T_{n_n'}\}.$$

Analogously to the article [18] we define the map $\mu(T_{\underline{n}'}^{(n)})$ as measure

$$\mu(T_{\underline{n}'}^{(n)}) = \frac{|T_{\underline{n}'}^{(n)}|}{|T^{(n)}|}, \quad \text{where} \quad T^{(n)} = \cup_{\underline{n}' \in I'} T_{\underline{n}'}^{(n)}. \tag{4.8}$$

We note $T_{\underline{m}'}^{(n)} \cap T_{\underline{n}'}^{(n)} = \emptyset$, if $\underline{n}' \neq \underline{m}'$, $\underline{n}', \underline{m}' \in I'$ and $1 < |I'| \leq M^n$ and

$$\sum_{\underline{n}' \in I'} \frac{|T_{\underline{n}'}^{(n)}|}{|T^{(n)}|} = 1. \tag{4.9}$$

We can derive the quantity $\nu(D(\underline{x}^{(n)}, r)$ on the set $T^{(n)}$ in the following way:

$$\nu(D(\underline{x}^{(n)}, r) = \sum_{\underline{n}' \in I', |\underline{x}^{(n)} - \underline{y}^{(n)}| \leq r} \frac{|T_{\underline{n}'}^{(n)}|}{|T^{(n)}|} = \sum_{\underline{n}' \in I', |\underline{x}^{(n)} - \underline{y}^{(n)}| \leq r} \mu(T_{\underline{n}'}^{(n)}),$$

where $a \leq r \leq L$. The generalized entropy (3.6) in this case:

$$K(q) = \limsup_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \frac{1}{1-q} \ln \sum_{\underline{n}' \in I'} \mu(T_{\underline{n}'}^{(n)})^q, \quad q \neq 1, \tag{4.10}$$

which is for $q = 1$ the Kolmogorov entropy:

$$K(1) = -\limsup_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \mu(T_{\underline{n}'}^{(n)}) \ln \mu(T_{\underline{n}'}^{(n)}), \tag{4.11}$$

because μ is a probability measure. It can be seen that $K(q) > K(q')$ for every $q' > q$ and $K(2)$ is numerically close to $K(1)$. It is difficult to determine $K(1)$ directly from the series of elements of trajectories, therefore we approximate this quantity by $K(2)$.

4.1.1. The Grassberger-Procaccia method

Let us investigate the correlations between points of the long-time trajectories. Choose the orbits $\underline{x}^{(n)}$of length n, points $x_i^{(n)}$, $(i = 1, \dots, n)$ are elements of the set K.

The correlation sum $C_2(r,n)$ is defined as

$$C_2(r,n) = \frac{2}{n(n-1)} \sum_{\underline{x}^{(n)} \in T^{(n)}} |D(\underline{x}^{(n)}, r)|, \tag{4.12}$$

which means the number of pair of the orbits, where the maximum distance between any two paths of length n is less than or equals to r. Let us suppose that $C_2(r,n)$ scales like $C_2(r) \sim r^\eta$ and that up to a factor of order unity $C_2(r,n) \simeq \sum_{\underline{n}' \in I'} \mu^2(T_{\underline{n}'}^{(n)})$, where η has been called the correlation exponent. It has been proved that η estimates the fractal dimension D of the set S as r goes to zero. Furthermore $C_2(r,n) \sim r^\eta \exp(-nK_n(2)\Delta t)$, where $K_n(2)$ is the order-2 Rényi entropy:

$$K_n(2) = \frac{1}{2}\big[\ln C_2(r,n+1) - \ln C_2(r,n)\big]. \tag{4.13}$$

In the next section we determine this quantity for a set $\widetilde{B}_\gamma$.

4.2. Application of the Grassberger-Procaccia algorithm to a finite set $\widetilde{B}_\gamma$

In this section we present the numerical results which is obtained for a generalised number systems in quadratic integers.

In the article [11] it was proved that $\theta \in \mathbb{C}$ is a root of the $f(x) = x^2 - ax + 2$ polinom, where $a = 0, \pm 1, \pm 2$. The smallest ring is $\Delta = \{1, \Theta\}$. Then

$$\cup_{\gamma \in \Delta}(H + \gamma) = \mathbb{C} \tag{4.14}$$

$$\lambda((H+\gamma_1) \cap (H+\gamma_2)) = 0, \quad \gamma_1 \neq \gamma_2, \quad \gamma_1, \gamma_2 \in \Delta. \tag{4.15}$$

Here $\gamma = \sum_{\nu=0}^{l} a_\nu \Theta^\nu$, $a_\nu \in \mathcal{A}$ holds. For all these Θ values $(\Theta, \mathcal{A})$ is a canonical number system in a quadratic field extensions.

First, we construct the transition graph $G(S)$ as can be seen in Figure 1. The base is chosen to be $\theta = -1 - i$ and the digit set $\mathcal{A} = \{0, 1\}$ and the edge is labelled by element of set $\mathcal{B} = \{-1, 0, 1\}$ according to the article [11].

The steps of graph construction are the following:

- Every $\gamma \in \mathbb{Z}[\Theta]$ which fulfils the condition $|\gamma| \leq \sqrt{2} + 1$, we calculate $\eta = \gamma\Theta - \delta$ for $\delta \in \mathcal{B}$. A directed edge fits from γ to η, if $|\eta| \leq \sqrt{2} + 1$ holds.

-That γ is deleted which has no edge from γ and remove all edges which are directed to γ.

This process results the graph $G(S)$.

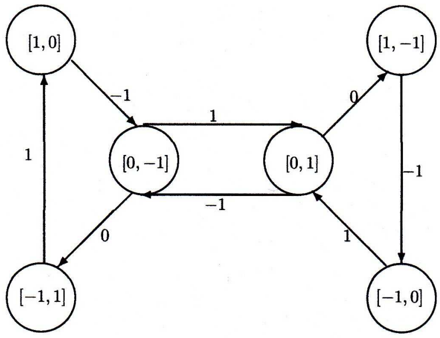

Figure 1. Transition graph for $\theta = -1 - i$ and $\mathcal{B} = \{-1, 0, 1\}$.

Each element of set $\widetilde{B}_\gamma$ which is introduced by the expression (4.5), corresponds to a walk of length finite k P along the transition graph, it is labelled by $(\delta_1, \delta_2, \ldots, \delta_k)$, $\delta_i \in \mathcal{B}$.

Let us take into consideration the process of the graph walking.

As a first step we choose one vertex along the graph $G(S)$ randomly. The basic idea of the graph walking is the minimal length orbit P, which contains all of possible edges at least ones $(\delta_1, \delta_2, \ldots \delta_k)$. Let us take in account that the outgoing degree of vertices q can be larger than 1 and the graph walking need to contain all edges, therefore the same node can appear more times resulting the perfect series of all directed ones. We consider all of series $f_1, \ldots, f_r$, $(f_i \in \mathcal{A})$ to the sequence of labels of edges accordance with section 2.1.

The correlation sum $C_2(r, n)$ is determined along the orbits of finite length n, where the distances between any trajectories of length n are smaller than r choosing for a fixed n. The quantity $\ln(C_2(r, n))$ vs $\ln(r)$ is plotted in Figure 2. for two cases $(n = 3, 9)$, where the slope of these lines means the correlation exponent η of the set $\widetilde{B}_\gamma$.

We present the entropy spectrum $K_n(2)$ vs n/N in Figure 3. on logarithmic scale, where N is maximal length of some orbits, in this case we choose 5 different lengths N. Increasing the value N of the graph walking, the entropy $K_n(2)$ decreases strictly monotonically with fix n/N and $K_n(2)$ converges to the theoretical value $K(2)$ for $N \to \infty$. The value of $K_n(2)$ goes to zero in the limit $N \to \infty$, therefore supposedly there is not chaos int this system.

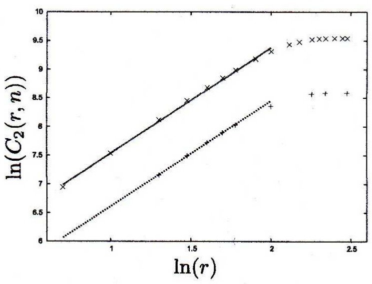

Figure 2. Correlation sum $\ln(C_2(r,n))$ vs $\ln(r)$ with $N = 20160$ and $n = 3, 9$.

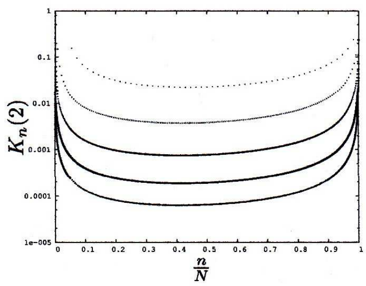

Figure 3. $K_n(2)$ vs n/N plotted on logarithm scale for 5 different length $N = 56, 336, 1680, 6720, 20160$ and $1 < n < N$.

5. Summary

In this paper we study the dynamical behaviour of generalized number systems in analogy to statistical physical phenomenon. We investigate a map on the transition graph to determine the entropy $K_n(2)$ using the correlation sum along the trajectories. The value $K(2)$ equals to 0 and this quantity is numerically close to $K(1)$, therefore we give a good approximation to the Kolmogorov-Sinai entropy $K(1)$. The conclusion of our calculation of the sandbox dimension [18] is correct, because this system does not show chaotic behaviour.

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