Annales Univ. Sci. Budapest., Sect. Comp. 40 (2013) 223–232

A NOTE ON AN ALTERNATIVE QUADRATIC EQUATION

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Dedicated to Professors Zoltán Daróczy and Imre Kátai on their 75th birthday

Communicated by Antal Járai

(Received March 21, 2013; accepted July 10, 2013)

Abstract. In this note we use the Ulam–Hyers stability for solving an alternative form of the quadratic equation.

1. Introduction

In the second half of 1970's several problems concerning alternative functional equations, mainly related to the Cauchy equation, have been proposed and solved by Roman Ger in [4] and [5] and by Marek Kuczma in [6].

In 1978, the author of the present note while investigating one of these problems concerning the alternative Cauchy equation, became aware of the existence of Hyers' theorem about stability of the additive equation and that stability result has been the main tool for solving that problem (see [2]).

Herein we consider a problem similar to Kuczma's one about the Cauchy equation, but concerning the quadratic equation. More precisely, we intend to describe the solutions of the following alternative quadratic functional equation

$$Qf(x,y) = f(xy) + f(xy^{-1}) - 2f(x) - 2f(y) \in \{0,1\} \tag{1}$$

Key words and phrases: Quadratic functional equation, stability, alternative equation.
2010 Mathematics Subject Classification: 39B55, 39B82.

where $f : G \to \mathbf{R}$ and G is a group with certain properties which will be specified below.

The first step is to recall the stability theorems concerning the quadratic equation: the first result is due to F. Skof (see [7]), later improved by P. Cholewa in [1]. The result which will be used herein is the following, proved by D. Yang in [8]:

Theorem 1.1. *Let $\mathcal{G}$ be an amenable group and assume that $f : G \to B$, where $(B, \|\cdot\|)$ is a Banach space, satisfies the inequality*

$$\|f(xy) + f(xy^{-1}) - 2f(x) - 2f(y)\| \le \delta \tag{2}$$

for some positive δ. Then there exists a unique quadratic function, i.e., solution of the equation

(3), $$q(xy) + q(xy^{-1}) - 2q(x) - 2q(y) = 0$$

such that

$$\|f(x) - q(x)\| \le \delta' \tag{4}$$

for every $x \in G$ and some δ' depending only on δ.

From now on we assume that the group G is amenable and e is its identity.

The next result we need is the analogue of that proved in [3, Th. 4] for the Cauchy functional equation. If

$$Qf(x, y) = f(xy) + f(xy^{-1}) - 2f(x) - 2f(y)$$

is bounded then, by the previous theorem, we have the decomposition $f(x) = q(x) + k(x)$, with q quadratic and k bounded. Our aim is to provide information on the range of the bounded function k.

Theorem 1.2. *Let $f : G \to B$, where B is a Banach space and let M be a bounded subset of B. If $Qf(x, y) \in M$, then $f(x) = q(x) + k(x)$, where q is quadratic and the range of k is contained in $\frac{1}{2}\overline{C(-M)}$, where $\overline{C(-M)}$ is the closure of the convex hull of $-M$.*

Proof. By the stability result, we have the decomposition

$$f(x) = q(x) + k(x)$$

with q quadratic and k bounded.

Since $q(e) = 0$, we have $f(e) = k(e) = -\frac{1}{2}m_0$, for certain $m_0 \in M$. Fix $x \in G$ and consider the value $k(x) =: u$. We have

$k(x^2) - 4k(x) = Qf(x, x) - f(e)$, hence $k(x^2) = 4u + \dfrac{1}{2}m_0 + m_1$ for some $m_1 \in M$.

By induction we obtain for every positive integer s

$$k(x^s) = s^2 u + \frac{s-1}{2} m_0 + \sum_{i=1}^{s-1} (s-i) m_i,$$

for some $m_i \in M$. By dividing by s^2 we have

$$\frac{k(x^s)}{s^2} = u + \frac{1}{2} \sum_{i=1}^{s-1} \frac{2(s-i)}{s^2} m_i + \frac{1}{2s} m_0 - \frac{1}{2s^2} m_0.$$

Clearly, $\sum_{i=1}^{s-1} \frac{2(s-i)}{s^2} m_i + \frac{1}{2s} m_0 \in C(M)$; taking the limit as $s \to \infty$ and remembering that k is bounded, we get

$$u + \frac{1}{2}\mu = 0, \text{ where } \mu = \lim_{s\to\infty} \sum_{i=1}^{s-1} \frac{2(s-i)}{s^2} m_i + \frac{1}{2s} m_0,$$

thus, $u \in \frac{1}{2}\overline{C(-M)}$. ■

Theorem 1.3. *In the hypotheses of Theorem* 1.2*, the range of* k *is contained in the set* $K = \Big\{ -\sum_{i=1}^{\infty} \frac{m_i}{4^1} - \frac{m_0}{6} : m_i \in M,\ m_0 = -2k(e) \Big\}$.

Proof. By Theorem 1.2 the range of k is contained in $\frac{1}{2}\overline{C(-M)}$ and we have $k(e) = -\frac{m_0}{2}$ for some $m_0 \in M$. From

$$Qk(x,x) = k(x^2) + k(e) - 4k(x) = k(x^2) - 4k(x) - \frac{m_0}{2} \in M,$$

we obtain

$$k(x^2) = 4k(x) + \frac{m_0}{2} + m_1 \in \frac{1}{2}\overline{C(-M)}, \text{ for some } m_1 \in M,$$

hence

$$k(x) \in \Big[\frac{1}{8}\overline{C(-M)} - \frac{m_0}{8} - \frac{m_1}{4}\Big] \cap \Big[\frac{1}{2}\overline{C(-M)}\Big].$$

It is easy to see that

$$\frac{1}{8}\overline{C(-M)} - \frac{m_0}{8} - \frac{m_1}{4} \subset \frac{1}{2}\overline{C(-M)},$$

thus

$$k(x) \in \bigcup_{m_1 \in M} \Big[\frac{1}{8}\overline{C(-M)} - \frac{m_0}{8} - \frac{m_1}{4}\Big].$$

We claim that

$$k(x) \in \bigcup_{m_1,m_2,\cdots,m_n \in M} \Big[\frac{1}{2\cdot 4^n}\overline{C(-M)} - \frac{m_0}{2}\sum_{i=1}^{n}\frac{1}{4^i} - \sum_{i=1}^{n}\frac{m_{n+1-i}}{4^i}\Big] =: K_n.$$

The proof is by induction. Consider $n+1$ and

$$4k(x) + \frac{m_0}{2} + m_{n+1} \in \frac{1}{2\cdot 4^n}\overline{C(-M)} - \frac{m_0}{2}\sum_{i=1}^{n}\frac{1}{4^i} - \sum_{i=1}^{n}\frac{m_{n+1-i}}{4^i}$$

for some $m_1, m_2, \cdots, m_{n+1} \in M$. Hence

$$k(x) \in \frac{1}{2\cdot 4^{n+1}}\overline{C(-M)} - \frac{m_0}{2}\sum_{i=1}^{n+1}\frac{1}{4^i} - \sum_{i=1}^{n}\frac{m_{n+1-i}}{4^i} - \frac{m_{n+1}}{4}$$

and

$$k(x) \in \bigcup_{m_1,m_2,\cdots,m_{n+1} \in M} \Big[\frac{1}{2\cdot 4^{n+1}}\overline{C(-M)} - \frac{m_0}{2}\sum_{i=1}^{n+1}\frac{1}{4^i} - \sum_{i=1}^{n+1}\frac{m_{n+2-i}}{4^i}\Big] = K_{n+1}.$$

It is not difficult to prove that $K_{n+1} \subset K_n$, then

$$k(x) \in \bigcap_{n=1}^{\infty} K_n =: K$$

and

$$K = \Big\{ -\sum_{i=1}^{\infty}\frac{m_i}{4^1} - \frac{m_0}{6} : m_i \in M,\ m_0 = -2k(e)\Big\}. \qquad \blacksquare$$

2. Alternative quadratic equation

We assume that $f : G \to \mathbf{R}$, where G is an amenable group and, as stated before, we intend to find the solutions of the following alternative equation:

$$Qf(x,y) = f(xy) + f(xy^{-1}) - 2f(x) - 2f(y) \in \{0,1\}$$

Thanks to Theorem 1.1, we transform the previous problem into the following

$$Qk(x,y) = k(xy) + k(xy^{-1}) - 2k(x) - 2k(y) \in \{0,1\}, \tag{5}$$

where the function k is bounded. By setting $x = y = 0$, we have $k(e) \in \{-\frac{1}{2}, 0\}$.

By setting $p(x) := -k(x) - \frac{1}{2}$ we see that $k(e) = -\frac{1}{2}$ implies $p(e) = 0$ and $Qp(x,y) \in \{0,1\}$. Thus, we can consider only the case $k(e) = 0$ and investigate the problem

$$k(xy) + k(xy^{-1}) - 2k(x) - 2k(y) \in \{0,1\}, \qquad k(e) = 0. \tag{6}$$

Theorem 1.3 applied to this situation gives that the range of k is contained in the set $K = \{-\sum_{n=1}^{\infty} \frac{\alpha_n}{4^n} : \ \alpha_n \in \{0,1\}\} \subset [-\frac{1}{2}, 0]$.

Writing the set K in the form

$$K = \{-\frac{1}{3} \sum_{n=1}^{\infty} \frac{3\alpha_n}{4^n} : \ \alpha_n \in \{0,1\}\}$$

we see that it is obtained by a procedure similar to that of the construction of the ternary Cantor set. In this case we take the unit interval, divide it in 4 equal parts, say $[0,1/4], [1/4,1/2], [1/2,3/4]$ and $[3/4,1]$ and eliminate the open central interval $(1/4,3/4)$. Proceeding in this way and multiplying the resulting set by $-\frac{1}{3}$ we obtain K.

It should be noted that the numbers in K have a unique representation in the form $-\sum_{n=1}^{\infty} \frac{\alpha_n}{4^n}$ with $\alpha_n \in \{0,1\}$.

Consider the set $Z_k = \{x \in G : k(x) = 0\}$ and put $x, y \in Z_k$ in equation (6): we have

$$k(xy) + k(xy^{-1}) \in \{0,1\}$$

and, since k cannot assume non negative values, this forces $k(xy) = k(xy^{-1}) = = 0$, i.e., Z_k is a subgroup of G. Since we are obviously looking for the non trivial solutions, we assume that Z_k is a proper subgroup of G.

Take now $x \notin Z_k$ and let $k(x) = -\sum_{n=1}^{\infty} \frac{\alpha_n}{4^n}$ for some sequence $\{\alpha_n\} \in \{0,1\}^{\mathbf{N}}$. Then

$$k(x^2) - 4k(x) \in \{0,1\} \ \Leftrightarrow k(x^2) \in \Big\{ -\sum_{n=1}^{\infty} \frac{\alpha_n}{4^{n-1}}, 1 - \sum_{n=1}^{\infty} \frac{\alpha_n}{4^{n-1}} \Big\}.$$

If $\alpha_1 = 0$, then

$$\sum_{n=1}^{\infty} \frac{\alpha_n}{4^{n-1}} = \sum_{n=2}^{\infty} \frac{\alpha_n}{4^{n-1}} \le \frac{1}{3}, \quad \text{hence} \quad 1 - \sum_{n=1}^{\infty} \frac{\alpha_n}{4^{n-1}} \ge \frac{2}{3}.$$

Thus,

$$k(x^2) = -\sum_{n=1}^{\infty} \frac{\alpha_n}{4^{n-1}} = -\sum_{n=2}^{\infty} \frac{\alpha_n}{4^{n-1}}.$$

If $\alpha_1 = 1$, then

$$\sum_{n=1}^{\infty} \frac{\alpha_n}{4^{n-1}} = 1 + \sum_{n=2}^{\infty} \frac{\alpha_n}{4^{n-1}} \geq 1, \quad \text{hence} \quad 1 - \sum_{n=1}^{\infty} \frac{\alpha_n}{4^{n-1}} \leq 0.$$

Thus,

$$k(x^2) = 1 - \sum_{n=1}^{\infty} \frac{\alpha_n}{4^{n-1}} = - \sum_{n=2}^{\infty} \frac{\alpha_n}{4^{n-1}}.$$

If we identify $k(x)$ with the sequence $\{\alpha_n\}_{n=1}^{\infty}$, then $k(x^2)$ is identified by $\{\alpha_{n+1}\}_{n=1}^{\infty}$.

Now we compute $k(x^3)$. From equation (6) with x^2 instead of x and x instead of y, we obtain

$$k(x^3) + k(x) - 2k(x^2) - 2k(x) = k(x^3) - 2k(x^2) - k(x) \in \{0, 1\}$$

whence

$$k(x^3) \in \Big\{ - \sum_{n=1}^{\infty} \frac{\alpha_n + 2\alpha_{n+1}}{4^n}, 1 - \sum_{n=1}^{\infty} \frac{\alpha_n + 2\alpha_{n+1}}{4^n} \Big\}.$$

If

$$k(x^3) = - \sum_{n=1}^{\infty} \frac{\alpha_n + 2\alpha_{n+1}}{4^n},$$

then for having $k(x^3) \in K$, by Theorem 3, we must have

$$\sum_{n=1}^{\infty} \frac{\alpha_n + 2\alpha_{n+1}}{4^n} = \sum_{n=1}^{\infty} \frac{a_n}{4^n}$$

for some sequence $\{a_n\}$ with $a_n = 0, 1$. We prove that this is possible if and only if $\alpha_n + 2\alpha_{n+1} \in \{0, 1\}$. If not, let n_0 be the first index such that $\alpha_n + 2\alpha_{n+1} \neq a_n$; we have two possibilities: either $a_{n_0} < \alpha_{n_0} + 2\alpha_{n_0+1}$ or $1 = a_{n_0} > \alpha_{n_0} + 2\alpha_{n_0+1} = 0$.

In the first case we have

$$\sum_{n=n_0}^{\infty} \frac{a_n}{4^n} \leq \frac{a_{n_0}}{4^{n_0}} + \frac{1}{3 \cdot 4^{n_0}} = \frac{a_{n_0} + 1/3}{4^{n_0}} < \frac{a_{n_0} + 1}{4^{n_0}} \leq$$

$$\leq \frac{\alpha_{n_0} + 2\alpha_{n_0+1}}{4^{n_0}} \leq \sum_{n=n_0}^{\infty} \frac{\alpha_n + 2\alpha_{n+1}}{4^n},$$

a contradiction.

In the second case we have

$$\sum_{n=n_0}^{\infty}\frac{\alpha_n+2\alpha_{n+1}}{4^n}\le\frac{1}{4^{n_0}}\le\frac{1}{4^{n_0}}+\sum_{n=n_0+1}^{\infty}\frac{a_n}{4^n}=\sum_{n=n_0}^{\infty}\frac{a_n}{4^n}.$$

This forces the equality and $a_n=0$, $\alpha_n+2\alpha_{n+1}=3$ for all $n>n_0$, i.e., $\alpha_n=1$ for all $n>n_0$. Hence $0=\alpha_{n_0}+2\alpha_{n_0+1}\ge 2$: a contradiction.

Thus $\alpha_n+2\alpha_{n+1}\in\{0,1\}$ and this is possible if and only if $\alpha_{n+1}=0$ for every $n\ge 1$. Thus, either $\alpha_n=0$ for every $n\ge 0$, i.e., $x\in Z_k$, impossible, or $\alpha_1=1$ and $\alpha_n=0$ for every $n\ge 2$. This means that $k(x)=-\frac{1}{4}$. In this case $k(x^2)=0$, i.e., $x^2\in Z_k$ and $k(x^3)=-\frac{1}{4}$.

The other possibility is

$$k(x^3)=1-\sum_{n=1}^{\infty}\frac{\alpha_n+2\alpha_{n+1}}{4^n}=\sum_{n=1}^{\infty}\frac{3}{4^n}-\sum_{n=1}^{\infty}\frac{\alpha_n+2\alpha_{n+1}}{4^n}=$$
$$=-\sum_{n=1}^{\infty}\frac{\alpha_n+2\alpha_{n+1}-3}{4^n}.$$

The condition $k(x^3)\in K$ implies $\alpha_n+2\alpha_{n+1}-3=0$, i.e., $\alpha_n+2\alpha_{n+1}=3$ for every $n\ge 1$, hence $\alpha_n=1$ for every $n\ge 1$.

This means that $k(x)=-\frac{1}{3}$. In this case $k(x^2)=-\frac{1}{3}$ and $k(x^3)=0$, i.e., $x^3\in Z_k$.

Thus, the group G is partitioned in three sets: the subgroup Z_k, and the sets $H_4:=\{x\in G : k(x)=-\frac{1}{4}\ \}$ and $H_3:=\{x\in G : k(x)=-\frac{1}{3}\ \}$, with $H_3\cup H_4\neq\emptyset$.

Assume that H_3 and H_4 are both non empty and take $x\in H_4$ and $y\in H_3$. Then

$$k(xy)+k(xy^{-1})+\frac{1}{2}+\frac{2}{3}\in\{0,1\}\ \Leftrightarrow k(xy)+k(xy^{-1})\in\{-\frac{7}{6},-\frac{1}{6}\}.$$

Since $k(xy),k(xy^{-1})\in\{-\frac{1}{3},-\frac{1}{4},0\}$, we can't obtain the values $-\frac{7}{6}$ and $-\frac{1}{6}$. Thus, we conclude that either $H_3=\emptyset$ or $H_4=\emptyset$.

Suppose $H_4\neq\emptyset$ and take $x,y\in H_4$, then $y^{-1}\in H_4$ and

$$k(xy)+k(xy^{-1})-2k(x)-2k(y)=k(xy)+k(xy^{-1})+1\in\{0,1\};$$

the only possibilities is $k(xy)=k(xy^{-1})=0$, i.e., $xy,xy^{-1}\in Z_k$. Moreover, from $x\in H_4$ and $y\in Z_k$, so $x^{-1}y\notin Z_k$, we obtain

$$k(x^{-1}yx)+k(x^{-1}yx^{-1})-2k(x^{-1}y)-2k(x)=k(x^{-1}yx)+k(x^{-1}yx^{-1})+1\in\{0,1\}$$

whence $k(x^{-1}yx) = k(x^{-1}yx^{-1}) = 0$, i.e., $x^{-1}yx, x^{-1}yx^{-1} \in Z_k$. Thus, Z_k is a normal subgroup of G of index 2 and $H_4 = xZ_k$ and G/Z_k is the cyclic group of order 2.

Suppose $H_3 \neq \emptyset$, from $x \in H_3$ and $y \in Z_k$, so $x^{-1}y \notin Z_k$, we obtain

$$\begin{aligned} &k(x^{-1}yx) + k(x^{-1}yx^{-1}) - 2k(x^{-1}y) - 2k(x) = \\ &= k(x^{-1}yx) + k(x^{-1}yx^{-1}) + \frac{4}{3} \in \{0, 1\} \end{aligned}$$

whence $k(x^{-1}yx)+k(x^{-1}yx^{-1}) \in \{-\frac{4}{3}, -\frac{1}{3}\}$. The only possibility is $k(x^{-1}yx)+$ $+k(x^{-1}yx^{-1}) = -\frac{1}{3}$, i.e., either $x^{-1}yx \in Z_k$ or $x^{-1}yx^{-1} \in Z_k$. This doesn't permit to conclude that Z_k is normal.

Take now $x, y \in H_3$, then

$$k(xy) + k(xy^{-1}) - 2k(x) - 2k(y) = k(xy) + k(xy^{-1}) + \frac{4}{3} \in \{0, 1\}$$

so $k(xy) + k(xy^{-1}) = -\frac{1}{3}$ and either $xy \in H_3$ and $xy^{-1} \in Z_k$, or vice–versa.

Hence the set $H_3 \times H_3$ is partitioned in two disjoint sets:

$$L_3 := \{(x, y) \in H_3 \times H_3 : xy \in H_3\}, \quad M_3 := \{(x, y) \in H_3 \times H_3 : xy \in Z_k\}.$$

Clearly we have

$$(x, y) \in L_3 \Leftrightarrow (x, y^{-1}) \in M_3.$$

From this follows that $(x, x) \in L_3$, $(x, x^2) \in M_3$.

The following theorem shows that the necessary conditions we have obtained assuming that k is a non trivial solution of equation (6), are also sufficient.

Theorem 2.1. *Non zero solutions of problem (6) exist only in this two cases:*

(i) *the group G has a normal subgroup Z of index 2; the solution k assume the value zero on Z and $-\frac{1}{4}$ on $G \setminus Z$;*

(ii) *the group G has a subgroup Z such that the set $(G \setminus Z) \times (G \setminus Z)$ can be split in two (disjoint) sets L and M with the property that $(x, y) \in L$ if $xy \notin Z$ and $(x, y) \in M$ if $(x, y^{-1}) \in L$; the solution k assume the value zero on Z and $-\frac{1}{3}$ on $G \setminus Z$.*

Proof. Let $x, y \in G$, we have the following possibilities:

Case (i):

a) $x, y \in Z$, then $xy, xy^{-1} \in Z$ and $Qk(x, y) = 0$.

b) $x \notin Z$, $y \in Z$, then $xy, xy^{-1} \notin Z$ and $Qk(x, y) = -\frac{1}{4} - \frac{1}{4} + \frac{1}{2} = 0$; the same if $x \in Z$ and $y \notin Z$.

c) $x, y \notin Z$, then $xy, xy^{-1} \in Z$ and $Qk(x,y) = \frac{1}{2} + \frac{1}{2} = 1$.

Case (ii):

d) $x, y \in Z$, then $xy, xy^{-1} \in Z$ and $Qk(x,y) = 0$.

e) $x \notin Z$, $y \in Z$, then $xy, xy^{-1} \notin Z$ and $Qk(x,y) = -\frac{1}{3} - \frac{1}{3} + \frac{2}{3} = 0$; the same if $x \in Z$ and $y \notin Z$.

f) $x, y \notin Z$; we have two possibilities. First, let $(x,y) \in L$, then $xy \notin Z$ and $xy^{-1} \in Z$ since $(x, y^{-1}) \in M$, so we obtain $Qk(x,y) = -\frac{1}{3} + \frac{2}{3} + \frac{2}{3} = 1$. Second case is $(x,y) \in M$, then $xy \in Z$ and $xy^{-1} \notin Z$, so again we have $Qk(x,y) = 1$. ■

Note that if G is Abelian, so Z is normal, the case (ii) of Theorem 2.1 becomes simply the following: *Z is a subgroup of G of index 3 and the solution k assume the value zero on Z and $-\frac{1}{3}$ on $G \setminus Z$.*

We have assumed that our functions f and k are real, but exactly in the same way we can treat the case $f : G \to B$ where B is a Banach space and equation (1) is substituted by the following

$$Qf(x,y) = f(xy) + f(xy^{-1}) - 2f(x) - 2f(y) \in \{0, \beta\}$$

where β is a fixed element in B which we can assume of norm 1. By Theorem 1.2 the range of f is contained in the segment having end points 0 and $-\frac{\beta}{2}$, thus we are reduced to the one dimensional case.

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